McMaster University

Mathematics and Statistics

$$\int_M d\omega = \int_{\partial M} \omega$$

Mathematics 3A03
Real Analysis I

Instructor: David Earn

Lecture 22
Metric Spaces
Monday 10 March 2025

Announcements

- New, exciting topic today. . .

Metric Spaces

The metric structure of $\mathbb{R}$

Definition (Absolute Value function)

For any $x \in \mathbb{R}$,

$$|x| \overset{\text{def}}{=} \begin{cases} x & \text{if } x \geq 0, \\ -x & \text{if } x < 0. \end{cases}$$

Theorem (Properties of the Absolute Value function)

For all $x, y \in \mathbb{R}$:

1. $-|x| \leq x \leq |x|$.
2. $|xy| = |x|\,|y|$.
3. $|x + y| \leq |x| + |y|$.
4. $|x| - |y| \leq |x - y|$.

The metric structure of $\mathbb{R}$

Definition (Distance function or metric)

The distance between two real numbers x and y is

$$d(x,y) = |x-y| \ .$$

Theorem (Properties of distance function or metric)

1. $d(x,y) \geq 0$ — *distances are positive or zero*
2. $d(x,y) = 0 \iff x = y$ — *distinct points have distance* > 0
3. $d(x,y) = d(y,x)$ — *distance is symmetric*
4. $d(x,y) \leq d(x,z) + d(z,y)$ — *the triangle inequality*

Note: Any function satisfying these properties can be considered a "distance" or "metric".

The metric structure of $\mathbb{R}$

Given $d(x, y) = |x - y|$, the properties of the distance function are equivalent to:

Theorem (Metric properties of the absolute value function)

For all $x, y \in \mathbb{R}$:

1. $|x| \geq 0$
2. $|x| = 0 \iff x = 0$
3. $|x| = |-x|$
4. $|x + y| \leq |x| + |y|$ *(the triangle inequality)*

Slick proof of the triangle inequality

Theorem (The Triangle Inequality for the standard metric on $\mathbb{R}$)

$|x + y| \leq |x| + |y|$ *for all* $x, y \in \mathbb{R}$.

Proof.

Let $s = \operatorname{sign}(x + y)$. Then

$$|x + y| \;=\; s(x + y) \;=\; sx + sy \;\leq\; |x| + |y| \;,$$

as required. □

A non-standard metric on $\mathbb{R}$

Example (finite distance between every pair of real numbers)

Let $f(x) = \frac{x}{1+x}$, and define $d(x,y) = f(|x-y|)$. Prove that $d(x,y)$ can be interpreted as a distance between x and y because it satisfies all the properties of a metric.

Proof: The only metric property that is non-trivial to prove is the triangle inequality. Note that $f(x)$ is an increasing function on $[0,\infty)$, so the usual triangle inequality, $|x-y| \leq |x-z| + |z-y|$, implies

$$\begin{aligned} f(|x-y|) \leq f(|x-z| + |z-y|) &= \frac{|x-z| + |z-y|}{1+|x-z| + |z-y|} \\ &= \frac{|x-z|}{1+|x-z| + |z-y|} + \frac{|z-y|}{1+|x-z| + |z-y|} \\ &\leq \frac{|x-z|}{1+|x-z|} + \frac{|z-y|}{1+|z-y|} = f(|x-z|) + f(|z-y|) \end{aligned}$$

i.e., $d(x,y) \leq d(x,z) + d(z,y)$. □

Poll

- Go to
  https://www.childsmath.ca/childsa/forms/main_login.php
- Click on Math 3A03
- Click on Take Class Poll
- Fill in poll **Metric spaces: Is "$=$ vs $\neq$" a metric?**
- `Submit`.

Discrete metric

Example (Discrete metric on $\mathbb{R}$)

Let $d(x,y) = \begin{cases} 0 & x = y, \\ 1 & x \neq y. \end{cases}$ Is d a metric on $\mathbb{R}$?

By definition, $d(x,y)$ is non-negative, zero iff $x = y$, and symmetric. For the triangle inequality, if $x = y$ then $d(x,y) = 0$ so the inequality holds for any z. If $x \neq y$ then $d(x,y) = 1$, and at least one of x and y must not equal z, so the inequality says either $1 \leq 1$ or $1 \leq 2$.

Example (Discrete metric on any set X)

The argument that $d(x,y)$ is a metric on $\mathbb{R}$ has nothing to do with $\mathbb{R}$ specifically. $d(x,y)$ is a metric on <u>any</u> set X.

General metric space (X, d)

Definition (Metric space)

A ***metric space*** (X, d) is a non-empty set X together with a distance function (or ***metric***) $d : X \times X \to \mathbb{R}$ satisfying

1. $d(x, y) \geq 0$ *distances are positive or zero*
2. $d(x, y) = 0 \iff x = y$ *distinct points have distance* > 0
3. $d(x, y) = d(y, x)$ *distance is symmetric*
4. $d(x, y) \leq d(x, z) + d(z, y)$ *the triangle inequality*

Much of our analysis of sequences of real numbers and topology of $\mathbb{R}$ generalizes to any metric space. Very often, definitions and proofs depend only on the the existence of a metric, not on $|x|$ specifically. Many useful inferences can be made by identifying a metric on a space of interest.

Examples of metric spaces

Example (Metric spaces (X, d))

- $X = \mathbb{Q}$, with the standard metric $d(x, y) = |x - y|$.
  As $\mathbb{Q} \subset \mathbb{R}$, each condition for d is satisfied in $\mathbb{Q}$.

  How different is $(\mathbb{Q}, d)$ from $(\mathbb{R}, d)$?

- $X = \mathbb{N}$, with the standard metric $d(x, y) = |x - y|$.
  As $\mathbb{N} \subset \mathbb{R}$, each condition for d is satisfied in $\mathbb{N}$.

- $X = \mathbb{R}^2$ with $d(x, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$, where we write the vectors $x = (x_1, x_2)$, $y = (y_1, y_2) \in \mathbb{R}^2$.

- $X = \mathbb{R}^n$ with $d(x, y) = \sqrt{\sum_{i=1}^{n}(x_i - y_i)^2}$, where $x = (x_1, \dots, x_n)$, $y = (y_1, \dots, y_n) \in \mathbb{R}^n$.

  This metric on $\mathbb{R}^n$ is called the ***Euclidean distance***.

Metrics from norms

The Euclidean metric on $\mathbb{R}^n$ is the (Euclidean) length of the difference of two vectors. This connection between length and distance generalizes to any vector space in which *length* is defined.

Definition (Norm)

A ***norm*** on a vector space X is a real-valued function on X such that if $x, y \in X$ and $\alpha \in \mathbb{R}$ then

1. $\|x\| \geq 0$ and $\|x\| = 0$ iff x is the zero element in X;
2. $\|\alpha x\| = |\alpha| \, \|x\|$;
3. $\|x + y\| \leq \|x\| + \|y\|$.

A vector space X equipped with a norm $\|\cdot\|$ is said to be a ***normed vector space***. Any norm $\|\cdot\|$ ***induces*** a metric d via

$$d(x, y) = \|x - y\| \, .$$

Proving that a function is a norm is not necessarily easy. Let's try for the Euclidean norm... To that end, recall the notion of inner product ...

Norms from inner products

Definition (Inner product)

An ***inner product*** on a vector space V over $\mathbb{R}$ is a function

$$\langle \cdot, \cdot \rangle : V \times V \to \mathbb{R}$$

such that for all $u, v, w \in V$ and all scalars $\alpha \in \mathbb{R}$:

1. $\langle u, v \rangle = \overline{\langle v, u \rangle}$ *conjugate symmetry*
2. $\langle \alpha u + v, w \rangle = \alpha \langle u, w \rangle + \langle v, w \rangle$ *linearity in 1st argument*
3. $\langle v, v \rangle \geq 0$ with equality iff $v = 0$ *positive definiteness*

A vector space equipped with an inner product is called an *inner product space*.

Definition (Inner Product Norm)

The norm induced by an inner product $\langle \cdot, \cdot \rangle$ is $\|u\| = \sqrt{\langle u, u \rangle}$.

Norms from inner products

Theorem (Cauchy-Schwarz inequality)

Let V be a (real) inner product space with inner product $\langle \cdot, \cdot \rangle$. For all vectors $u, v \in V$, we have

$$|\langle u, v \rangle| \;\le\; \|u\| \, \|v\|$$

where $\|u\| = \sqrt{\langle u, u \rangle}$ is the norm induced by the inner product.

Proof.

The standard proof begins with an idea that probably took someone a long time to think of: Since $\langle v, v \rangle \ge 0$ for any $v \in V$, for any $t \in \mathbb{R}$ we have

$$\begin{aligned} 0 \;\le\; \langle u + tv, u + tv \rangle \;&=\; \langle u, u \rangle + t \langle u, v \rangle + t \langle v, u \rangle + t^2 \langle v, v \rangle \\ &=\; \langle u, u \rangle + 2t \langle u, v \rangle + t^2 \langle v, v \rangle \end{aligned}$$

This is a quadratic polynomial in t, which is non-negative for all $t \in \mathbb{R}$. Hence, this quadratic has at most one real root. Consequently, its discriminant is non-positive, *i.e.*, $(2 \langle u, v \rangle)^2 - 4 \langle u, u \rangle \langle v, v \rangle \;\le\; 0$.

continued...

Norms from inner products

Proof of Cauchy-Schwarz inequality (continued).

Simplifying the non-positive discriminant condition, we have

$$(\langle u, v\rangle)^2 \;\le\; \langle u, u\rangle \langle v, v\rangle \;.$$

Taking square roots, we have

$$|\langle u, v\rangle| \;\le\; \|u\| \, \|v\| \;,$$

as required. □

How might you come up with such a proof?

Perhaps by guessing the result (based on knowing it in $\mathbb{R}^2$) and then working backwards.

McMaster University

Mathematics and Statistics

$$\int_M d\omega = \int_{\partial M} \omega$$

Mathematics 3A03
Real Analysis I

Instructor: David Earn

Lecture 23
Metric Spaces II
Wednesday 12 March 2025

Announcements

- Participation deadline for Assignment 4 was at 11:25am today.

Last time. . .

- Introduction to metric spaces
  - Critical ingredient: the triangle inequality
- Cauchy-Schwarz inequality
  (proved for real inner product spaces)

Norms from inner products

If X is an inner product space, then Cauchy-Schwarz allows us to prove that the induced norm really is a norm (*i.e.*, satisfies the triangle inequality). For $x, y \in X$, we have

$$\begin{aligned} \|x+y\|^2 &= \langle x+y, x+y\rangle \\ &= \langle x,x\rangle + \langle x,y\rangle + \langle y,x\rangle + \langle y,y\rangle \\ &= \|x\|^2 + \|y\|^2 + 2\langle x,y\rangle \\ &\le \|x\|^2 + \|y\|^2 + 2\|x\|\,\|y\| \\ &= (\|x\| + \|y\|)^2 \end{aligned}$$

$$\implies \quad \|x+y\| \le \|x\| + \|y\| \,.$$

In particular, $\mathbb{R}^n$ with the usual "dot product" $\langle x, y\rangle = \sum_{i=1}^n x_i y_i$ induces the Euclidean norm $\|\cdot\|_2$, which therefore really is a norm, and $d(x,y) = \|x-y\|_2$ really is a metric (the Euclidean distance).

What about other norms induced by inner products?

Other metric spaces induced by inner products

We are accustomed to finite vectors:

$$x = (x_1, x_2) \in X = \mathbb{R}^2$$
$$x = (x_1, x_2, x_3) \in X = \mathbb{R}^3$$
$$x = (x_1, x_2, \ldots, x_n) \in X = \mathbb{R}^n$$

We can think of a sequence as an infinite vector:

$$x = (x_1, x_2, \ldots) \in X = \big\{\{x_n\} : n \in \mathbb{N}\big\}$$

The points in this space (X) are infinite-dimensional vectors.

We can think of an infinite vector as a function:

$$x_n = f(n) \quad \Longrightarrow \quad (x_1, x_2, \ldots) = \big(f(1), f(2), \ldots\big)$$

The points in this space are functions: $X = \big\{f \,\big|\, f : \mathbb{N} \to \mathbb{R}\big\}$.

So we can generalize to other spaces via functions, *e.g.*,

$$C[0,1] = \big\{f : [0,1] \to \mathbb{R} \,\big|\, f \text{ continuous}\big\}$$

All of the above spaces have a natural inner product, and hence a natural norm and metric.

