McMaster University

Mathematics and Statistics

$$\int_M d\omega = \int_{\partial M} \omega$$

Mathematics 3A03
Real Analysis I

Instructor: David Earn

Lecture 22
Metric Spaces
Monday 10 March 2025

Announcements

- New, exciting topic today. . .

Metric Spaces

The metric structure of $\mathbb{R}$

Definition (Absolute Value function)

For any $x \in \mathbb{R}$,

$$|x| \overset{\text{def}}{=} \begin{cases} x & \text{if } x \geq 0, \\ -x & \text{if } x < 0. \end{cases}$$

Theorem (Properties of the Absolute Value function)

For all $x, y \in \mathbb{R}$:

1. $-|x| \leq x \leq |x|$.
2. $|xy| = |x|\,|y|$.
3. $|x + y| \leq |x| + |y|$.
4. $|x| - |y| \leq |x - y|$.

The metric structure of $\mathbb{R}$

Definition (Distance function or metric)

The distance between two real numbers x and y is

$$d(x,y) = |x - y| \ .$$

Theorem (Properties of distance function or metric)

1. $d(x,y) \geq 0$ — *distances are positive or zero*
2. $d(x,y) = 0 \iff x = y$ — *distinct points have distance > 0*
3. $d(x,y) = d(y,x)$ — *distance is symmetric*
4. $d(x,y) \leq d(x,z) + d(z,y)$ — *the triangle inequality*

Note: Any function satisfying these properties can be considered a "distance" or "metric".

The metric structure of $\mathbb{R}$

Given $d(x,y) = |x-y|$, the properties of the distance function are equivalent to:

Theorem (Metric properties of the absolute value function)

For all $x, y \in \mathbb{R}$:

1. $|x| \geq 0$
2. $|x| = 0 \iff x = 0$
3. $|x| = |-x|$
4. $|x+y| \leq |x| + |y|$ *(the triangle inequality)*

Slick proof of the triangle inequality

Theorem (The Triangle Inequality for the standard metric on $\mathbb{R}$)

$|x+y| \le |x| + |y|$ *for all* $x, y \in \mathbb{R}$.

Proof.

Let $s = \operatorname{sign}(x+y)$. Then

$$|x+y| \;=\; s(x+y) \;=\; sx + sy \;\le\; |x| + |y| \;,$$

as required. □

A non-standard metric on $\mathbb{R}$

Example (finite distance between every pair of real numbers)

Let $f(x) = \frac{x}{1+x}$, and define $d(x,y) = f(|x-y|)$. Prove that $d(x,y)$ can be interpreted as a distance between x and y because it satisfies all the properties of a metric.

Proof: The only metric property that is non-trivial to prove is the triangle inequality. Note that $f(x)$ is an increasing function on $[0,\infty)$, so the usual triangle inequality, $|x-y| \le |x-z| + |z-y|$, implies

$$\begin{aligned} f(|x-y|) &\le f(|x-z| + |z-y|) = \frac{|x-z| + |z-y|}{1 + |x-z| + |z-y|} \\ &= \frac{|x-z|}{1 + |x-z| + |z-y|} + \frac{|z-y|}{1 + |x-z| + |z-y|} \\ &\le \frac{|x-z|}{1+|x-z|} + \frac{|z-y|}{1+|z-y|} = f(|x-z|) + f(|z-y|) \end{aligned}$$

i.e., $d(x,y) \le d(x,z) + d(z,y)$. □

Poll

- Go to
  https://www.childsmath.ca/childsa/forms/main_login.php
- Click on Math 3A03
- Click on Take Class Poll
- Fill in poll **Metric spaces: Is "$=$ vs $\neq$" a metric?**
- Submit.

Discrete metric

Example (Discrete metric on $\mathbb{R}$)

Let $d(x,y) = \begin{cases} 0 & x = y, \\ 1 & x \neq y. \end{cases}$ Is d a metric on $\mathbb{R}$?

