McMaster University

Mathematics and Statistics

$$\int_M d\omega = \int_{\partial M} \omega$$

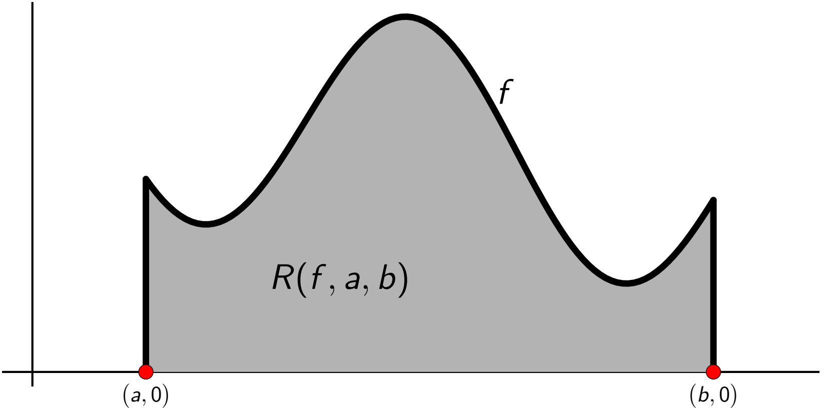
Mathematics 3A03
Real Analysis I

Instructor: David Earn

Lecture 26
Integration
Friday 8 November 2019

Integration

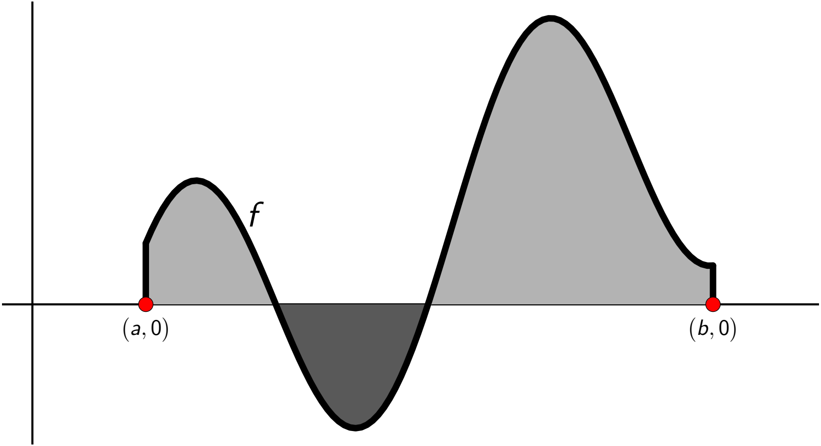
Integration



- "Area of region $R(f, a, b)$" is actually a very subtle concept.
- We will only scratch the surface of it.
- Textbook presentation of integral is different (but equivalent).

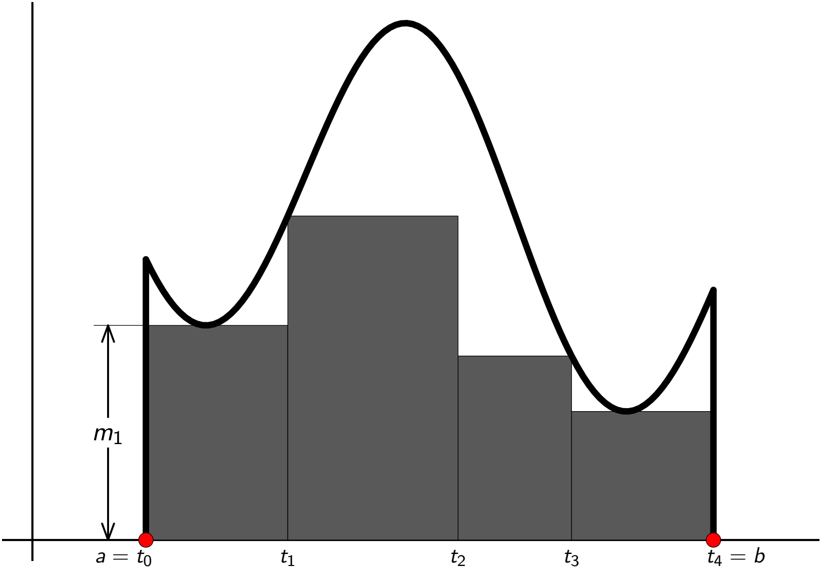
Our treatment is closer to that in M. Spivak "Calculus" (2008).