Other metric spaces induced by inner products

Inner products that convert the spaces on the previous slide into (Euclidean) metric spaces:

$\mathbb{R}^n$: $\quad \langle x, y\rangle = \sum_{i=1}^{n} x_i y_i$

$\ell^2(\mathbb{R})$: $\quad \langle x, y\rangle = \sum_{i=1}^{\infty} x_i y_i$

$C[a, b]$: $\quad \langle f, g\rangle = \int_a^b f(x)g(x)\,dx$

Note: ℓ^2 includes only ***square-summable*** sequences: $\sum_{n=1}^{\infty} x_n^2 < \infty$

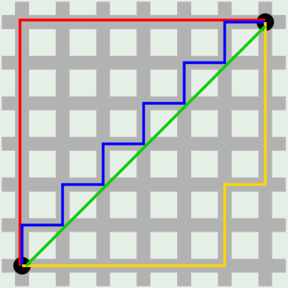
*Do we need to specify that $C[a, b]$ contains only **square-integrable** functions?*

Metrics from norms

Example (Taxicab distance)

Taxicab norm on $\mathbb{R}^n$

$$\|x\|_1 = \sum_{i=1}^{n} |x_i|$$



In taxicab geometry, the lengths of the **red**, **blue**, **green**, and **yellow** paths all equal 12, the taxicab distance between the opposite corners, and all four paths are shortest paths. Instead, in Euclidean geometry, the **red**, **blue**, and **yellow** paths still have length 12 but the **green** path is the unique shortest path, with length equal to the Euclidean distance between the opposite corners, $6\sqrt{2} \approx 8.49$.

Image and caption from Wikipedia article on "Taxicab geometry".

Note: The **green** path can be followed. All the points of $\mathbb{R}^2$ are still present when we measure distance with the taxicab metric. Any monotonic curve path that joins the two points can be approximated by an arbitrarily fine grid, and will have the same length. The metric does not impose a particular grid.

Metrics from norms

Example (p-metric)

p-norm on $\mathbb{R}^n$ (for $p \geq 1$) $\|x\|_p = \left(\sum_{i=1}^{n} |x_i|^p \right)^{\frac{1}{p}}$.

- $p = 1$ is the taxicab norm.
- $p = 2$ is the Euclidean norm.

What happens as $p \to \infty$? For any $x \in \mathbb{R}^n$, we have

$$\|x\|_p = \left(\sum_{i=1}^{n} |x_i|^p \right)^{\frac{1}{p}} = |x_k| \left(\sum_{i=1}^{n} \left| \frac{x_i}{x_k} \right|^p \right)^{\frac{1}{p}} \qquad (|x_k| > |x_i| \ \ \forall i \neq k)$$

$$= |x_k| \left(1 + \sum_{i \neq k} \left| \frac{x_i}{x_k} \right|^p \right)^{\frac{1}{p}} \xrightarrow{p \to \infty} |x_k|$$

What further work is required if $\nexists k \ni |x_k| > |x_i| \ \ \forall i \neq k$?

Therefore, we define $\|\cdot\|_\infty$ to be

Max norm on $\mathbb{R}^n$ $\|x\|_\infty = \max_{1 \leq i \leq n} |x_i|$

Poll

- Go to
  https://www.childsmath.ca/childsa/forms/main_login.php
- Click on Math 3A03
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- Fill in poll **Metric spaces: Which p-norms are induced?**
- Submit.

Metrics from norms

Example (p-metric)

Proving that $p = 1$ and $p = \infty$ yield norms is a *good exercise*.

Only $p = 2$ is induced by an inner product.

That the p-norms for $p \neq 1, 2, \infty$ are norms is harder to prove (but true), so

$$d_p(x, y) = \|x - y\|_p$$

is a metric on $\mathbb{R}^n$ for any $p \geq 1$.

Topology of metric spaces

We can generalize the notion of "neighbourhood of a point" to any metric space:

Definition (Open ball)

Let (X, d) be a metric space. If $x_0 \in X$ and $r > 0$ then the ***open ball of radius*** r ***about*** x_0 is

$$B_r(x_0) = \{x \in X : d(x, x_0) < r\}\,.$$

x_0 is said to be the ***centre*** of $B_r(x_0)$.

Note: The notation $B(x_0, r)$ is also common (and used in TBB).

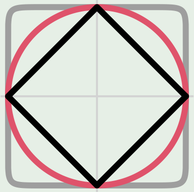
Definition (Neighbourhood)

A ***neighborhood*** of x is any set that contains an open ball $B_r(x)$ for some $r > 0$.

Topology of metric spaces

Example (Open balls in metric spaces)

- In the metric space $(\mathbb{R}, \mathit{standard})$, *i.e.*, $\mathbb{R}$ with $d(x,y) = |x-y|$, $B_r(x) = (x-r, x+r)$, an open interval of length $2r$ centred at x.
- In $\mathbb{R}^n$ with Euclidean metric $d(x,y) = \|x-y\|_2$, $B_r(x)$ has a spherical boundary (circular boundary if $n=2$).
- In $\mathbb{R}^n$ with a p-norm $\|\cdot\|_p$, the ball is not spherical. For $\mathbb{R}^2$ with the Taxicab metric $d(x,y) = \|x-y\|_1$, $B_r(x)$ is diamond shaped, and for $\mathbb{R}^2$ with the Max norm $\|\cdot\|_\infty$, $B_r(x)$ is a square. We write $B_r^p(x)$ for balls in the p-norm ("***p-balls***").



$B_r^p(x)$

- $p = 1$
- $p = 2$
- $p = 16$

Topology of metric spaces

Example (p-norm inequalities and containments)

The following inequalities relate the various p-norms on $\mathbb{R}^n$,

$$\|x\|_\infty \le \|x\|_2 \le \|x\|_1 \le n\,\|x\|_\infty\,, \quad \text{and} \quad \|x\|_1 \le \sqrt{n}\,\|x\|_2\,.$$

A good exercise.

Balls in the norm $\|\cdot\|_p$ are often written $B_r^p(x)$. The inequalities above imply that the following sets are *nested:*

$$B^2_{r/n}(x) \subset B^\infty_{r/n}(x) \subset B^1_r(x) \subset B^2_r(x) \subset B^\infty_r(x).$$

Another good exercise.

Example (Balls in the discrete metric)

For any set X, in the *discrete metric* the balls are simple, but strange. If $0 < r \le 1$ then $B_r(x) = \{x\}$, a single point! If $r > 1$ then $B_r(x) = X$, the whole space! You can't be "close" to a point x unless you are at x itself!

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Mathematics and Statistics

$$\int_M d\omega = \int_{\partial M} \omega$$

Mathematics 3A03
Real Analysis I

Instructor: David Earn

Lecture 25
Metric Spaces III
Monday 17 March 2025

Announcements

Discussion of metrics so far. . .

- Metrics induced by norms, *e.g.*, p-norms
- Metrics from norms induced by inner products
- Metric on any set: discrete metric
- Balls (p-balls, discrete balls)

Note:

- I added a note to the slide on the taxicab metric.

Topology of metric spaces

Definition (Convergence of a sequence in a metric space)

Let (X, d) be any metric space. A sequence $(x_n)_{n\in\mathbb{N}}$ ***converges*** to $x \in X$, written $x_n \xrightarrow{n\to\infty} x$ or $\lim_{n\to\infty} x_n = x$, if:

$$\forall \varepsilon > 0,\ \exists N \in \mathbb{N} \ \ni\ d(x_n, x) < \varepsilon \quad \forall n \geq N.$$

If the sequence does not converge to any $x \in X$, we say it ***diverges***.

Equivalently, $x_n \xrightarrow{n\to\infty} x$ if, for any ball $B_\varepsilon(x)$ centered at x, the sequence (x_n) lies inside that ball eventually ($\exists N \in \mathbb{N}$ such that $x_N \in B_\varepsilon(x)$), and stays inside it ($x_n \in B_\varepsilon(x)\ \forall n > N$), *i.e.*,

$$\forall \varepsilon > 0,\ \exists N \in \mathbb{N} \quad \ni \quad x_n \in B_\varepsilon(x) \quad \forall n \geq N.$$

Topology of metric spaces

Definition (Boundedness in a metric space)

In any metric space (X, d), a sequence (x_n) is ***bounded*** if there exists $x_0 \in X$ and $r > 0$ such that $x_n \in B_r(x_0)$ for all $n \in \mathbb{N}$.

Theorem

In any metric space (X, d), any convergent sequence is bounded.

Note: The converse is `FALSE`. *e.g.*, $x_n = (-1)^n$ in $(\mathbb{R}, \text{standard})$.

Proof.

Taking $\varepsilon = 1$ (or any particular value) in the definition of convergence, since $x_n \xrightarrow{n\to\infty} x$ in (X, d), $\exists N \in \mathbb{N}$ with $d(x_n, x) < \varepsilon = 1$, $\forall n \geq N$, *i.e.*, $x_n \in B_1(x)$, $\forall n \geq N$. The earlier elements of the sequence, $x_1, \ldots, x_{N-1}$, are a finite collection, so we can choose $r > 0$ so that

$$r > \max\{d(x_1, x), d(x_2, x), \ldots, d(x_{N-1}, x), 1).$$

With this r, $x_n \in B_r(x)$ holds $\forall n \in \mathbb{N}$. So (x_n) is bounded. □

Topology of metric spaces

Definition (Interior point)

$x \in E \subseteq X$ is an ***interior point*** of the set E in the metric space (X, d) if x lies in an open ball that is contained in E, *i.e.*,

$$\exists \varepsilon > 0 \quad \ni \quad B_\varepsilon(x) \subset E.$$

Definition (Interior of a set)

If $E \subseteq X$ then the ***interior*** of E, denoted $\mathrm{int}(E)$ or E°, is the set of all interior points of E.

Note:

- $x \in E^\circ$ means not only that $x \in E$, but that there is an entire open ball $B_\varepsilon(x) \subset E$.
- For any *smaller* radius, $0 < r < \varepsilon$, we have $x \in B_r(x) \subseteq B_\varepsilon(x) \subseteq E$, so the choice of ε is not unique for an interior point.

Topology of metric spaces

Definition (Open set)

A set $E \subseteq X$ is ***open*** if every point of E is an interior point.

Example (Upper half-plane in $(\mathbb{R}^2, \text{Euclidean})$)

Let $W = \{x = (x_1, x_2) : x_2 > 0\}$. Then $\forall x \in W$, if $0 < \varepsilon < x_2$ then $B_\varepsilon(x) \subset W$. Thus all $x \in W$ are interior points. So W is an open set.

Example (Any set in $(X, \text{discrete})$)

$(X, \text{discrete})$ means X is a non-empty set and d is the discrete metric d. So, $\forall x \in X$,

$$B_\varepsilon(x) = \begin{cases} \{x\}, & \text{if } 0 < \varepsilon \leq 1, \\ X, & \text{if } \varepsilon > 1. \end{cases}$$

Therefore, for any subset $E \subseteq X$ and for any point $x \in E$, $B_1(x) = \{x\} \subseteq E$, so every point of any set E is an interior point, so **all** sets in the discrete metric are open sets!

Topology of metric spaces

Definition (Closed set)

A set $F \subseteq X$ is ***closed*** if F^c is open.

Example (Lower half-plane in $(\mathbb{R}^2, \text{Euclidean})$)

Let $Z = \{x = (x_1, x_2) : x_2 \leq 0\}$. Then $Z = W^c$, where W is the (open) upper half-plane. So Z is a closed set.

Note: In $(\mathbb{R}^2, \text{Euclidean})$, a half-plane is open or closed depending on whether none or all of the x-axis is included. If neither none nor all of the x-axis is included then the half-plane is neither open nor closed.

Example (Any set in $(X, \text{discrete})$)

Any set $E \subseteq X$ is open. So given any $F \subseteq X$, $F^c \subseteq X$ so F^c is open. Hence F is closed. So **all** sets are *both open and closed* with respect to the discrete metric.

Topology of metric spaces

When studying $X = \mathbb{R}$, we defined closed sets differently.
Could we have used the same definition for a general metric space?