By definition, $d(x,y)$ is non-negative, zero iff $x = y$, and symmetric. For the triangle inequality, if $x = y$ then $d(x,y) = 0$ so the inequality holds for any z. If $x \neq y$ then $d(x,y) = 1$, and at least one of x and y must not equal z, so the inequality says either $1 \leq 1$ or $1 \leq 2$.

Example (Discrete metric on any set X)

The argument that $d(x,y)$ is a metric on $\mathbb{R}$ has nothing to do with $\mathbb{R}$ specifically. $d(x,y)$ is a metric on <u>any</u> set X.

General metric space (X, d)

Definition (Metric space)

A ***metric space*** (X, d) is a non-empty set X together with a distance function (or ***metric***) $d : X \times X \to \mathbb{R}$ satisfying

1. $d(x, y) \geq 0$ — *distances are positive or zero*
2. $d(x, y) = 0 \iff x = y$ — *distinct points have distance > 0*
3. $d(x, y) = d(y, x)$ — *distance is symmetric*
4. $d(x, y) \leq d(x, z) + d(z, y)$ — *the triangle inequality*

Much of our analysis of sequences of real numbers and topology of $\mathbb{R}$ generalizes to any metric space. Very often, definitions and proofs depend only on the the existence of a metric, not on $|x|$ specifically. Many useful inferences can be made by identifying a metric on a space of interest.

Examples of metric spaces

Example (Metric spaces (X, d))

- $X = \mathbb{Q}$, with the standard metric $d(x, y) = |x - y|$.
  As $\mathbb{Q} \subset \mathbb{R}$, each condition for d is satisfied in $\mathbb{Q}$.

  How different is $(\mathbb{Q}, d)$ from $(\mathbb{R}, d)$?

- $X = \mathbb{N}$, with the standard metric $d(x, y) = |x - y|$.
  As $\mathbb{N} \subset \mathbb{R}$, each condition for d is satisfied in $\mathbb{N}$.

- $X = \mathbb{R}^2$ with $d(x, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$, where we write the vectors $x = (x_1, x_2)$, $y = (y_1, y_2) \in \mathbb{R}^2$.

- $X = \mathbb{R}^n$ with $d(x, y) = \sqrt{\sum_{i=1}^{n}(x_i - y_i)^2}$, where $x = (x_1, \dots, x_n)$, $y = (y_1, \dots, y_n) \in \mathbb{R}^n$.

  This metric on $\mathbb{R}^n$ is called the ***Euclidean distance***.

Metrics from norms

The Euclidean metric on $\mathbb{R}^n$ is the (Euclidean) length of the difference of two vectors. This connection between length and distance generalizes to any vector space in which *length* is defined.

Definition (Norm)

A ***norm*** on a vector space X is a real-valued function on X such that if $x, y \in X$ and $\alpha \in \mathbb{R}$ then

1. $\|x\| \geq 0$ and $\|x\| = 0$ iff x is the zero element in X;
2. $\|\alpha x\| = |\alpha| \, \|x\|$;
3. $\|x + y\| \leq \|x\| + \|y\|$.

A vector space X equipped with a norm $\|\cdot\|$ is said to be a ***normed vector space***. Any norm $\|\cdot\|$ ***induces*** a metric d via

$$d(x, y) = \|x - y\| .$$

Proving that a function is a norm is not necessarily easy. Let's try for the Euclidean norm... To that end, recall the notion of inner product ...

Norms from inner products

Definition (Inner product)

An ***inner product*** on a vector space V over $\mathbb{R}$ is a function

$$\langle \cdot, \cdot \rangle : V \times V \to \mathbb{R}$$

such that for all $u, v, w \in V$ and all scalars $\alpha \in \mathbb{R}$:

1. $\langle u, v \rangle = \overline{\langle v, u \rangle}$ *conjugate symmetry*
2. $\langle \alpha u + v, w \rangle = \alpha \langle u, w \rangle + \langle v, w \rangle$ *linearity in 1st argument*
3. $\langle v, v \rangle \geq 0$ with equality iff $v = 0$ *positive definiteness*

A vector space equipped with an inner product is called an *inner product space*.