Integration

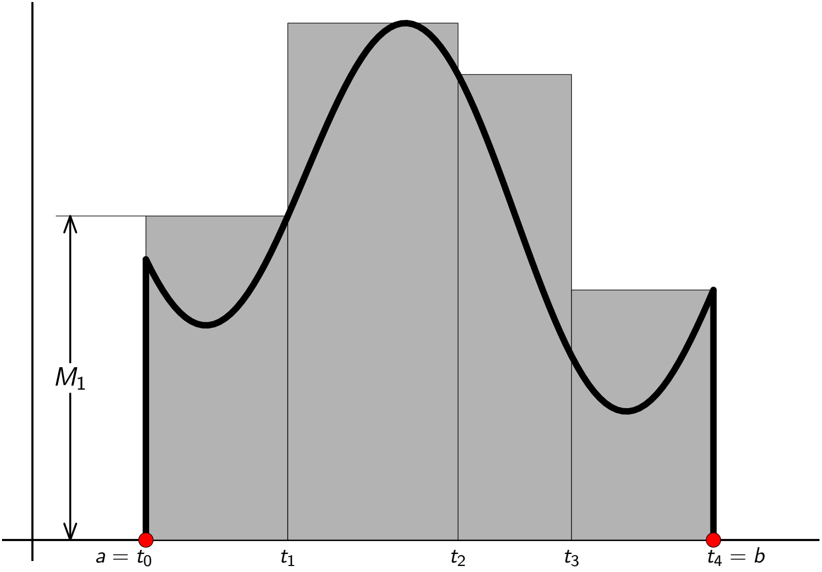


- Contribution to "area of $R(f, a, b)$" is positive or negative depending on whether f is positive or negative.