Definition (Accumulation Point or Limit Point or Cluster Point)

If $E \subseteq X$ in a metric space (X, d) then x is an ***accumulation point*** of E if every neighbourhood of x contains infinitely many points of E,

$$i.e., \qquad \forall \varepsilon > 0 \qquad B_\varepsilon(x) \cap (E \setminus \{x\}) \neq \varnothing .$$

Equivalently, $x \in X$ is an accumulation point of the set E if and only if there exists a sequence $(x_n)_{n\in\mathbb{N}}$ with $x_n \in E$ $\forall n \in \mathbb{N}$, such that $x_n \neq x$ $\forall n$ and $x_n \xrightarrow{n\to\infty} x$.

Topology of metric spaces

Theorem (Closed sets in a metric space)

A set $F \subseteq X$ in a metric space (X, d) is closed if and only if F contains all its accumulation points.

Proof.

First, suppose F is closed, and that $x \in X$ is an accumulation point of F. If $x \notin F$, then $x \in F^c$, which is open. Therefore, $\exists \varepsilon > 0$ such that $B_\varepsilon(x) \subset F^c$, *i.e.*, $B_\varepsilon(x) \cap F = \varnothing$. But this contradicts x being an accumulation point of F. So we must have $x \in F$.

Conversely, suppose F contains all its accumulation points. Then, if $z \in F^c$, z cannot be an accumulation point of F. But for any $\varepsilon > 0$, $z \in B_\varepsilon(z)$, so $\exists \varepsilon > 0$ for which $B_\varepsilon(z) \cap F = \varnothing$, *i.e.*, $B_\varepsilon(z) \subset F^c$. Hence F^c is open, so F is closed. □

Topology of metric spaces

Theorem (Properties of open sets in a metric space (X, d))

1. *The sets X and $\varnothing$ are open.*
2. *Any intersection of a finite number of open sets is open.*
3. *Any union of an arbitrary collection of open sets is open.*
4. *The complement of an open set is closed.*

Theorem (Properties of closed sets in a metric space (X, d))

1. *The sets X and $\varnothing$ are closed.*
2. *Any union of a finite number of closed sets is closed.*
3. *Any intersection of an arbitrary collection of closed sets is closed.*
4. *The complement of a closed set is open.*

Poll

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- Click on Math 3A03
- Click on Take Class Poll
- Fill in poll **Metric spaces: Set differences of balls**
- Submit.

Topology of metric spaces

Example (Set differences of balls)

Suppose $0 < r_1 < r_2$ and (X, d) is a metric space. If $x \in X$, is it necessarily true that $B_{r_2}(x) \setminus B_{r_1}(x)$ is open or closed?

No, it depends on the metric d. If $(X, d) = (\mathbb{R}^n, \text{Euclidean})$ then $B_{r_2}(x) \setminus B_{r_1}(x)$ is neither open nor closed, *e.g.*, in $\mathbb{R}^1$,

$$\begin{aligned} B_{r_2}(x) \setminus B_{r_1}(x) &= (x - r_2, x + r_2) \setminus (x - r_1, x_+ r_1) \\ &= (x - r_2, x - r_1] \cup [x + r_1, x + r_2). \end{aligned}$$

In contrast, if $(X, d) = (\mathbb{R}^n, \text{discrete})$ then $B_{r_2}(x) \setminus B_{r_1}(x)$ is open since any set in (X, d) is both open and closed.

In general, if A and B are sets then

$$A \setminus B = A \cap B^{\text{c}},$$

so if A and B are both open, and B is also closed, then $A \setminus B$ is open.

Topology of metric spaces

Definition (Isolated point)

If $x \in E \subseteq X$ in a metric space (X, d) then x is an ***isolated point*** of E if there is a neighbourhood of x for which the only point in E is x itself, *i.e.*,

$$\exists \varepsilon > 0 \quad \ni \quad B_\varepsilon(x) \cap E \ = \ \{x\}\,.$$

Example

Consider the metric space $(X, d) = ([0, 1], \text{standard})$.
What are the isolated points of (X, d)?

There are no isolated points!

Now suppose $(X, d) = ([0, 1], \text{discrete})$.
What are the isolated points of (X, d)?

All points are isolated!

McMaster University

Mathematics and Statistics

$$\int_M d\omega = \int_{\partial M} \omega$$

Mathematics 3A03
Real Analysis I

Instructor: David Earn

Lecture 26
Metric Spaces IV
Wednesday 19 March 2025

Announcements

Last time. . .

- More definitions and examples related to metric topology

Topology of metric spaces

Definition (Closure of a set)

If (X, d) is a metric space and $E \subseteq X$ then the ***closure*** of E, denoted $\overline{E}$, is the smallest closed set that contains E.

$\therefore$ $\overline{E}$ is closed, $E \subseteq \overline{E}$, and if F is closed and $E \subseteq F$, then $\overline{E} \subseteq F$.

Theorem

$x \in \overline{E} \iff x$ is either an element of E or an accumulation point of E.
i.e., $x \in \overline{E} \iff \forall \varepsilon > 0,\ B_\varepsilon(x) \cap E \neq \varnothing$.

Proof.

$(\Longleftarrow)$ Suppose $\forall \varepsilon > 0$ $B_\varepsilon(x) \cap E \neq \varnothing$. In order to derive a contradiction, assume $x \notin \overline{E}$. Then $x \in (\overline{E})^c$, which is open, so $\exists \varepsilon > 0$ such that $B_\varepsilon(x) \subseteq (\overline{E})^c$, *i.e.*, $B_\varepsilon(x) \cap \overline{E} = \varnothing$. But $E \subset \overline{E}$, so $B_\varepsilon(x) \cap E = \varnothing$. $\Rightarrow\Leftarrow$

$(\Longrightarrow)$ Suppose $x \in \overline{E}$ and, in order to derive a contradiction, suppose $\exists \varepsilon > 0$ $\ni$ $B_\varepsilon(x) \cap E = \varnothing$. Then $E \subseteq (B_\varepsilon(x))^c$. Hence $\overline{E} \subseteq \overline{(B_\varepsilon(x))^c} = (B_\varepsilon(x))^c$. But $x \in \overline{E}$ and $x \notin (B_\varepsilon(x))^c$. $\Rightarrow\Leftarrow$. $\therefore \forall \varepsilon > 0,\ B_\varepsilon(x) \cap E \neq \varnothing$. □

Topology of metric spaces

Example (Cubic balls in $\mathbb{R}^n$)

In $\mathbb{R}^n$ with the max norm $\|\cdot\|_\infty$, the distance between $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ is

$$d(x, y) = \|x - y\|_\infty = \max\{|x_i - y_i| : 1 \le i \le n\}.$$

Consider the set $E \subset \mathbb{R}^n$, where

$$E = \Big\{x \in \mathbb{R}^n : 0 < x_i < 1, \ \forall i \in \{1, \ldots, n\}\Big\}.$$

What are the interior E° and closure $\overline{E}$ of the set E?

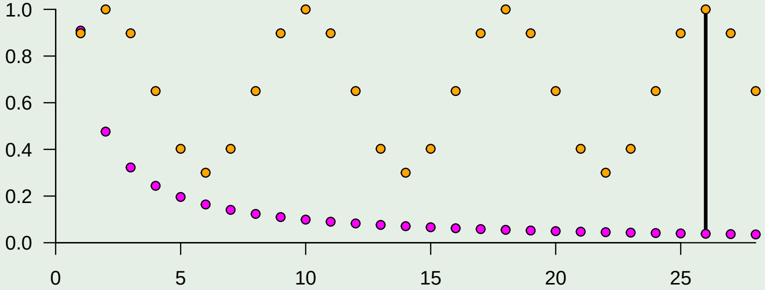
- $\boldsymbol{n = 1}$: $E = (0, 1)$, an open interval. $E^\circ = (0, 1) = E$. $\overline{E} = [0, 1]$.
- $\boldsymbol{n = 2}$: $E = (0, 1) \times (0, 1)$, an open square (an open ball in this metric). $E^\circ = (0, 1)^2 = E$. $\overline{E} = [0, 1]^2$.
- $\boldsymbol{n > 2}$: $E = (0, 1)^n$, an open n-cube (an open ball in this metric). $E^\circ = (0, 1)^n = E$. $\overline{E} = [0, 1]^n$.

  When we imagine an n-cube for $n > 3$, we are probably thinking about $n = 3$ in our minds. But we can easily represent individual points in $\mathbb{R}^n$ in the plane. *How?*

Topology of metric spaces

Example (Cubic balls in $\mathbb{R}^n$)

Suppose $x, y \in E \subset \mathbb{R}^{28}$, $\quad d(x, y) = \|x - y\|_\infty$.



The vertical **black line** indicates the distance between x and y.

Note: The same picture works for $\ell^\infty(\mathbb{R})$, the space of sequences that are bounded, and in which the norm is defined by sup rather than max.

Topology of metric spaces

For the space of continuous functions on a closed interval $[a, b]$, we defined the Euclidean norm via the standard inner product. We can also define a p-norm on this space for any $p \geq 1$,

$$\|f\|_p = \left(\int_a^b |f(x)|^p \, dx \right)^{1/p} .$$

As in finite dimensions, $\lim_{p\to\infty} \|f\|_p = \|f\|_\infty$, where

$$\|f\|_\infty \equiv \sup\{|f(x)| : a \leq x \leq b\} = \max\{|f(x)| : a \leq x \leq b\}.$$

Note: In the metric space $(C[a, b], d)$ with $d(f, g) = \|f - g\|_\infty$, convergence of sequences of "points", $f_n \xrightarrow{n\to\infty} f$, implies $f_n \xrightarrow[\text{unif}]{n\to\infty} f \in C[a, b]$.

Example

In the metric space $C([a, b])$, with distance given by the sup-norm,

$$d(f, g) = \|f - g\|_\infty = \sup\{|f(x) - g(x)| : a \leq x \leq b\},$$

let

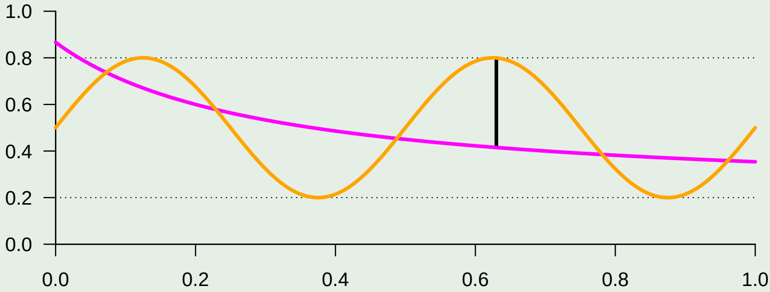
$$E = \Big\{ f \in C([a, b]) \, : \, 0 < f(x) < 1, \; \forall x \in [a, b] \Big\}.$$

What are the interior E° and closure $\overline{E}$ of the set E?

Topology of metric spaces

Example (Cubic balls in $C[0,1]$)

Suppose $f, g \in E \subset C[0,1]$, $\quad d(f,g) = \|f - g\|_\infty$.

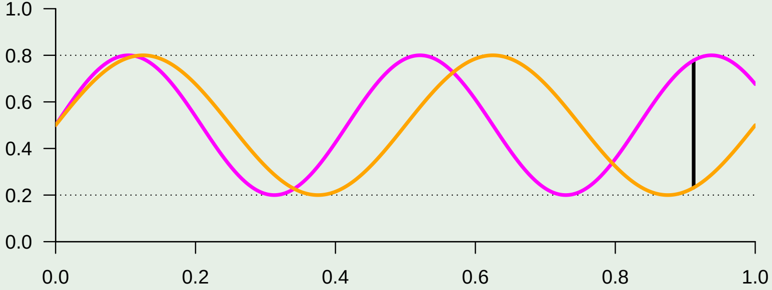


The vertical **black line** indicates the distance between f and g.

Topology of metric spaces

Example (Cubic balls in $C[0,1]$)

Suppose $f, g \in E \subset C[0,1]$, $\quad d(f,g) = \|f - g\|_\infty$.

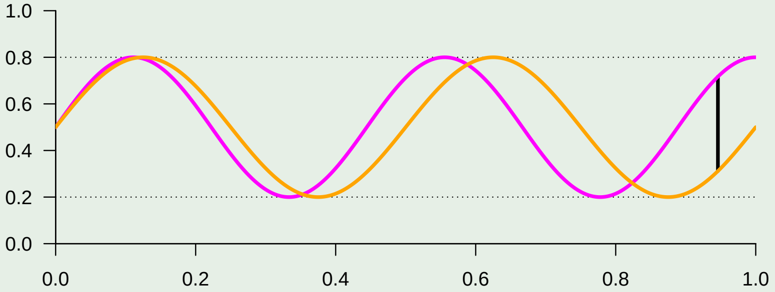


The vertical **black line** indicates the distance between f and g.