Definition (Inner Product Norm)

The norm induced by an inner product $\langle \cdot, \cdot \rangle$ is $\|u\| = \sqrt{\langle u, u \rangle}$.

Norms from inner products

Theorem (Cauchy-Schwarz inequality)

Let V be a (real) inner product space with inner product $\langle\cdot,\cdot\rangle$. For all vectors $u, v \in V$, we have

$$|\langle u, v\rangle| \ \leq \ \|u\| \, \|v\|$$

where $\|u\| = \sqrt{\langle u, u\rangle}$ *is the norm induced by the inner product.*

Proof.

The standard proof begins with an idea that probably took someone a long time to think of: Since $\langle v, v\rangle \geq 0$ for any $v \in V$, for any $t \in \mathbb{R}$ we have

$$\begin{aligned} 0 \ \leq \ \langle u + tv, u + tv\rangle \ &= \ \langle u, u\rangle + t\,\langle u, v\rangle + t\,\langle v, u\rangle + t^2\,\langle v, v\rangle \\ &= \ \langle u, u\rangle + 2t\,\langle u, v\rangle + t^2\,\langle v, v\rangle \end{aligned}$$

This is a quadratic polynomial in t, which is non-negative for all $t \in \mathbb{R}$. Hence, this quadratic has at most one real root. Consequently, its discriminant is non-positive, *i.e.*, $(2\,\langle u, v\rangle)^2 - 4\,\langle u, u\rangle\,\langle v, v\rangle \ \leq \ 0$.

continued...

Norms from inner products

Proof of Cauchy-Schwarz inequality (continued).

Simplifying the non-positive discriminant condition, we have

$$(\langle u, v\rangle)^2 \le \langle u, u\rangle \langle v, v\rangle \ .$$

Taking square roots, we have

$$|\langle u, v\rangle| \le \|u\| \, \|v\| \ ,$$

as required. □

How might you come up with such a proof?

Perhaps by guessing the result (based on knowing it in $\mathbb{R}^2$) and then working backwards.

McMaster University

Mathematics and Statistics

$$\int_M d\omega = \int_{\partial M} \omega$$

Mathematics 3A03
Real Analysis I

Instructor: David Earn

Lecture 23
Metric Spaces II
Wednesday 12 March 2025

Announcements

- Participation deadline for Assignment 4 was at 11:25am today.

Last time...

- Introduction to metric spaces
  - Critical ingredient: the triangle inequality
- Cauchy-Schwarz inequality
  (proved for real inner product spaces)

Norms from inner products

If X is an inner product space, then Cauchy-Schwarz allows us to prove that the induced norm really is a norm (*i.e.*, satisfies the triangle inequality). For $x, y \in X$, we have

$$\begin{aligned} \|x+y\|^2 &= \langle x+y, x+y \rangle \\ &= \langle x,x\rangle + \langle x,y\rangle + \langle y,x\rangle + \langle y,y\rangle \\ &= \|x\|^2 + \|y\|^2 + 2\langle x,y\rangle \\ &\le \|x\|^2 + \|y\|^2 + 2\|x\|\,\|y\| \\ &= (\|x\| + \|y\|)^2 \end{aligned}$$

$$\implies \quad \|x+y\| \le \|x\| + \|y\| \,.$$

In particular, $\mathbb{R}^n$ with the usual "dot product" $\langle x, y\rangle = \sum_{i=1}^n x_i y_i$ induces the Euclidean norm $\|\cdot\|_2$, which therefore really is a norm, and $d(x,y) = \|x-y\|_2$ really is a metric (the Euclidean distance).

What about other norms induced by inner products?