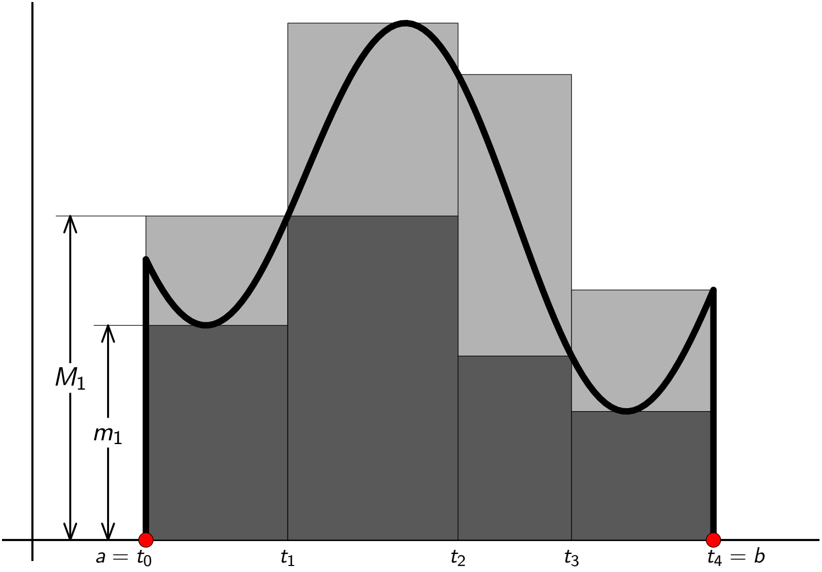
Lower sum



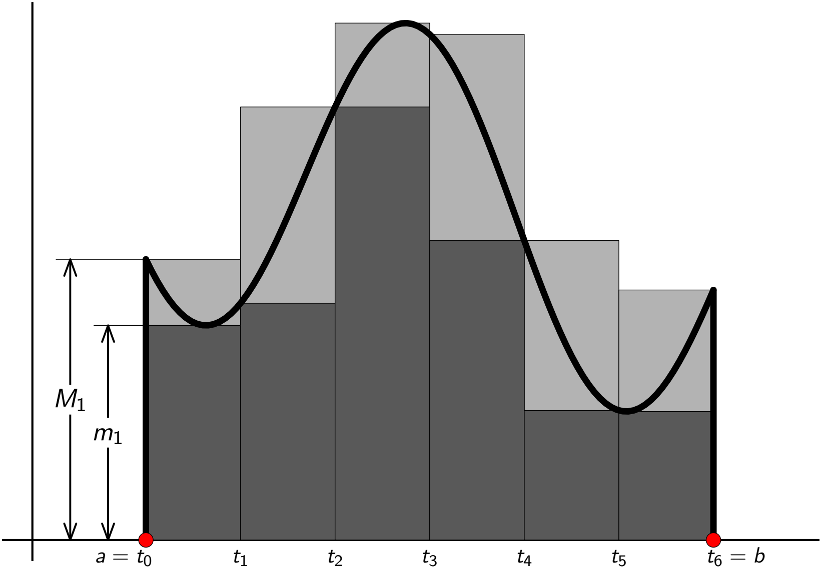
Upper sum



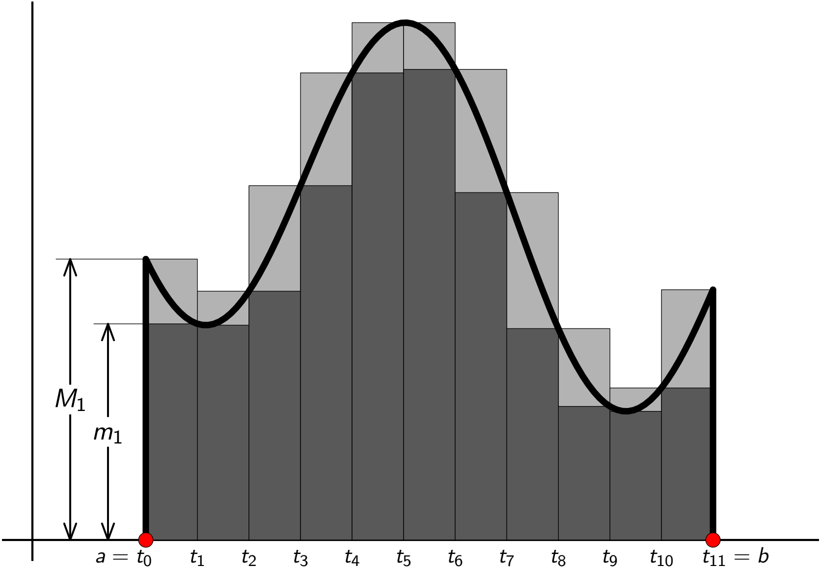
Lower and upper sums



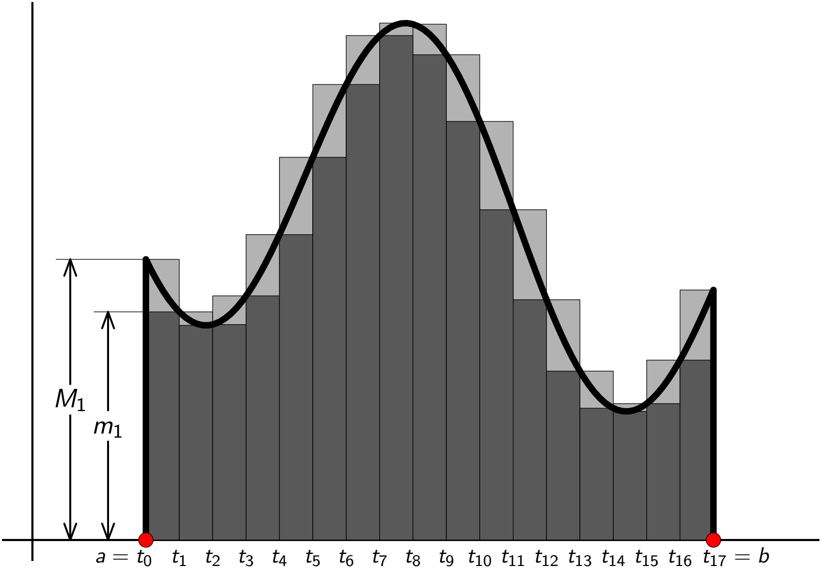
Lower and upper sums



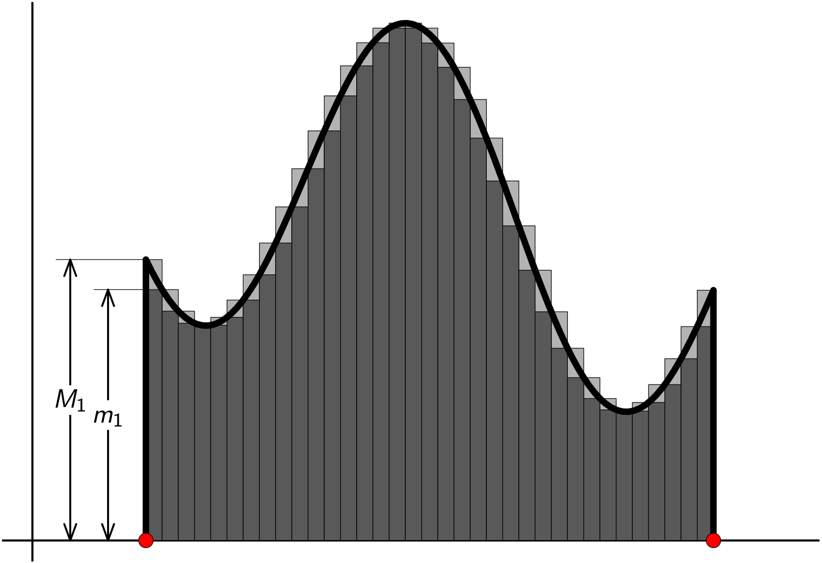
Lower and upper sums



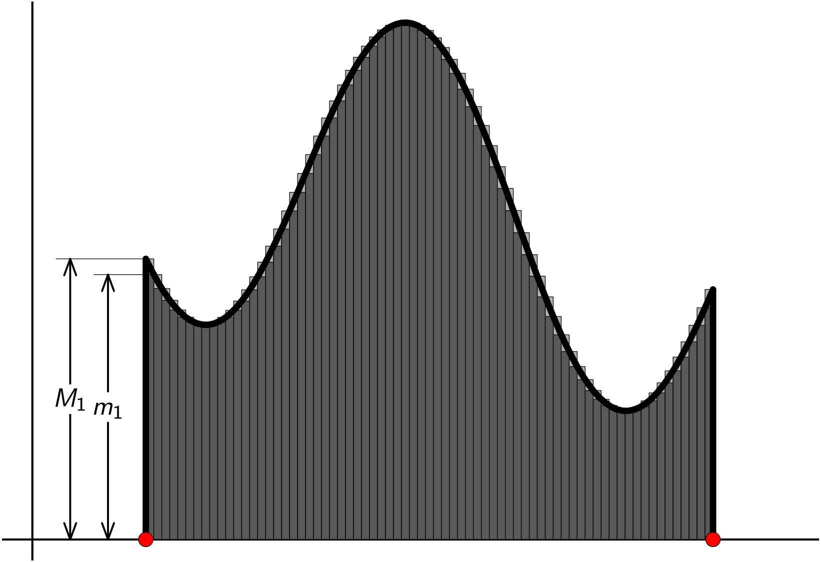
Lower and upper sums



Lower and upper sums



Lower and upper sums



Rigorous development of the integral

Definition (Partition)

Let $a < b$. A ***partition*** of the interval $[a, b]$ is a finite collection of points in $[a, b]$, one of which is a, and one of which is b.

We normally label the points in a partition

$$a = t_0 < t_1 < \cdots < t_{n-1} < t_n = b \,,$$

so the ith subinterval in the partition is

$$[t_{i-1}, t_i] \,.$$

Rigorous development of the integral

Definition (Lower and upper sums)