Topology of metric spaces

Example (Cubic balls in $C[0,1]$)

Suppose $f, g \in E \subset C[0,1]$, $\quad d(f,g) = \|f-g\|_\infty$.

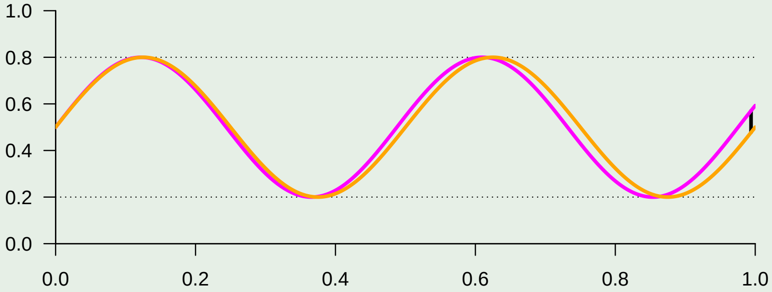


The vertical **black line** indicates the distance between f and g.

Topology of metric spaces

Example (Cubic balls in $C[0,1]$)

Suppose $f, g \in E \subset C[0,1]$, $\quad d(f,g) = \|f - g\|_\infty$.



The vertical **black line** indicates the distance between f and g.

Topology of metric spaces

Example (E° and $\overline{E}$ for $E = \{f \in C([a,b]) : 0 < f(x) < 1 \quad \forall x \in [a,b]\}$)

If $f \in C([a,b])$ then f is continuous on $[a,b]$ so, by the Extreme Value Theorem, f attains its minimum and maximum values on $[a,b]$, say at $u, v \in [a,b]$. Since $f(x) \in (0,1)$ for all $x \in [a,b]$, the extreme values, $f(u)$ and $f(v)$, must also be in the open interval $(0,1)$. Therefore,

$$0 < f(u) \le f(x) \le f(v) < 1, \quad \forall x \in [a,b]. \qquad (\heartsuit)$$

Thus, the range of f is $[f(u), f(v)] \subset (0,1)$. There is a gap (a finite interval) that "insulates" f from the extreme values (0 and 1).

Let $\varepsilon = \min\{f(u), 1 - f(v)\}$. Then $\varepsilon > 0$. Now consider $g \in B_\varepsilon(f) \subset C([a,b])$.

$$\begin{aligned}
\text{Then:} \qquad & \|g - f\|_\infty < \varepsilon \\
& \implies \max\{|g(x) - f(x)| : 0 \le x \le 1\} < \varepsilon \\
& \implies |g(x) - f(x)| < \varepsilon \quad \forall x \in [0,1] \\
& \implies -\varepsilon < g(x) - f(x) < \varepsilon \quad \forall x \in [0,1] \\
& \implies f(x) - \varepsilon < g(x) < f(x) + \varepsilon \quad \forall x \in [0,1]
\end{aligned}$$

Topology of metric spaces

Example (E° and $\overline{E}$ for $E = \{f \in C([a,b]) : 0 < f(x) < 1 \quad \forall x \in [a,b]\}$)

Now using (♡), we have

$$0 \le f(u) - \varepsilon \le f(x) - \varepsilon < g(x) < f(x) + \varepsilon \le f(v) + \varepsilon \le 1, \quad \forall x \in [a,b],$$

from which we conclude that $g \in E$.

Since g was an arbitrary "point" in $B_\varepsilon(f)$, it follows that $B_\varepsilon(f) \subseteq E$, so any $f \in E$ is an interior point, so E is open and $E^\circ = E$.

What about $\overline{E}$, the closure of E?

We will show that the closure of E is the set

$$F = \{f \in C([a,b]) : 0 \le f(x) \le 1, \ \forall x \in [a,b]\} . \qquad (♠)$$

We will show $\overline{E} \subseteq F$ and then $F \subseteq \overline{E}$.

Topology of metric spaces

Example (E° and $\overline{E}$ for $E = \{f \in C([a,b]) : 0 < f(x) < 1 \quad \forall x \in [a,b]\}$)

($\overline{E} \subseteq F$) First, consider any sequence $(f_n)_{n\in\mathbb{N}}$ with $f_n \in E$, $\forall n \in \mathbb{N}$, and suppose $f_n \xrightarrow{n\to\infty} f$ in the sup-norm, *i.e.*, $f_n \xrightarrow[\text{unif}]{n\to\infty} f$. Since uniform convergence implies pointwise convergence, and $f_n(x) \in (0,1)$, we must have $0 \le f(x) \le 1$, $\forall x \in \mathbb{N}$, and since $f_n \xrightarrow[\text{unif}]{n\to\infty} f$, we know $f \in C([a,b])$, so $f \in F$. Since f is a limit point of E and any limit point of E must lie in $\overline{E}$, we must have $\overline{E} \subseteq F$.

($F \subseteq \overline{E}$) Suppose $f \in F$, and define the sequence $f_n \in E$ by

$$f_n(x) = \begin{cases} 1 - \frac{1}{n}, & \text{if } f(x) > 1 - \frac{1}{n}, \\ f(x), & \text{if } \frac{1}{n} \le f(x) \le 1 - \frac{1}{n}, \\ \frac{1}{n}, & \text{if } f(x) < \frac{1}{n}. \end{cases}$$

By construction, $f_n \xrightarrow[\text{unif}]{n\to\infty} f$, and so f is a limit point of E so (by the theorem about closures again) $F \subseteq \overline{E}$. *(Note: $f_n \to f$ is illustrated in the next few slides.)*

Therefore, $F = \overline{E}$. □

Illustration of $f_n \to f$ for $f_n \in E$, $f \notin E$, $f \in F$

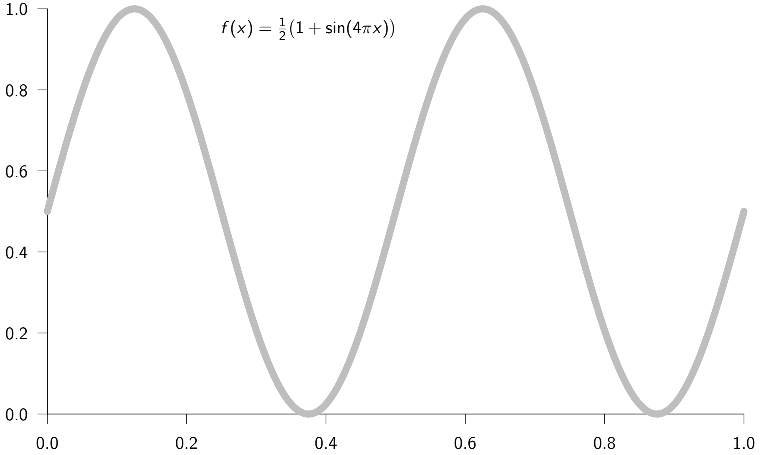


Illustration of $f_n \to f$ for $f_n \in E$, $f \notin E$, $f \in F$

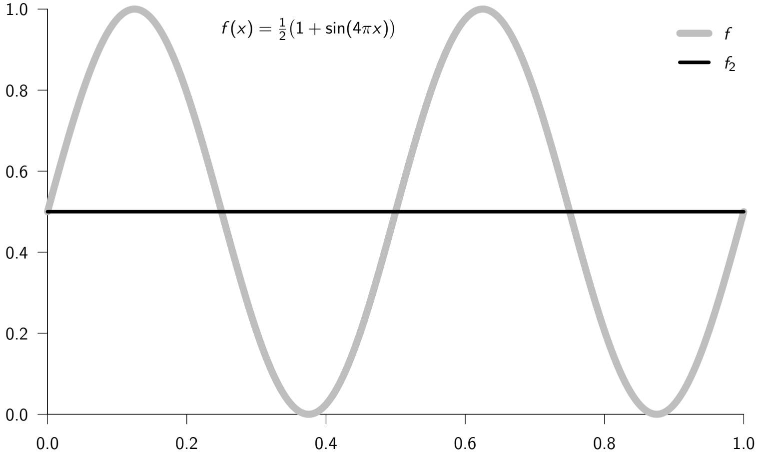


Illustration of $f_n \to f$ for $f_n \in E$, $f \notin E$, $f \in F$

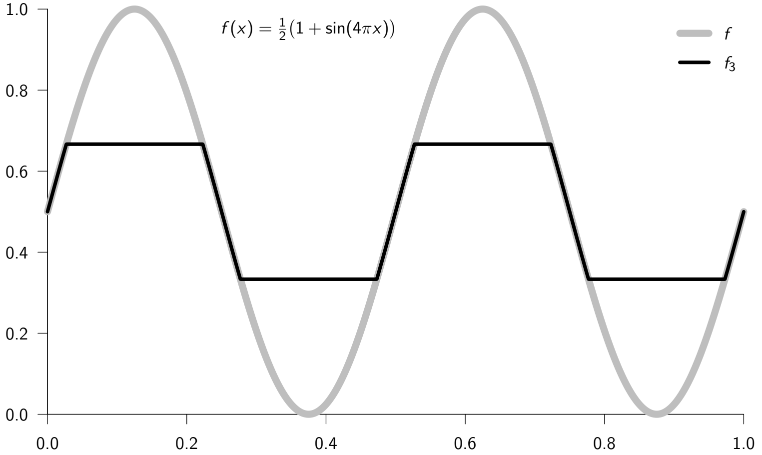


Illustration of $f_n \to f$ for $f_n \in E$, $f \notin E$, $f \in F$

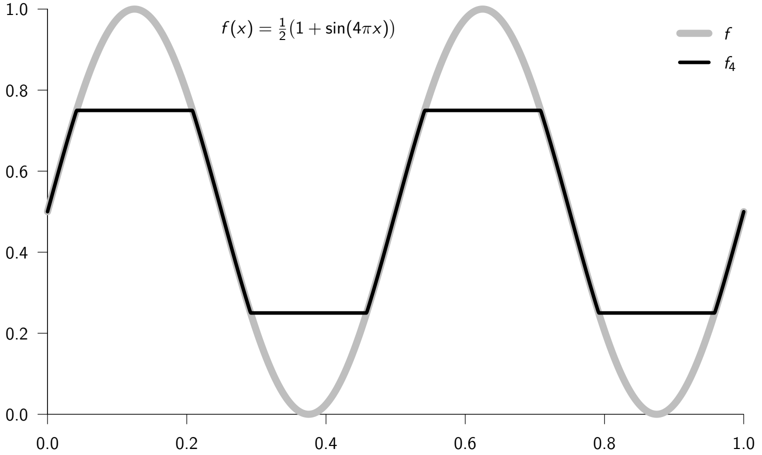
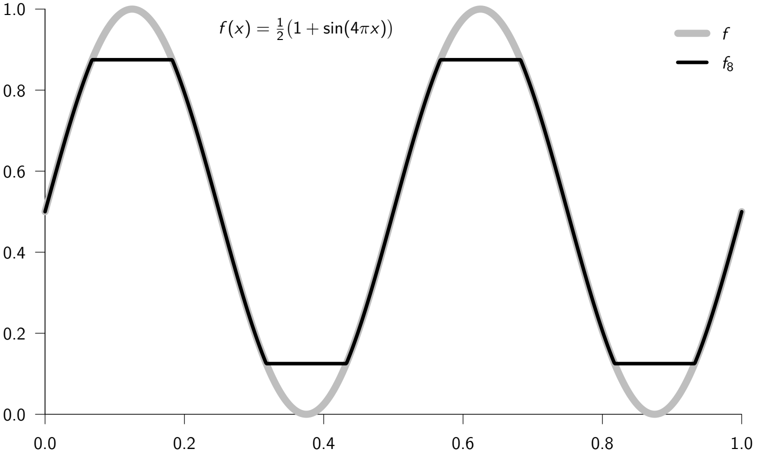