Other metric spaces induced by inner products

We are accustomed to finite vectors:

$$x = (x_1, x_2) \in X = \mathbb{R}^2$$
$$x = (x_1, x_2, x_3) \in X = \mathbb{R}^3$$
$$x = (x_1, x_2, \dots, x_n) \in X = \mathbb{R}^n$$

We can think of a sequence as an infinite vector:

$$x = (x_1, x_2, \dots) \in X = \big\{\{x_n\} : n \in \mathbb{N}\big\}$$

The points in this space (X) are infinite-dimensional vectors.

We can think of an infinite vector as a function:

$$x_n = f(n) \quad \Longrightarrow \quad (x_1, x_2, \dots) = \big(f(1), f(2), \dots\big)$$

The points in this space are functions: $X = \big\{f \,\big|\, f : \mathbb{N} \to \mathbb{R}\big\}$.

So we can generalize to other spaces via functions, *e.g.*,

$$C[0,1] = \big\{f : [0,1] \to \mathbb{R} \,\big|\, f \text{ continuous}\big\}$$

All of the above spaces have a natural inner product, and hence a natural norm and metric.

Other metric spaces induced by inner products

Inner products that convert the spaces on the previous slide into (Euclidean) metric spaces:

$$\mathbb{R}^n: \quad \langle x, y \rangle = \sum_{i=1}^{n} x_i y_i$$

$$\ell^2(\mathbb{R}): \quad \langle x, y \rangle = \sum_{i=1}^{\infty} x_i y_i$$

$$C[a,b]: \quad \langle f, g \rangle = \int_a^b f(x) g(x)\, dx$$

Note: ℓ^2 includes only ***square-summable*** sequences: $\sum_{n=1}^{\infty} x_n^2 < \infty$

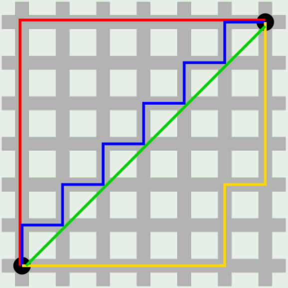
*Do we need to specify that $C[a,b]$ contains only **square-integrable** functions?*

Metrics from norms

Example (Taxicab distance)

Taxicab norm on $\mathbb{R}^n$

$$\|x\|_1 = \sum_{i=1}^{n} |x_i|$$



In taxicab geometry, the lengths of the **red**, **blue**, **green**, and **yellow** paths all equal 12, the taxicab distance between the opposite corners, and all four paths are shortest paths. Instead, in Euclidean geometry, the **red**, **blue**, and **yellow** paths still have length 12 but the **green** path is the unique shortest path, with length equal to the Euclidean distance between the opposite corners, $6\sqrt{2} \approx 8.49$.

Image and caption from Wikipedia article on "Taxicab geometry".

Note: The **green** path can be followed. All the points of $\mathbb{R}^2$ are still present when we measure distance with the taxicab metric. Any monotonic curve path that joins the two points can be approximated by an arbitrarily fine grid, and will have the same length. The metric does not impose a particular grid.

Metrics from norms

Example (p-metric)

p-norm on $\mathbb{R}^n$ (for $p \geq 1$) $$\|x\|_p = \Big(\sum_{i=1}^{n} |x_i|^p\Big)^{\frac{1}{p}}.$$

- $p = 1$ is the taxicab norm.
- $p = 2$ is the Euclidean norm.

What happens as $p \to \infty$? For any $x \in \mathbb{R}^n$, we have

$$\begin{aligned}\|x\|_p &= \Big(\sum_{i=1}^{n} |x_i|^p\Big)^{\frac{1}{p}} = |x_k| \Big(\sum_{i=1}^{n} \Big|\frac{x_i}{x_k}\Big|^p\Big)^{\frac{1}{p}} \qquad (|x_k| > |x_i| \;\; \forall i \neq k) \\ &= |x_k| \Big(1 + \sum_{i \neq k} \Big|\frac{x_i}{x_k}\Big|^p\Big)^{\frac{1}{p}} \xrightarrow{p \to \infty} |x_k|\end{aligned}$$

What further work is required if $\nexists k \ni |x_k| > |x_i| \;\; \forall i \neq k$?