Suppose f is bounded on $[a, b]$ and $P = \{t_0, \ldots, t_n\}$ is a partition of $[a, b]$. Let

$$m_i = \inf \{ f(x) : x \in [t_{i-1}, t_i] \},$$

$$M_i = \sup \{ f(x) : x \in [t_{i-1}, t_i] \}.$$

The lower sum of f for P, denoted by $L(f, P)$, is defined as

$$L(f, P) = \sum_{i=1}^{n} m_i(t_i - t_{i-1}).$$

The upper sum of f for P, denoted by $U(f, P)$, is defined as

$$U(f, P) = \sum_{i=1}^{n} M_i(t_i - t_{i-1}).$$

Rigorous development of the integral

Relationship between motivating sketch and rigorous definition of lower and upper sums:

- The lower and upper sums correspond to the total areas of rectangles lying below and above the graph of f in our motivating sketch.
- However, these sums have been defined precisely without any appeal to a concept of "area".
- The requirement that f be bounded on $[a, b]$ is essential in order that all the m_i and M_i be well-defined.
- It is also essential that the m_i and M_i be defined as inf's and sup's (rather than maxima and minima) because f was not assumed continuous.

Rigorous development of the integral

Relationship between motivating sketch and rigorous definition of lower and upper sums:

- Since $m_i \le M_i$ for each i, we have

$$m_i(t_i - t_{i-1}) \le M_i(t_i - t_{i-1}) . \qquad i = 1, \ldots, n.$$

$\therefore$ For <u>any</u> partition P of $[a, b]$ we have

$$L(f, P) \le U(f, P),$$

because

$$L(f, P) = \sum_{i=1}^{n} m_i(t_i - t_{i-1}) ,$$

$$U(f, P) = \sum_{i=1}^{n} M_i(t_i - t_{i-1}) .$$

Poll

- Go to https://www.childsmath.ca/childsa/forms/main_login.php
- Click on Math 3A03
- Click on Take Class Poll
- Fill in poll **Lecture 26: Lower and Upper Sums**
- Submit.

Rigorous development of the integral

Relationship between motivating sketch and rigorous definition of lower and upper sums:

- More generally, if P_1 and P_2 are <u>any</u> two partitions of $[a, b]$, it <u>*ought*</u> to be true that

  $$L(f, P_1) \le U(f, P_2),$$

  because $L(f, P_1)$ should be $\le$ area of $R(f, a, b)$, and $U(f, P_2)$ should be $\ge$ area of $R(f, a, b)$.

- But "ought to" and "should be" prove nothing, especially since we haven't yet even defined "area of $R(f, a, b)$".

- Before we can *define* "area of $R(f, a, b)$", we need to prove that $L(f, P_1) \le U(f, P_2)$ for any partitions P_1, P_2 ...

McMaster University

Mathematics and Statistics

$$\int_M d\omega = \int_{\partial M} \omega$$

Mathematics 3A03
Real Analysis I

Instructor: David Earn

Lecture 27
Integration II
Tuesday 12 November 2019

Poll

- Go to https://www.childsmath.ca/childsa/forms/main_login.php
- Click on Math 3A03
- Click on Take Class Poll
- Fill in poll **Lecture 27: Assignment 3 Awaremess**
- $\boxed{\texttt{Submit}}$.

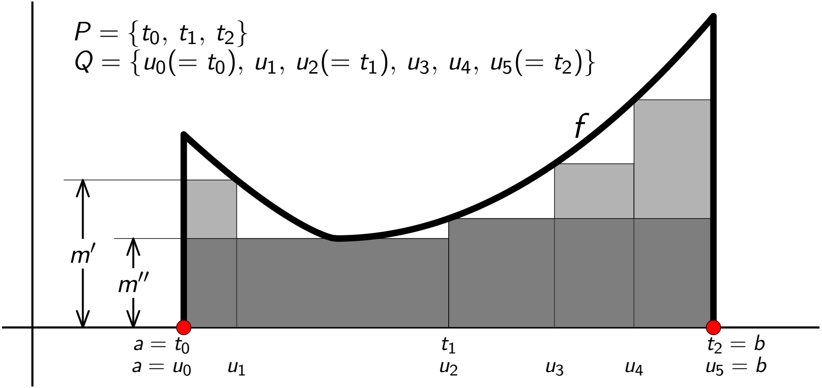
Announcements

- Assignment 4 was due before class today.
- Assignment 5 is due on
  Thursday 21 November 2019 @ 2:25pm via crowdmark.
- Math 3A03 **Test #2**
  Tuesday 26 November 2019, 5:30–7:00pm, in JHE 264
- Assignment 6 will be due on **Tuesday 3 December 2019 @ 2:25pm via crowdmark**.
- Math 3A03 **Final Exam**: Fri 6 Dec 2019, 9:00am–11:30am
  Location: MDCL 1105

Rigorous development of the integral

Lemma

If partition $P \subseteq$ partition Q (i.e., if every point of P is also in Q), then $\quad L(f,P) \le L(f,Q) \quad$ *and* $\quad U(f,P) \ge U(f,Q)$.