Illustration of $f_n \to f$ for $f_n \in E$, $f \notin E$, $f \in F$



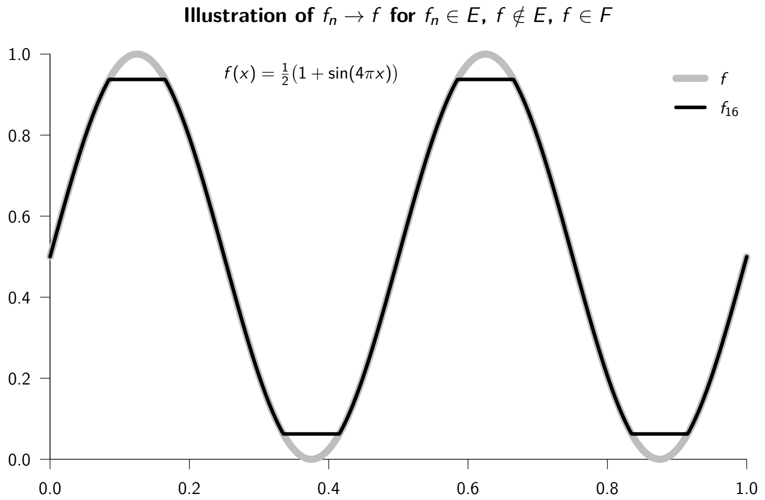
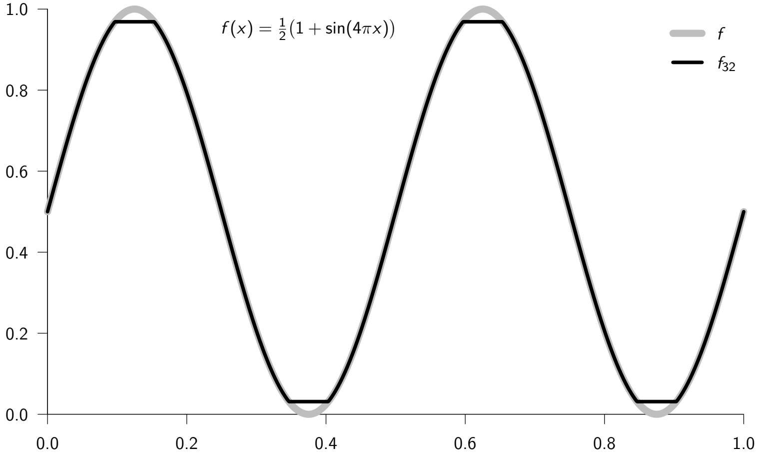


Illustration of $f_n \to f$ for $f_n \in E$, $f \notin E$, $f \in F$
$f(x) = \frac{1}{2}(1 + \sin(4\pi x))$
f
f_{16}
1.0
0.8
0.6
0.4
0.2
0.0
0.0
0.2
0.4
0.6
0.8
1.0

Illustration of $f_n \to f$ for $f_n \in E$, $f \notin E$, $f \in F$



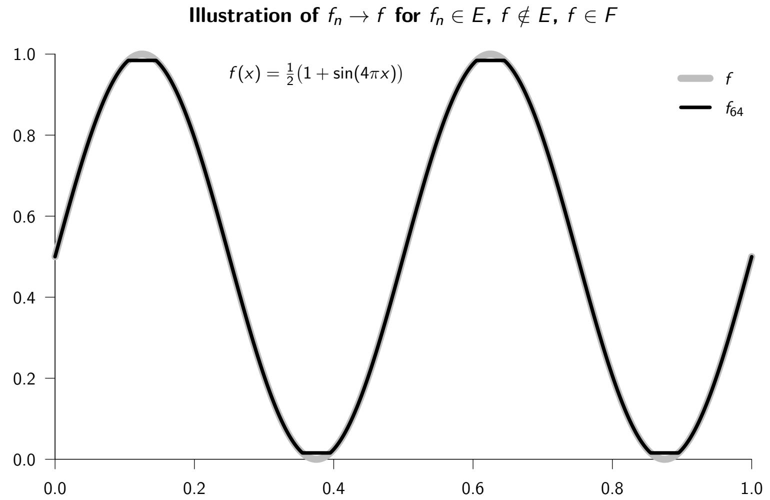
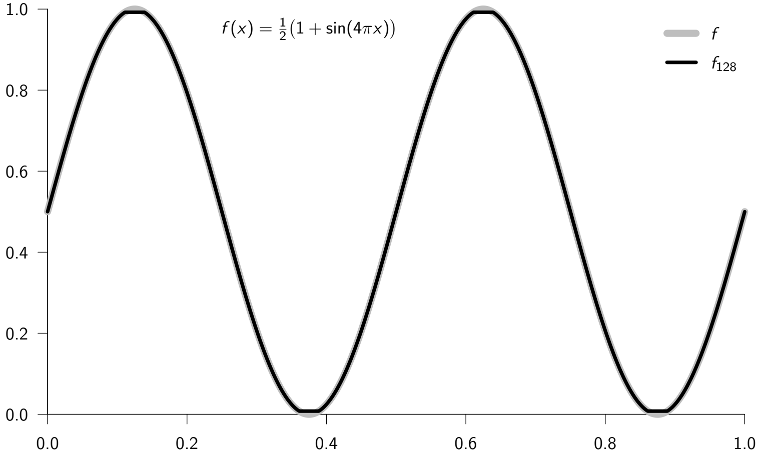


Illustration of $f_n \to f$ for $f_n \in E$, $f \notin E$, $f \in F$
$f(x) = \frac{1}{2}(1 + \sin(4\pi x))$
f
f_{64}
1.0
0.8
0.6
0.4
0.2
0.0
0.0
0.2
0.4
0.6
0.8
1.0

Illustration of $f_n \to f$ for $f_n \in E$, $f \notin E$, $f \in F$



Mathematics
and Statistics

$$\int_M d\omega = \int_{\partial M} \omega$$

Mathematics 3A03
Real Analysis I

Instructor: David Earn

Lecture 27
Metric Spaces V
Friday 21 March 2025

Announcements

- *Last time:* interior and closure
- I added some graphs illustrating $f_n \to f$ in the construction that proved $F \subseteq \overline{E}$ at the end of the discussion of E° and $\overline{E}$ for $E \subset C[a,b]$.

Boundary

Definition (Boundary Point)

If $E \subseteq X$ in a metric space (X, d), then x is a ***boundary point*** of E if every neighbourhood of x contains at least one point of E and at least one point not in E, *i.e.*,

$$\forall \varepsilon > 0 \qquad B_\varepsilon(x) \cap E \neq \varnothing \\ \wedge \quad B_\varepsilon(x) \cap (X \setminus E) \neq \varnothing \,.$$

Definition (Boundary)

If $E \subseteq X$ then the ***boundary*** of E, denoted ∂E, is the set of all boundary points of E.

Topology of metric spaces

Example (Properties of boundary of E in a metric space (X, d))

For any $E \subset X$

- $\partial E = \overline{E} \setminus E^\circ$;
- ∂E is a closed set;
- E is closed if and only if $\partial E \subseteq E$.

Excellent exercises...

Topology of metric spaces

Example (E°, $\overline{E}$, ∂E for $E = (0,1)^\infty \subset \ell^\infty$)

In the metric space ℓ^∞, *i.e.*, bounded sequences (x_n) with distance given by the sup-norm,

$$d(x,y) \;=\; \|x-y\|_\infty \;=\; \sup\{|x_n - y_n| : n \in \mathbb{N}\},$$

let

$$E \;=\; \big\{(x_n) \in \ell^\infty \;:\; 0 < x_n < 1,\; \forall n \in \mathbb{N}\big\}.$$

What are the interior E°, closure $\overline{E}$, and boundary ∂E of the set E?

Please do poll: Metric spaces: ℓ^∞

Consider the following points in ℓ^∞.
Are they in E? Are they in E°? Are they in $\overline{E}$? Are they in ∂E?

- $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \dots)$
- $(0,0,0,\dots)$
- $(0,1,0,1,\dots)$
- $((-1)^n)_{n\in\mathbb{N}}$
- $(\frac{1}{n})_{n\in\mathbb{N}}$
- $(\frac{1}{n+1})_{n\in\mathbb{N}}$

Continuity in metric spaces

We have previously considered continuous functions $f : \mathbb{R} \to \mathbb{R}$.
What does it mean for a function to map one metric space to another *continuously*?

Definition (Continuous function)

Suppose $(\mathcal{M}, d)$ and $(\mathcal{N}, \rho)$ are metric spaces, $f : \mathcal{M} \to \mathcal{N}$, and $x \in \mathcal{M}$. The function f is ***continuous*** at x if:

$$\forall \varepsilon > 0,\ \exists \delta > 0 \ \ni \ \big(y \in \mathcal{M} \ \wedge \ d(x,y) < \delta\big) \implies \rho\big(f(x), f(y)\big) < \varepsilon.$$

If f is continuous at every $x \in \mathcal{M}$, we say that f is ***continuous on*** $\mathcal{M}$.

- Continuity is determined point-by-point, but it is not enough to know f at a point; we must know how f behaves in a neighborhood of the point.
- While $(\mathcal{M}, d)$ and $(\mathcal{N}, \rho)$ can be any metric spaces, the most common situations are $(\mathcal{N}, \rho) = (\mathbb{R}, \text{standard})$ (so $f : \mathcal{M} \to \mathbb{R}$) and $(\mathcal{M}, d) = (\mathcal{N}, \rho)$ (so $f : \mathcal{M} \to \mathcal{M}$).
- If $\mathcal{M}$ is a function space (*e.g.*, $C[a, b]$) then $f : \mathcal{M} \to \mathbb{R}$ is often called a ***functional*** and $f : \mathcal{M} \to \mathcal{M}$ is often called an ***operator***.

Continuity in metric spaces

It is usually helpful to rephrase the definition of continuity more geometrically in terms of balls. If we write the ball of radius δ in the distance d on $\mathcal{M}$ as

$$B_\delta^d(x) = \{y \in \mathcal{M} : d(x,y) < \delta\},$$

then

$$y \in \mathcal{M} \ \wedge \ d(x,y) < \delta \quad \Longleftrightarrow \quad y \in B_\delta^d(x).$$

To rephrase the second part of the definition, for any subset $E \subseteq \mathcal{M}$, we write the ***image*** of E by f as

$$f(E) \ = \ \Big\{f(x) : x \in E\Big\} \subseteq \mathcal{N}.$$

Then, we can write "$f(x) \in B$, $\forall x \in A$" as "$f(A) \subseteq B$", so the definition of continuity can be expressed concisely as

$$\forall \varepsilon > 0, \ \exists \delta > 0 \ \ni \ f\Big(B_\delta^d(x)\Big) \subseteq B_\varepsilon^\rho\Big(f(x)\Big).$$

So a function is continuous at $x \in \mathcal{M}$ if any ball about $f(x) \in \mathcal{N}$ is the image of a ball about $x \in \mathcal{M}$.

McMaster University

Mathematics and Statistics

$$\int_M d\omega = \int_{\partial M} \omega$$

Mathematics 3A03
Real Analysis I

Instructor: David Earn

Lecture 28
Metric Spaces VI
Monday 24 March 2025

Announcements

Assignment 5 is posted on the course web site.

Last time...

- boundary
- $(0,1)^\infty$ in ℓ^∞
- continuous functions from one metric space to another
- continuity expressed with balls

Continuity in metric spaces

As in $\mathbb{R}$, in a general metric space we can express continuity using sequences.

Theorem (Continuity via sequences)

Let $(\mathcal{M}, d)$ *and* $(\mathcal{N}, \rho)$ *be metric spaces, and suppose* $f : \mathcal{M} \to \mathcal{N}$. *Then* f *is continuous at* $x \in \mathcal{M}$ *if and only if for any sequence* $(x_n)_{n\in\mathbb{N}} \subseteq \mathcal{M}$

$$x_n \xrightarrow{n\to\infty} x \quad \Longrightarrow \quad f(x_n) \xrightarrow{n\to\infty} f(x).$$

Thus, f is continuous on $\mathcal{M}$ iff, for every convergent sequence (x_n),

$$\lim_{n\to\infty} f(x_n) = f\Big(\lim_{n\to\infty} x_n\Big).$$

Proof.

($\Longrightarrow$) Suppose f is continuous at $x \in \mathcal{M}$, and $x_n \xrightarrow{n\to\infty} x \in \mathcal{M}$. By definition, given any $\varepsilon > 0$, $\exists\delta > 0$ such that

$$y \in \mathcal{M} \ \wedge \ d(x,y) < \delta \quad \Longrightarrow \quad \rho\big(f(x), f(y)\big) < \varepsilon. \qquad (*)$$

But $x_n \xrightarrow{n\to\infty} x$ implies $\exists N \in \mathbb{N} \ni \forall n \geq N$, $d(x, x_n) < \delta$. So $(*)$ implies $\rho\big(f(x), f(x_n)\big) < \varepsilon$, *i.e.*, $f(x_n) \xrightarrow{n\to\infty} f(x)$.