Therefore, we define $\|\cdot\|_\infty$ to be

Max norm on $\mathbb{R}^n$ $$\|x\|_\infty = \max_{1 \leq i \leq n} |x_i|$$

Poll

- Go to
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- Fill in poll **Metric spaces: Which p-norms are induced?**
- Submit.

Metrics from norms

Example (p-metric)

Proving that $p = 1$ and $p = \infty$ yield norms is a *good exercise*.

Only $p = 2$ is induced by an inner product.

That the p-norms for $p \neq 1, 2, \infty$ are norms is harder to prove (but true), so

$$d_p(x, y) = \|x - y\|_p$$

is a metric on $\mathbb{R}^n$ for any $p \geq 1$.

Topology of metric spaces

We can generalize the notion of "neighbourhood of a point" to any metric space:

Definition (Open ball)

Let (X, d) be a metric space. If $x_0 \in X$ and $r > 0$ then the ***open ball of radius r about x_0*** is

$$B_r(x_0) = \{x \in X : d(x, x_0) < r\} .$$

x_0 is said to be the ***centre*** of $B_r(x_0)$.

Note: The notation $B(x_0, r)$ is also common (and used in TBB).

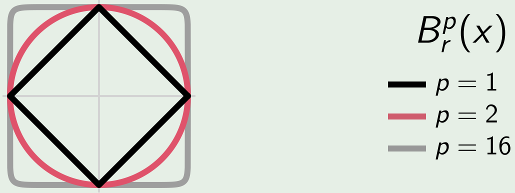
Definition (Neighbourhood)

A ***neighborhood*** of x is any set that contains an open ball $B_r(x)$ for some $r > 0$.

Topology of metric spaces

Example (Open balls in metric spaces)

- In the metric space $(\mathbb{R}, \textit{standard})$, *i.e.*, $\mathbb{R}$ with $d(x,y) = |x-y|$, $B_r(x) = (x-r, x+r)$, an open interval of length $2r$ centred at x.
- In $\mathbb{R}^n$ with Euclidean metric $d(x,y) = \|x-y\|_2$, $B_r(x)$ has a spherical boundary (circular boundary if $n=2$).
- In $\mathbb{R}^n$ with a p-norm $\|\cdot\|_p$, the ball is not spherical. For $\mathbb{R}^2$ with the Taxicab metric $d(x,y) = \|x-y\|_1$, $B_r(x)$ is diamond shaped, and for $\mathbb{R}^2$ with the Max norm $\|\cdot\|_\infty$, $B_r(x)$ is a square. We write $B_r^p(x)$ for balls in the p-norm ("***p-balls***").



Topology of metric spaces

Example (p-norm inequalities and containments)

The following inequalities relate the various p-norms on $\mathbb{R}^n$,

$$\|x\|_\infty \le \|x\|_2 \le \|x\|_1 \le n\,\|x\|_\infty\,, \quad \text{and} \quad \|x\|_1 \le \sqrt{n}\,\|x\|_2\,.$$

A good exercise.

Balls in the norm $\|\cdot\|_p$ are often written $B_r^p(x)$. The inequalities above imply that the following sets are *nested:*

$$B^2_{r/n}(x) \subset B^\infty_{r/n}(x) \subset B^1_r(x) \subset B^2_r(x) \subset B^\infty_r(x).$$

Another good exercise.

Example (Balls in the discrete metric)

For any set X, in the *discrete metric* the balls are simple, but strange. If $0 < r \le 1$ then $B_r(x) = \{x\}$, a single point! If $r > 1$ then $B_r(x) = X$, the whole space! You can't be "close" to a point x unless you are at x itself!