Rigorous development of the integral

Proof of Lemma

As a first step, consider the special case in which the finer partition Q contains only one more point than P:

$$P = \{t_0, \dots, t_n\},$$
$$Q = \{t_0, \dots, t_{k-1}, u, t_k, \dots, t_n\},$$

where

$$a = t_0 < t_1 < \cdots < t_{k-1} < u < t_k < \cdots < t_n = b.$$

Let

$$m' = \inf\{\, f(x) \,:\, x \in [t_{k-1}, u] \,\},$$
$$m'' = \inf\{\, f(x) \,:\, x \in [u, t_k] \,\}.$$

. . . continued. . .

Rigorous development of the integral

Proof of Lemma (cont.)

Then $$L(f,P)=\sum_{i=1}^{n} m_i(t_i-t_{i-1})\,,$$

and $$L(f,Q)=\sum_{i=1}^{k-1} m_i(t_i-t_{i-1})+m'(u-t_{k-1}) + m''(t_k-u)+\sum_{i=k+1}^{n} m_i(t_i-t_{i-1})\,.$$

$\therefore$ To prove $L(f,P)\le L(f,Q)$, it is enough to show

$$m_k(t_k-t_{k-1})\le m'(u-t_{k-1})+m''(t_k-u)\,.$$

. . . continued. . .

Rigorous development of the integral

Proof of Lemma (cont.)

Now note that since

$$\{\, f(x) \,:\, x \in [t_{k-1}, u] \,\} \quad \subseteq \quad \{\, f(x) \,:\, x \in [t_{k-1}, t_k] \,\}\,,$$

the RHS might contain some additional *smaller* numbers, so we must have

$$\begin{aligned} m_k \quad &= \quad \inf \{\, f(x) \,:\, x \in [t_{k-1}, t_k] \,\} \\ &\leq \quad \inf \{\, f(x) \,:\, x \in [t_{k-1}, u] \,\} \quad = \quad m'\,. \end{aligned}$$

Thus, $m_k \leq m'$, and, similarly, $m_k \leq m''$.

$$\begin{aligned} \therefore \quad m_k(t_k - t_{k-1}) \quad &= \quad m_k(t_k - u + u - t_{k-1}) \\ &= \quad m_k(u - t_{k-1}) + m_k(t_k - u) \\ &\leq \quad m'(u - t_{k-1}) + m''(t_k - u)\,, \end{aligned}$$

...continued...

Rigorous development of the integral

Proof of Lemma (cont.)

which proves (in this special case where Q contains only one more point than P) that $L(f,P) \le L(f,Q)$.

We can now prove the general case by adding one point at a time.

If Q contains ℓ more points than P, define a sequence of partitions

$$P = P_0 \subset P_1 \subset \cdots \subset P_\ell = Q$$

such that P_{j+1} contains exactly one more point that P_j. Then

$$L(f,P) = L(f,P_0) \le L(f,P_1) \le \cdots \le L(f,P_\ell) = L(f,Q)\,,$$

so $L(f,P) \le L(f,Q)$.

(Proving $U(f,P) \ge U(f,Q)$ is similar: check!) $\square$

Rigorous development of the integral

Theorem (Partition Theorem)

Let P_1 and P_2 be any two partitions of $[a, b]$. If f is bounded on $[a, b]$ then

$$L(f, P_1) \leq U(f, P_2).$$

Proof.

This is a straightforward consequence of the partition lemma.

Let $P = P_1 \cup P_2$, *i.e.*, the partition obtained by combining all the points of P_1 and P_2.