...continued...

Continuity in metric spaces

Proof of continuity via sequences (continued).

($\Longleftarrow$) Suppose that $x_n \xrightarrow{n\to\infty} x \implies f(x_n) \xrightarrow{n\to\infty} f(x)$ but f is not continuous at x. Then, inverting the definition, $\exists \varepsilon_0 > 0$ such that

for any $\delta > 0$ we can find $y = y(\delta) \in \mathcal{M}$
with $d(x,y) < \delta$ and $\rho\big(f(x), f(y)\big) \geq \varepsilon_0$.

In particular, for $\delta = \frac{1}{n}$, where $n \in \mathbb{N}$, $\exists y_n \in \mathcal{M}$ with

$$d(x, y_n) < \frac{1}{n} \quad \text{and} \quad \rho\big(f(x), f(y_n)\big) \geq \varepsilon_0.$$

This sequence (y_n) converges to x, so by hypothesis $f(y_n) \xrightarrow{n\to\infty} f(x)$. But this contradicts $\rho\big(f(y_n), f(x)\big) \geq \varepsilon_0$. $\Rightarrow\Leftarrow$ Hence f is continuous at x. □

Continuity in metric spaces

Example (Continuity in $(\mathbb{R}, \text{standard})$ (refesher exercises))

1. Suppose $f : \mathbb{R} \to \mathbb{R}$, f is continuous at all $x \in \mathbb{R}$, and $f(q) = 0$ for all $q \in \mathbb{Q}$. Prove that $f(x) = 0$, $\forall x \in \mathbb{R}$.

2. Let $f : \mathbb{R} \to \mathbb{R}$,

$$f(x) = \cos\frac{1}{x}, \quad \forall x \neq 0, \qquad f(0) = a.$$

Show that no matter how we choose the value of $a \in \mathbb{R}$, f is never continuous at $x = 0$.

Continuity in metric spaces

Another equivalent characterization of continuity depends on the notion of:

Definition (Preimage or inverse image of a set)

Let $f : \mathcal{M} \to \mathcal{N}$ and $A \subseteq \mathcal{N}$. The ***preimage*** or ***inverse image*** of a A with respect to f is the set

$$f^{-1}(A) = \{x \in \mathcal{M} : f(x) \in A\}.$$

Note: The function f in this definition does not need to be invertible. There may be many points in $\mathcal{M}$ that f maps to a single point in $\mathcal{N}$.

Continuity in metric spaces

Theorem (Continuity via inverse images)

Let $f : \mathcal{M} \to \mathcal{N}$. The following are equivalent:

1. *f is continuous on $\mathcal{M}$.*
2. *If $U \subset \mathcal{N}$ is an open set in $\mathcal{N}$, then $f^{-1}(U)$ is an open set in $\mathcal{M}$.*
3. *If $F \subset \mathcal{N}$ is a closed set in $\mathcal{N}$, then $f^{-1}(F)$ is a closed set in $\mathcal{M}$.*

Note: f continuous does not imply that f maps open sets to open sets. $f(U)$ need not be open if U is open, and $f(F)$ need not be closed if F is closed.

Example

Consider $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 1 \; \forall x$.
$U = (0, 1)$ is an open set, but $f(U) = \{1\}$ is not open.

Consider $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = e^x \; \forall x$.
$F = \mathbb{R}$ is a closed set, but $f(F) = (0, \infty)$ is not closed.

Note: There is no mention of the metrics in the theorem above. If there is no metric on a space then we define continuity via inverse images.

Continuity in metric spaces

Proof of continuity via inverse images.

(1 $\implies$ 3) Let $E \subset \mathcal{N}$ be any closed set, and take any sequence $(x_n)_{n\in\mathbb{N}}$ with $x_n \in f^{-1}(E)$, $\forall n \in \mathbb{N}$, and $x_n \to x$ in $(\mathcal{M}, d)$. Since f is continuous, it follows that $f(x_n) \xrightarrow{n\to\infty} f(x)$ in $(\mathcal{N}, \rho)$. Since E is closed, $f(x) \in E$, which is the same as saying $x \in f^{-1}(E)$. Therefore, $f^{-1}(E)$ is closed, so 3 holds.

(3 $\implies$ 2) Let $V \subseteq \mathcal{N}$ be any open set. Then V^{c} is closed in $(\mathcal{N}, \rho)$. But

$$f^{-1}(V^{\mathrm{c}}) = \{x \in \mathcal{M} : f(x) \in V^{\mathrm{c}}\} = \{x \in \mathcal{M} \mid f(x) \notin V\} = \left(f^{-1}(V)\right)^{\mathrm{c}},$$

so by 3 $f^{-1}(V)$ is open, *i.e.*, 2 holds.

(2 $\implies$ 1) Let $\varepsilon > 0$ and $x \in \mathcal{M}$ be given. Since $V = B^{\rho}_{\varepsilon}(f(x))$ is open in $(\mathcal{N}, \rho)$, $f^{-1}(V)$ is open in $(\mathcal{M}, d)$. Therefore, x is an interior point of $f^{-1}(V)$, so $\exists\delta > 0$ for which $B^{d}_{\delta}(x) \subseteq f^{-1}(V)$, or, $f(B^{d}_{\delta}(x)) \subseteq V = B^{\rho}_{\varepsilon}(f(x))$. Hence, we conclude that f is continuous at any $x \in \mathcal{M}$, *i.e.*, 1 holds. $\square$

Continuity in metric spaces

Example (Using inverse images to prove continuity)

Let's revisit the first refresher exercise, but prove it using the inverse image of closed sets (3) characterization of continuity.

Suppose $f : \mathbb{R} \to \mathbb{R}$ is continuous at all $x \in \mathbb{R}$, and $f(q) = 0 \;\forall\, q \in \mathbb{Q}$.

Which closed set in $\mathcal{N} = \mathbb{R}$ should we focus on?

The range of f contains 0, and that's all we know about the range of f. So consider $F = \{0\}$.

F is closed, and $f^{-1}(F)$ contains $\mathbb{Q}$, since $f(q) = 0 \;\forall q \in \mathbb{Q}$. But f is continuous. So $f^{-1}(F)$ is closed. Therefore, $f^{-1}(F)$ contains all the accumulation points of $\mathbb{Q}$, *i.e.*, $f^{-1}(F)$ contains all of $\mathbb{R}$.

But $\mathbb{R} = f^{-1}(F) \implies f(\mathbb{R}) = f(f^{-1}(F)) = F = \{0\}$, *i.e.*, $f(x) = 0$ for all $x \in \mathbb{R}$. □

Continuity in metric spaces

Example (Good exercises involving the discrete metric)

1. Consider $f : \mathcal{M} \to \mathcal{N}$ with domain ($\mathcal{M}$,discrete), and $\mathcal{N}$ any metric space $(\mathcal{N}, \rho)$. Show that any such f is continuous.

2. Now suppose $f : \mathcal{M} \to \mathcal{N}$ but the range is ($\mathcal{N}$,discrete), and $(\mathcal{M}, d)$ is a metric space that is not discrete (where "discrete" means $A \subset \mathcal{M}$ is both open and closed iff $A = \mathcal{M}$ or $A = \varnothing$). Show that if f is continuous then f is a constant function.

Continuity in metric spaces

Example ($f : [0,1] \to \mathbb{R}$)

Let $f : [0,1] \to \mathbb{R}$, with the usual metric on $\mathcal{M} = \mathbb{R} = \mathcal{N}$, and $f(x) = x^2$ for all $x \in [0,1]$. Then $V = (\frac{1}{4}, 2)$ is open in $\mathbb{R}$, and

$$f^{-1}(V) \; = \; \left\{ x \in [0,1] : x^2 \in \left(\frac{1}{4}, 2 \right) \right\} \; = \; \left(\frac{1}{2}, 1 \right],$$

which—as a subset of $\mathbb{R}$ with the usual metric—is neither open nor closed! *What went wrong?*

Nothing is wrong! The domain is the metric space $[0,1]$, not $\mathbb{R}$, and the set $\left(\frac{1}{2}, 1\right]$ is open in $[0,1]$. Since the domain space is $[0,1]$, no subset can contain points that are not in $[0,1]$. Balls in $[0,1]$ mean the intersection of balls in $\mathbb{R}$ with $[0,1]$. This is called the ***relative topology*** (or ***subset topology*** or ***induced topology***).

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Mathematics 3A03
Real Analysis I

Instructor: David Earn

Lecture 29
Metric Spaces VII
Wednesday 26 March 2025

Announcements

Assignment 5 is posted on the course web site.

Last time. . .

- continuity via sequences
- continuity via inverse images

Continuity in metric spaces

Definition (Open or closed relative to $A \subset \mathcal{M}$)

1. $U \subset A$ is ***open relative to*** A if $U = V \cap A$ for an open set V of $\mathcal{M}$.
2. $F \subset A$ is ***closed relative to*** A if $F = E \cap A$ for a closed set E of $\mathcal{M}$.

So the set $U = (\frac{1}{2}, 1]$ is open relative to $A = [0, 1]$ in $\mathbb{R}$, because (for example) $U = (\frac{1}{2}, 2) \cap A$, with $(\frac{1}{2}, 2)$ an open set in $\mathbb{R}$.

To distinguish a function $f : \mathcal{M} \to \mathcal{N}$ with domain $A \subset \mathcal{M}$ from a function defined on all of $\mathcal{M}$, we use the notation $f|_A$ for the restricted function (with domain A) that agrees with $f(x)$ for $x \in A$.

Example (Restriction of $f : \mathbb{R} \to \mathbb{R}$ to $A \subset \mathbb{R}$)

$$\text{Let} \qquad f(x) = \begin{cases} -x, & x \le 0, \\ 1, & x > 0, \end{cases} \qquad \text{and} \qquad A = (-\infty, 0].$$

Then $f|_A = -x$ is continuous on A, and, in particular, $f|_A$ is continuous on the boundary of A, *i.e.*, $f|_A$ is continuous at $x = 0$.

Continuity in metric spaces

Continuity in subspaces of metric spaces:

In general, the equivalence of the inverse image definition and the sequence definition of continuity applies for functions $f|_A$ that are restricted to the domain $A \subset \mathcal{M}$ provided we define "open" and "closed" in the relative topology on A, and restrict to sequences $(x_n)_{n\in\mathbb{N}}$ with $x_n \in A \; \forall n \in \mathbb{N}$.

Continuity of operators on metric spaces of functions:

Remember that continuity of an operator $T : \mathcal{M} \to \mathcal{N}$ is distinct from the question of whether elements (*i.e.*, functions) in $\mathcal{M}$ and $\mathcal{N}$ are continuous.

Continuity in metric spaces

Example (Is the integral operator continuous?)

Consider the operator $T : C[a,b] \to C[a,b]$ with $T(f)$ defined by

$$T(f)(x) \;=\; \int_a^x f(x)\,dx\,.$$

Is T continuous?

Note: If $g = T(f)$, then g is continuous by the "integrals are continuous" theorem or the FFTC (since f is continuous).

The question is not whether such a g is continuous. It is whether T is continuous. Phrased in terms of sequences, the question is:

If $f_n \to f \in C[a,b]$ does it follow that $T(f_n) \to T(f) \in C[a,b]$?

Does the answer depend the metric?

Continuity in metric spaces

Example (Is the integral operator continuous?)

The answer is yes, for the sup norm on $C[a,b]$, $\quad \|f\|_\infty = \sup_{x\in[a,b]} |f(x)|$.

Proof: For any $f_n, f \in C[a,b]$, and any $x \in [a,b]$, we have

$$\begin{aligned} |T(f_n)(x) - T(f)(x)| &= \left| \int_a^x (f_n(t) - f(t))\,dt \right| \\ &\le \int_a^x |f_n(t) - f(t)|\,dt \\ &\le \int_a^x \sup_{t\in[a,b]} |f_n(t) - f(t)|\,dt \qquad (\clubsuit) \end{aligned}$$

now apply the triangle inequality

If $f_n \xrightarrow{n\to\infty} f$ wrt $\|\cdot\|_\infty$, *i.e.*, $f_n \xrightarrow[\text{unif}]{n\to\infty} f$, then $\forall \varepsilon > 0\ \exists N \in \mathbb{N}$ ∍ $\forall n \ge N$

$$\|f_n - f\|_\infty = \sup_{t\in[a,b]} |f_n(t) - f(t)| < \varepsilon. \qquad (\spadesuit)$$

. . . continued . . .

