31 What is $\mathbb{R}$?

McMaster University

Mathematics and Statistics

$$\int_M d\omega = \int_{\partial M} \omega$$

Mathematics 3A03
Real Analysis I

Instructor: David Earn

Lecture 31
What is $\mathbb{R}$?
Thursday 21 November 2019

Please consider. . .

5 minute *Student Respiratory Illness Survey:*

https://surveys.mcmaster.ca/limesurvey/index.php/893454

> *Please complete this anonymous survey to help us monitor the patterns of respiratory illness, over-the-counter drug use, and social contact within the McMaster community. There are no risks to filling out this survey, and your participation is voluntary. You do not need to answer any questions that make you uncomfortable, and all information provided will be kept strictly confidential. Thanks for participating.*
>
> *–Dr. Marek Smieja (Infectious Diseases)*

Last time. . .

Convergence of sequences of functions:

- Pointwise convergence
- Uniform convergence
- Theorem about integrability and uniform convergence
- Theorem about continuity and uniform convergence
- Theorem about differentiability and uniform convergence

Test 2 on Tuesday (26 November 2019), 5:30pm, JHE 264

- All material covered until today (up to but not including construction of $\mathbb{R}$).
- Emphasis on material since the first test, but the subject is cumulative.
- Remove the staple carefully, without damaging your test, when you hand it in. Bring a staple remover if that helps you.

What exactly is $\mathbb{R}$?

Construction of the Real Numbers

- Recall that we defined the natural numbers $\mathbb{N}$ as sets: $0 \equiv \varnothing$, $1 \equiv \{0\}$, $2 \equiv \{0, 1\}$, *etc.*
- For $m, n \in \mathbb{N}$ we defined $m < n$ to mean $m \subset n$.
- We defined the rational numbers $\mathbb{Q}$ to be ordered pairs of integers (more precisely, $\mathbb{Q}$ is a set of equivalence classes of $\mathbb{Z} \times \mathbb{N}$).
- In the same spirit, we can define real numbers not by determining what they "really are" but instead by settling for a definition that determines their mathematical properties completely.
- So, just as $\mathbb{Z}$ can be built from $\mathbb{N}$, and $\mathbb{Q}$ can be built from $\mathbb{Z}$, we can build $\mathbb{R}$ from $\mathbb{Q}$.
- Richard Dedekind's idea was to construct a real number α as a set of rational numbers, in a way that naturally yields the one property of $\mathbb{R}$ that $\mathbb{Q}$ does not have: least upper bounds...

Construction of the Real Numbers

Dedekind's stroke of genius (on 24 Nov 1858) was to define α as "*the set of rational numbers less than α*" in a way that is not circular.

Definition (Real number)

A ***real number*** is a set $\alpha \subseteq \mathbb{Q}$, with the following four properties:

1. $\forall x \in \alpha$, if $y \in \mathbb{Q}$ and $y < x$, then $y \in \alpha$;
2. $\alpha \neq \varnothing$;
3. $\alpha \neq \mathbb{Q}$;
4. there is no greatest element in α,
   i.e., if $x \in \alpha$ then $\exists\, y \in \alpha$ such that $y > x$.

The set of all real numbers is denoted by $\mathbb{R}$.

Historical note: Dedekind originally defined a real number α as the pair of sets (L, R) where L is the set of rationals $< \alpha$ and R is the set of rationals $\geq \alpha$. A real number is then described as a ***Dedekind cut***.

Construction of the Real Numbers

Example: $\sqrt{2} = \{q \in \mathbb{Q} : q^2 < 2 \text{ or } q < 0\}$.

With real numbers defined, we can easily define an ordering on $\mathbb{R}$.

Definition (Order of real numbers)

If $\alpha, \beta \in \mathbb{R}$ then $\alpha < \beta$ iff $\alpha \subset \beta$. (Similarly for $>$, $\leq$, and $\geq$.)

We now have enough to prove:

Theorem ($\mathbb{R}$ is complete)

If $A \subset \mathbb{R}$, $A \neq \varnothing$, and A is bounded above, then A has a least upper bound.

We also need to define $+$, $\cdot$, 1 and α^{-1}. (See Assignment 6.)

Then we can prove that $\mathbb{R}$ *is a complete ordered field* and, moreover, it is the *unique* such field (up to isomorphism).

Construction of the Real Numbers

Proof that $\mathbb{R}$ is complete.

Let $\beta = \{x : x \in \alpha \text{ for some } \alpha \in A\} = \bigcup_{\alpha \in A} \alpha$.
Then, for any $\alpha \in A$ we have $\alpha \subseteq \beta$.

Since each $x \in \beta$ is in some set $\alpha \subseteq \mathbb{Q}$, we have $\beta \subseteq \mathbb{Q}$. To verify that $\beta \in \mathbb{R}$, we check the four defining properties:

1. Suppose (i) $x \in \beta$ and (ii) $y < x$. (i) $\implies$ $x \in \alpha$ for some $\alpha \in A$. But α is a real number, so (ii) $\implies$ $y \in \alpha$. Hence $y \in \beta$.
2. Since $A \neq \varnothing$, $\exists \alpha \in A$. Since α is a real number, $\exists x \in \alpha$. This implies $x \in \beta$, so $\beta \neq \varnothing$.
3. Since A is bounded above (by hypothesis), there is some real number γ such that $\alpha < \gamma$ for every $\alpha \in A$. Since γ is a real number, there is some rational number $x \notin \gamma$. But $\alpha < \gamma$ means that $\alpha \subset \gamma$, so it follows that $x \notin \alpha$ for any $\alpha \in A$. This implies $x \notin \beta$, so $\beta \neq \mathbb{Q}$.

. . . continued. . .

Construction of the Real Numbers

Proof that $\mathbb{R}$ is complete (*continued*).

4. Suppose $x \in \beta$. Then $x \in \alpha$ for some $\alpha \in A$. Since α does not have a greatest element, $\exists y \in \mathbb{Q}$ with $x < y$ and $y \in \alpha$. But this implies $y \in \beta$; thus β does not have a greatest element.

These four points establish that β is a real number. It remains to show that β is the ***least upper bound*** of A.

If $\alpha \in A$, then $\alpha \subseteq \beta$, *i.e.*, $\alpha \leq \beta$, so β is an upper bound for A. On the other hand, if γ is an upper bound for A, then $\alpha \leq \gamma$ for every $\alpha \in A$; this implies $\alpha \subseteq \gamma$, for every $\alpha \in A$, and hence $\beta \subseteq \gamma$, *i.e.*, $\beta \leq \gamma$. Thus β is the <u>least</u> upper bound of A. □

Poll

- Go to https://www.childsmath.ca/childsa/forms/main_login.php
- Click on Math 3A03
- Click on Take Class Poll
- Fill in poll **Lecture 31: construction of reals**
- Submit.

Next time...

An alternative construction of $\mathbb{R}$...

And much much more...

Surreal numbers...

...Guest lecture by Dr. Jonathan Dushoff

...Check out Hackenbush on Wikipedia and Hackenbush app (http://geometer.org/hackenbush).

See also: Michael Cronin, Combinatorial games and surreal numbers

For the very serious: Combinatorial Game Theory