Then

$$L(f, P_1) \leq L(f, P) \leq U(f, P) \leq U(f, P_2).$$

$\square$

Rigorous development of the integral

Important inferences that follow from the partition theorem:

- For any partition P', the upper sum $U(f,P')$ is an upper bound for the set of all lower sums $L(f,P)$.

  $$\therefore \quad \sup\{L(f,P) : P \text{ a partition of } [a,b]\} \le U(f,P') \qquad \forall P'$$

  $$\therefore \quad \sup\{L(f,P)\} \le \inf\{U(f,P)\}$$

  $\therefore$ For any partition P',

  $$L(f,P') \le \sup\{L(f,P)\} \le \inf\{U(f,P)\} \le U(f,P')$$

- If $\sup\{L(f,P)\} = \inf\{U(f,P)\}$ then we can define "area of $R(f,a,b)$" to be this number.
  - Is it possible that $\sup\{L(f,P)\} < \inf\{U(f,P)\}$?

Rigorous development of the integral

Example

$\exists ?\ f : [a,b] \to \mathbb{R}$ such that $\sup\{L(f,P)\} < \inf\{U(f,P)\}$

Let

$$f(x) = \begin{cases} 1 & x \in \mathbb{Q} \cap [a,b], \\ 0 & x \in \mathbb{Q}^{c} \cap [a,b]. \end{cases}$$

If $P = \{t_0, \ldots, t_n\}$ then $m_i = 0 \qquad (\because \quad [t_{i-1}, t_i] \cap \mathbb{Q}^{c} \neq \varnothing),$
and $M_i = 1 \qquad (\because \quad [t_{i-1}, t_i] \cap \mathbb{Q} \neq \varnothing).$

$\therefore \quad L(f,P) = 0$ and $U(f,P) = b - a$ for <u>any</u> partition P.

$\therefore \quad \sup\{L(f,P)\} = 0 < b - a = \inf\{U(f,P)\}.$ □

Can we define "area of $R(f,a,b)$" for such a weird function?
Yes, but not in this course!

Rigorous development of the integral

Definition (Integrable)

A function $f : [a, b] \to \mathbb{R}$ is said to be ***integrable*** on $[a, b]$ if it is bounded on $[a, b]$ and

$$\begin{aligned} &\sup\{L(f, P) : P \text{ a partition of } [a, b]\} \\ &\quad = \inf\left\{U(f, P) : P \text{ a partition of } [a, b]\right\}. \end{aligned}$$

In this case, this common number is called the ***integral*** of f on $[a, b]$ and is denoted

$$\int_a^b f$$

Note: If f is integrable then for any partition P we have

$$L(f, P) \leq \int_a^b f \leq U(f, P),$$

and $\int_a^b f$ is the unique number with this property.

Rigorous development of the integral

- *Notation:*

$$\int_a^b f(x)\,dx \qquad \text{means precisely the same as} \qquad \int_a^b f\,.$$

  - The symbol "dx" has no meaning in isolation just as "$x \to$" has no meaning except in $\lim_{x\to a} f(x)$.

- It is not clear from the definition which functions are integrable.

- The definition of the integral does not itself indicate how to compute the integral of any given integrable function. So far, without a lot more effort we can't say much more than these two things:
  1. If $f(x) \equiv c$ then f is integrable on $[a, b]$ and $\int_a^b f = c \cdot (b - a)$.
  2. The weird example function is not integrable.

Rigorous development of the integral

- A function that is integrable according to our definition is usually said to be ***Riemann integrable***, to distinguish this definition from other definitions of integrability.
- In Math 4A03 you will define "Lebesgue integrable", a more subtle concept that makes it possible to attach meaning to "area of $R(f, a, b)$" for the weird example function (among others), and to precisely characterize functions that are Riemann integrable.

Rigorous development of the integral

Theorem (Equivalent condition for integrability)

A bounded function $f : [a, b] \to \mathbb{R}$ is integrable on $[a, b]$ iff for all $\varepsilon > 0$ there is a partition P of $[a, b]$ such that

$$U(f, P) - L(f, P) < \varepsilon\,.$$

Proof.

2016 Assignment 5. ☐

Note: This theorem is just a restatement of the definition of integrability. It is often more convenient to work with $\varepsilon > 0$ than with sup's and inf's.