Continuity in metric spaces

Example (Is the integral operator continuous?)

Inserting (♠) in (♣), we have, for any $x \in [a,b]$,

$$|T(f_n)(x) - T(f)(x)| \ < \ \int_a^x \varepsilon \, dt \ = \ \varepsilon(x-a) \ \leq \ \varepsilon(b-a)$$

$$\therefore \qquad \|T(f_n) - T(f)\|_\infty \ = \ \sup_{x\in[a,b]} |T(f_n)(x) - T(f)(x)| \ \leq \ (b-a)\varepsilon.$$

Now, recognizing that in (♠) we can replace ε with $\frac{\varepsilon}{2(b-a)}$, we can conclude that $\forall \varepsilon > 0 \ \exists N \in \mathbb{N} \ \ni \ \forall n \geq N, \ \|T(f_n) - T(f)\|_\infty < \varepsilon$,

i.e., $T(f_n) \xrightarrow[\text{unif}]{n\to\infty} T(f)$.

Thus, the integral operator (T) is continuous on the metric space $C[a,b]$ with norm $\|\cdot\|_\infty$. □

Completeness

Completeness of metric spaces

The concept of a Cauchy sequence generalizes to any metric space.

Definition (Cauchy sequence in a metric space)

In a metric space $(\mathcal{M}, d)$, a sequence $(x_n)_{n\in\mathbb{N}}$ is called a ***Cauchy sequence*** iff

$$\forall \varepsilon > 0, \ \exists N \in \mathbb{N} \quad \ni \quad m, n \geq N \implies d(x_n, x_m) < \varepsilon.$$

Theorem (Convergent implies Cauchy)

In a metric space $(\mathcal{M}, d)$, if (x_n) converges then (x_n) is a Cauchy sequence.

Proof.

Given $x_n \to x$, for any $\varepsilon > 0$, we can find $N \in \mathbb{N}$ such that $\forall n \geq N$, $d(x_n, x) < \frac{\varepsilon}{2}$. Suppose both $m \geq N$ and $n \geq N$. Then

$$d(x_n, x_m) \ \leq \ d(x_n, x) + d(x, x_m) \ \leq \ \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

So (x_n) is Cauchy. □

Completeness of metric spaces

In $(\mathbb{R}, \text{standard})$, in addition to convergent $\implies$ Cauchy the converse Cauchy $\implies$ convergent is also true. *This converse does not hold in general in every metric space.* If it does hold, then we say the metric space is complete.

Definition (Complete metric space)

A metric space $(\mathcal{M}, d)$ is said to be ***complete*** iff every Cauchy sequence in $\mathcal{M}$ converges (to a point in $\mathcal{M}$).

Example ($\mathbb{Q}$)

Let $q_n = \sum_{k=0}^{n} \frac{1}{k!}$. Then $q_n \in \mathbb{Q}$ for all n. But $q_n \to e = \sum_{k=0}^{\infty} \frac{1}{k!}$, and we showed $e \notin \mathbb{Q}$. So $\mathbb{Q}$ is not complete.

In $\mathbb{R}$, the existence of least upper bounds is equivalent to Cauchy sequences always converging. So $(\mathbb{R}, \text{standard})$ is complete. But $\mathbb{R}$ has much more structure. It is a field and it has an order. In fact, *$\mathbb{R}$ is the unique, complete, ordered, field.*

Completeness of metric spaces

Example (Euclidean n-space)

- $\mathbb{R}^n$ with the Euclidean norm is complete.

  Proof: Let $(x_k)_{k\in\mathbb{N}}$ be a Cauchy sequence in $(\mathbb{R}^n,\text{Euclidean})$, say $x_k = (x_{k,1}, x_{k,2}, \ldots, x_{k,n})$. For each fixed $j = 1, 2, \ldots, n$, the sequence of real numbers $(x_{k,j})_{k\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{R}$, and hence converges in $\mathbb{R}$, say $x_{k,j} \xrightarrow{k\to\infty} \hat{x}_j \in \mathbb{R}$, for each $j = 1, 2, \ldots, n$. Define $\hat{x} = (\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_n) \in \mathbb{R}^n$. Since each component of the vector converges, it follows (from A5 Q1) that $x_k \xrightarrow{k\to\infty} \hat{x}$. □

  Note: $(\mathbb{R}^n, \text{Euclidean})$ is an example of a ***Hilbert Space***, an inner product space that is complete wrt the induced metric.

- $\mathbb{R}^n$ is also complete wrt the Taxicab ($\|\cdot\|_1$) and Maximum ($\|\cdot\|_\infty$) norms (*check!*). In fact, for any $p \geq 1$, $\mathbb{R}^n$ with norm $\|\cdot\|_p$ is complete wrt to the metric $d(x,y) = \|x-y\|_p$ induced by the norm.

  Any complete, normed, vector space is said to be a ***Banach Space***.

Poll

- Go to https://www.childsmath.ca/childsa/forms/main_login.php
- Click on Math 3A03
- Click on Take Class Poll
- Fill in poll **Metric spaces: completeness**
- `Submit`.

Completeness of metric spaces

Example (More examples related to completeness)

- For any interval $I \subset \mathbb{R}$, the space $C_{\mathrm{b}}(I)$ of bounded continuous functions, with distance determined by the sup-norm, is a complete metric space. *A good exercise.*

  Hint: If (f_n) is a Cauchy sequence in $C_{\mathrm{b}}(I)$ with the sup norm, what does that imply about sequences $(f_n(x))$ in $\mathbb{R}$ for $x \in I$?

- Consider the subspace of $C[0,1]$ consisting of polynomials. We call this $P[0,1]$. *Is $P[0,1]$ complete?* Can we construct a sequence of polynomials that does not converge in $P[0,1]$? The sequence

$$P_n(x) \ = \ 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \cdots \frac{x^n}{n!}$$

  converges as $n \to \infty$ (uniformly, *i.e.,* in the sup norm) to e^x, which is not a polynomial $\implies$ $P[0,1]$ is not complete.

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Mathematics 3A03
Real Analysis I

Instructor: David Earn

Lecture 30
Metric Spaces VIII
Friday 28 March 2025

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Announcements

Please complete the **Student Experience Survey**.

Last time...

- Continuity of the integral as an operator on $C[a, b]$.
- Completeness of metric spaces.
- Completeness of $C_{\mathrm{b}}(I)$
- Incompleteness of $P[0, 1]$.

Completeness of metric spaces

Example (Completeness of $C[a, b]$)

Consider $C[a, b]$, the space of continuous functions on $[a, b]$.

With the sup norm $\|\cdot\|_\infty$ it is complete, since convergence in $\|\cdot\|_\infty$ is equivalent to uniform convergence, and sequences of continuous functions that converge uniformly, converge to continuous functions.

What about the Taxicab norm $\|\cdot\|_1$? Is $C[a, b]$ complete wrt $\|\cdot\|_1$?

No, it is not complete wrt $\|\cdot\|_1$. Construct a sequence of continuous functions on $[a, b]$ that converges wrt $\|\cdot\|_1$ to a discontinuous function.

Contractions

Much of applied mathematics consists of solving equations of various sorts. Often we cannot find solutions but—with the help of metric space theory—we can prove that solutions exist and/or are unique. Such proofs can suggest useful methods for approximating solutions, even if we cannot find them exactly.

Many problems can be expressed in the form of the question

$$\text{Is there a solution to} \quad f(x) = x \ ?$$

Equivalently, does the function f have a ***fixed point***?

One way to address this question uses the notion of a *contraction*.

Definition (Contraction mapping)

Let $f : \mathcal{M} \to \mathcal{M}$ be a function that maps some metric space $(\mathcal{M}, d)$ to itself. We say that f is a ***contraction*** on $\mathcal{M}$ if $\exists\ \alpha \in [0, 1)$ such that

$$d(f(x), f(y)) \ \leq \ \alpha\, d(x, y) \quad \forall x, y \in \mathcal{M}.$$

Contractions

Theorem (Banach Fixed Point Thm or Contraction Mapping Principle)

Let $(\mathcal{M}, d)$ be a complete metric space, and $f : \mathcal{M} \to \mathcal{M}$ a contraction mapping. Then there exists a unique fixed point $x_* \in \mathcal{M}$, i.e., a unique point in $\mathcal{M}$ where $f(x_*) = x_*$.

Note:

- The theorem guarantees that a fixed point x_* exists and is unique, but does not tell us what the fixed point is. In some situations, we can solve the equation $f(x_*) = x_*$ for x_*, and hence determine which point is fixed.
- Finding fixed points seems like a very special type of problem. But note that if we are trying to solve $f(x) = g(x)$ then that is equivalent to finding a fixed point of the equation $f(x) - g(x) + x = x$, *i.e.*, finding a fixed point of $h(x) = f(x) - g(x) + x$.
- $h(x)$ can be defined as above iff addition is defined in $\mathcal{M}$.

Contractions

Example (Contraction mapping)

Suppose $f : [a,b] \to [a,b]$ is differentiable and, for some $\alpha \in (0,1)$, $|f'(x)| < \alpha \ \forall x \in [a,b]$. Then f has a unique fixed point $x_* \in [a,b]$.

Proof: $[a,b]$ is a closed subset of the complete metric space $(\mathbb{R}, \text{standard})$, so $([a,b], \text{standard})$ is a complete metric space (see A5 Q9). Consequently, it is sufficient to show that f is a contraction mapping on $[a,b]$.

Let $u, x \in [a,b]$. By the Mean Value Theorem, $\exists c$ between u and x—and hence between a and b—such that

$$\begin{aligned} d(f(x), f(u)) &= |f(x) - f(u)| = |f'(c)(x-u)| \\ &\le \alpha\,|x-u| = \alpha\, d(x,u). \end{aligned}$$

Thus, f is a contraction, so the contraction mapping principle applies. □

Contractions

Lemma (To be used in proving the Contraction Mapping Principle)

Let (x_n) be a sequence in a metric space $(\mathcal{M}, d)$, and suppose there exist constants $C > 0$ and $0 < \alpha < 1$ such that

$$d(x_{n+1}, x_n) < C\,\alpha^n \quad \text{for all } n \in \mathbb{N}.$$

Then (x_n) is a Cauchy sequence.

Proof.

Given $\varepsilon > 0$, we must find $N \in \mathbb{N}$ such that $\forall\, m, n \geq N$, $d(x_n, x_m) < \varepsilon$. Without loss of generality, assume $m > n$. By the triangle inequality,

$$\begin{aligned} d(x_n, x_m) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{m-1}, x_m) \\ &= \sum_{k=n}^{m-1} d(x_k, x_{k+1}) < \sum_{k=n}^{m-1} C\alpha^k < \sum_{k=n}^{\infty} C\alpha^k = C\sum_{k=n}^{\infty} \alpha^k \\ &= C\alpha^n \sum_{k=0}^{\infty} \alpha^k = \left(\frac{C}{1-\alpha}\right)\alpha^n < \varepsilon \quad \text{for sufficiently large } n. \end{aligned}$$

Hence, (x_n) is a Cauchy sequence. □

Contractions

Proof of the Contraction Mapping Principle.

For any $x_0 \in \mathcal{M}$, define a sequence (x_n) by iteration of $f(x)$:

$$x_n = f(x_{n-1}), \qquad \forall n \in \mathbb{N}.$$

$$\text{Thus,} \qquad (x_n) = (x_0, x_1, x_2, x_3, \ldots) = (x_0, f(x_0), f(f(x_0)), f(f(f(x_0))), \ldots).$$

Since $f : \mathcal{M} \to \mathcal{M}$ is a contraction, $\exists\, \alpha \in [0, 1)$ such that

$$\begin{aligned} d(x_{n+1}, x_n) &= d(f(x_n), f(x_{n-1})) \\ &\leq \alpha\, d(x_n, x_{n-1}), \quad \forall n \geq 1, \\ &= \alpha\, d(f(x_{n-1}), f(x_{n-2})), \quad \forall n \geq 2, \\ &\leq \alpha^2\, d(x_{n-1}, x_{n-2}), \quad \forall n \geq 2, \\ &\vdots \\ &\leq \alpha^n\, d(x_1, x_0). \end{aligned}$$

. . . continued . . .

Contractions

Proof of the Contraction Mapping Principle (continued).

Thus, defining $C = d(x_1, x_0)$, we have

$$d(x_{n+1}, x_n) \leq C\,\alpha^n, \qquad \forall n \in \mathbb{N}.$$

So, by the lemma, (x_n) is a Cauchy sequence in $(\mathcal{M}, d)$. Since the metric space is complete, $\exists x_* \in \mathcal{M}$ with $x_n \xrightarrow{n\to\infty} x_*$. Notice that

$$\begin{aligned} d(f(x_*), x_*) &\leq d(f(x_*), x_{n+1}) + d(x_{n+1}, x_*) \\ &= d(f(x_*), f(x_n)) + d(x_{n+1}, x_*) \\ &\leq \alpha\, d(x_*, x_n) + d(x_{n+1}, x_*) \quad \xrightarrow{n\to\infty} \quad 0, \end{aligned}$$

so $f(x_*) = x_*$, *i.e.*, x_* is a fixed point.

Finally, if y_* is another fixed point, so $f(y_*) = y_*$, then

$$d(x_*, y_*) = d(f(x_*), f(y_*)) \leq \alpha\, d(x_*, y_*),$$

which is impossible for $0 < \alpha < 1$, unless $x_* = y_*$, so the fixed point is unique. □

Applications of the contraction mapping principle

- The construction of the sequence (x_n) in the proof of the contraction mapping principle can be interpreted as a ***dynamical system*** in which f maps the state at the current time to the state a certain time in the future.
    - The meaning of a fixed point of f is then an ***equilibrium*** of the dynamical system.
    - If the hypotheses of the theorem are satisfied then, not only is there a unique fixed point, but all initial states converge onto that fixed point, in which case we say that the equilibrium is ***globally asymptotically stable***.
- An alternative interpretation of a contraction mapping f is that as we iterate it, we obtain a better and better approximation to the exact solution of an equation (which might represent the full time-evolution of a dynamical system, perhaps an ODE or PDE, for example). Then, if the hypotheses of the theorem are satisfied in an appropriate metric space, then it follows that there exists a unique solution to the equation(s).

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Mathematics 3A03
Real Analysis I

Instructor: David Earn

Lecture 31
Metric Spaces IX
Monday 31 March 2025

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Announcements

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Last time...

- Contraction mappings
- Contraction Mapping Principle

Compactness

Compactness of metric spaces

When we studied the topology of the real line, we defined a compact set to be a set satisfying any one of three properties:

Definition (Compact Set in $\mathbb{R}$)

A set $E \subseteq \mathbb{R}$ is said to be ***compact*** if it has any of the following equivalent properties:

1. E is closed and bounded.
2. E has the Bolzano-Weierstrass property.
3. E has the Heine-Borel property.

This definition made sense in $\mathbb{R}$ because these properties are equivalent in $\mathbb{R}$ with the standard metric.

What about $\mathbb{R}$ with other metrics?

Compactness of metric spaces

Before answering that question, let's introduce some terminology.

Definition (Sequentially compact *i.e.,* Bolzano-Weierstrass property)

Let $(\mathcal{M}, d)$ be a metric space. A set $K \subseteq \mathcal{M}$ is ***sequentially compact*** if every sequence in K has a subsequence that converges in K.

Thus, if K is sequentially compact then given any sequence $(x_n)_{n\in\mathbb{N}}$ in K, $\exists x \in K$ and a subsequence $(x_{n_k})_{k\in\mathbb{N}}$, such that $x_{n_k} \xrightarrow{k\to\infty} x$.

Definition (Compact or covering compact *i.e.,* Heine-Borel property)

Let $(\mathcal{M}, d)$ be a metric space. A set $K \subseteq \mathcal{M}$ is ***compact*** or ***covering compact*** if every open cover of K contains a finite subcover.

Thus, if K is (covering) compact and $\{U_\alpha\}_{\alpha\in\mathcal{I}}$ is an open cover of K then the index set $\mathcal{I}$ contains a finite subset, say $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$, such that

$$K \subseteq \bigcup_{i=1}^{n} U_{\alpha_i}.$$

Compactness of metric spaces

Example (Compactness in $\mathbb{R}$ with the discrete metric)

With the discrete metric, every singleton $\{x\}$ is an open set. Consequently, if K is any set then $\{\{x\} : x \in K\}$ is an open cover of K. If K is compact, then the open cover must contain a finite subcover, so K must be a finite set. Thus, a set is compact iff it is finite.

However, with the discrete metric, every set is closed, and every set is also bounded, since the maximum distance between any two points is 1.

Thus, with the discrete metric, "compact" and "closed and bounded" are distinct properties.

Note: This argument made no reference to $\mathbb{R}$. In any discrete metric space $(\mathcal{M}, d)$, "compact" and "closed and bounded" are distinct properties (*provided* $\mathcal{M}$ is not just a finite set).

Poll

- Go to https://www.childsmath.ca/childsa/forms/main_login.php
- Click on Math 3A03
- Click on Take Class Poll
- Fill in poll **Metric spaces: compactness**
- Submit.

Compactness of metric spaces

Theorem (Compactness equivalents in metric spaces)

In any metric space $(\mathcal{M}, d)$, a set $K \subseteq M$ is covering compact if and only if it is sequentially compact.

This is a very important theorem. In any metric space, sequential and covering compactness are equivalent, so whichever is more convenient can be used in any context.

The equivalence of sequential and covering compactness in metric spaces is not trivial to prove (TBB §13.12.3). This emphasizes the importance of the theorem. You can always prove compactness using either equivalent concept, but what you are proving may be much easier if you make the "right" choice. The other choice may effectively force you to re-prove the equivalence theorem.

Compactness of metric spaces

Example (Compactness in $\mathbb{R}$ with non-standard non-discrete metrics?)

The equivalence theorem implies that with any metric d a set $K \subseteq \mathbb{R}$ is sequentially compact iff it is covering compact. We saw that if d is the discrete metric then sets can be closed and bounded yet not compact. *Is this situation particular to the discrete metric?*

No. For example, consider $d(x, y) = \min\{1, |x - y|\}$.

- d is a valid metric on $\mathbb{R}$ (*check*).
- d restricts large distances to 1.
- Every bounded set wrt the usual metric is also bounded wrt d.
- The set $H = [0, \infty)$ is closed and bounded wrt d, since its "diameter" is 1. But H is not compact: consider the open cover by intervals $\{(n, n+2)\}_{n\in\mathbb{N}} \cup \{[0, 2)\}$; these intervals are open in (H, d), but no finite collection of these intervals can cover all of $[0, \infty)$.

Can you construct other metrics on $\mathbb{R}$ wrt which "closed and bounded" is distinct from "compact"?

Compactness of metric spaces

Example (Compactness in ℓ^∞)

In the space ℓ^∞ of bounded sequences of real numbers with the sup norm, consider the points $e_n \in \ell^\infty$ defined by $e_n(i) = \delta_{in} = \begin{cases} 1, & i = n, \\ 0, & i \neq n, \end{cases}$ so

$$e_1 = (1, 0, 0, 0, \ldots), \quad e_2 = (0, 1, 0, 0, \ldots), \quad e_3 = (0, 0, 1, 0, \ldots), \quad \ldots$$

The set $S = \{e_n : n \in \mathbb{N}\}$ is the generalization—to infinite dimensions—of the set of standard basis vectors in $\mathbb{R}^n$.

Now consider the sequence (of sequences!) $(e_n)_{n\in\mathbb{N}} \subset \ell^\infty$.

$\|e_n\|_\infty = 1 \ \forall n$, and $d(e_n, e_m) = \|e_n - e_m\|_\infty = 1$ if $n \neq m$. So (e_n) is not a Cauchy sequence, and hence does not converge. Moreover, (e_n) has no convergent subsequence, since $d(e_n, e_m) = 1$ if $n \neq m$ implies each point e_n is isolated. (So S has no limit points.)

The set S is *bounded* in ℓ^∞, since $d(x, y) \leq 1 \ \forall x, y \in S$. S is also *closed* in ℓ^∞, since it has no limit points. But S not sequentially compact!

Thus, in ℓ^∞, there are closed and bounded sets that are not compact. This is typical for metric spaces that are *infinite dimensional* vector spaces.

Compactness of metric spaces

Example (Compactness in $C[0,1]$ with the sup norm)

Consider $C[0,1]$ with distance defined by the sup-norm, and

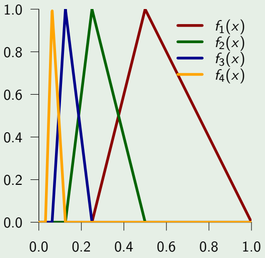
$$S = \left\{ f \in C([0,1]) : 0 \leq f(x) \leq 1 \ \forall x \in [0,1] \right\},$$

which is a *closed and bounded* set (as we showed in a previous example).

Consider the sequence $f_n \in S$,

$$f_n(x) = \begin{cases} 1 + 2^{n+1}(x - \frac{1}{2^n}), & \frac{1}{2^{n+1}} < x \leq \frac{1}{2^n} \\ 2^n(\frac{1}{2^{n-1}} - x), & \frac{1}{2^n} < x < \frac{1}{2^{n-1}} \\ 0, & \text{otherwise}, \end{cases}$$

so f_n has a tent of height 1 over $x = \frac{1}{2^n}$, on the interval $[\frac{1}{2^{n+1}}, \frac{1}{2^{n-1}}]$, $n \in \mathbb{N}$.



$d(f_n, f_m) = \|f_n - f_m\|_\infty = 1$ for all $n \neq m$, so this sequence has no convergent subsequence in the sup norm. Hence S is *not compact*.

Continuity and compactness

Theorem (Continuous functions map compact sets to compact sets)

Let $f : \mathcal{M} \to \mathcal{N}$ be continuous. If K is compact in $\mathcal{M}$ then $f(K)$ is compact in $\mathcal{N}$.

Proof.

Consider an open cover of $f(K)$, say $\mathcal{U} = \{U_\alpha\}_{\alpha \in \mathcal{I}}$. We will show that $\mathcal{U}$ contains a finite subcover of $f(K)$.

Since $\mathcal{U}$ covers $f(K)$, $f(K) \subseteq \bigcup_{\alpha \in \mathcal{I}} U_\alpha$. So, $\forall x \in K$, $\exists \alpha \in \mathcal{I}$ such that $f(x) \in U_\alpha$. Therefore, $\forall x \in K$, $\exists \alpha \in \mathcal{I}$ such that $x \in f^{-1}(U_\alpha)$. But f is continuous, so $f^{-1}(U_\alpha)$ is an open set in $\mathcal{M}$. Now observe that

$$K \subseteq f^{-1}\big(f(K)\big) \subseteq \bigcup_{\alpha \in \mathcal{I}} f^{-1}(U_\alpha),$$

which is an open cover of K. But K is compact, so there are finitely many $\alpha \in \mathcal{I}$, say $\alpha_1, \ldots, \alpha_N$, such that $K \subseteq f^{-1}(U_{\alpha_1}) \cup \cdots \cup f^{-1}(U_{\alpha_N})$, a finite subcover. But then

$$f(K) \subseteq U_{\alpha_1} \cup \cdots \cup U_{\alpha_N},$$

a finite subcover of the original cover $\{U_\alpha\}_{\alpha \in \mathcal{I}}$ of $f(K)$. Since the original cover was arbitrary, by definition $f(K)$ is compact. □

Continuity and compactness

Corollary (Extreme Value Theorem in $\mathbb{R}$ (EVT))

If $K \subset \mathbb{R}$ is compact and $f : K \to \mathbb{R}$ is continuous then f attains its maximum and minimum values on K.

Note: Since closed intervals $[a, b]$ are compact, this includes the standard EVT.

Proof.

Since f is continuous and K is compact, the image $f(K) \subset \mathbb{R}$ is compact. But compact subsets of $\mathbb{R}$ are exactly the closed and bounded subsets.
Therefore, $f(K)$ is a bounded set, so f is a bounded function on K. Moreover, since $f(K)$ is closed—and the supremum and infimum of a bounded set in $\mathbb{R}$ are either in the set or limit points of the set—$f(K)$ contains its supremum and infimum. That is, there exist real numbers $m = \inf f(K)$ and $M = \sup f(K)$ such that $m, M \in f(K)$. Thus, there exist points $x_{\min}, x_{\max} \in K$ such that

$$f(x_{\min}) = m \qquad \text{and} \qquad f(x_{\max}) = M.$$

Hence, f attains its minimum and maximum on K. □