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Mathematics and Statistics

$$\int_M d\omega = \int_{\partial M} \omega$$

Mathematics 3A03
Real Analysis I

Instructor: David Earn

Lecture 25
Metric Spaces III
Monday 17 March 2025

Announcements

Discussion of metrics so far...

- Metrics induced by norms, *e.g.*, p-norms
- Metrics from norms induced by inner products
- Metric on any set: discrete metric
- Balls (p-balls, discrete balls)

Note:

- I added a note to the slide on the taxicab metric.

Topology of metric spaces

Definition (Convergence of a sequence in a metric space)

Let (X, d) be any metric space. A sequence $(x_n)_{n\in\mathbb{N}}$ ***converges*** to $x \in X$, written $x_n \xrightarrow{n\to\infty} x$ or $\lim_{n\to\infty} x_n = x$, if:

$$\forall\varepsilon > 0,\ \exists N \in \mathbb{N} \ \ni\ d(x_n, x) < \varepsilon \quad \forall n \geq N.$$

If the sequence does not converge to any $x \in X$, we say it ***diverges***.

Equivalently, $x_n \xrightarrow{n\to\infty} x$ if, for any ball $B_\varepsilon(x)$ centered at x, the sequence (x_n) lies inside that ball eventually ($\exists N \in \mathbb{N}$ such that $x_N \in B_\varepsilon(x)$), and stays inside it ($x_n \in B_\varepsilon(x)$ $\forall n > N$), *i.e.*,

$$\forall\varepsilon > 0,\ \exists N \in \mathbb{N} \quad\ni\quad x_n \in B_\varepsilon(x) \quad \forall n \geq N.$$

Topology of metric spaces

Definition (Boundedness in a metric space)

In any metric space (X, d), a sequence (x_n) is ***bounded*** if there exists $x_0 \in X$ and $r > 0$ such that $x_n \in B_r(x_0)$ for all $n \in \mathbb{N}$.

Theorem

In any metric space (X, d), any convergent sequence is bounded.

Note: The converse is FALSE. *e.g.*, $x_n = (-1)^n$ in $(\mathbb{R}, \text{standard})$.

Proof.

Taking $\varepsilon = 1$ (or any particular value) in the definition of convergence, since $x_n \xrightarrow{n \to \infty} x$ in (X, d), $\exists N \in \mathbb{N}$ with $d(x_n, x) < \varepsilon = 1$, $\forall n \geq N$, *i.e.*, $x_n \in B_1(x)$, $\forall n \geq N$. The earlier elements of the sequence, $x_1, \ldots, x_{N-1}$, are a finite collection, so we can choose $r > 0$ so that

$$r > \max\{d(x_1, x), d(x_2, x), \ldots, d(x_{N-1}, x), 1\}.$$

With this r, $x_n \in B_r(x)$ holds $\forall n \in \mathbb{N}$. So (x_n) is bounded. □

Topology of metric spaces

Definition (Interior point)

$x \in E \subseteq X$ is an ***interior point*** of the set E in the metric space (X, d) if x lies in an open ball that is contained in E, *i.e.*,

$$\exists \varepsilon > 0 \quad \ni \quad B_\varepsilon(x) \subset E.$$

Definition (Interior of a set)

If $E \subseteq X$ then the ***interior*** of E, denoted $\text{int}(E)$ or E°, is the set of all interior points of E.

Note:

- $x \in E^\circ$ means not only that $x \in E$, but that there is an entire open ball $B_\varepsilon(x) \subset E$.
- For any *smaller* radius, $0 < r < \varepsilon$, we have $x \in B_r(x) \subseteq B_\varepsilon(x) \subseteq E$, so the choice of ε is not unique for an interior point.

Topology of metric spaces

Definition (Open set)

A set $E \subseteq X$ is ***open*** if every point of E is an interior point.

Example (Upper half-plane in $(\mathbb{R}^2, \text{Euclidean})$)

Let $W = \{x = (x_1, x_2) : x_2 > 0\}$. Then $\forall x \in W$, if $0 < \varepsilon < x_2$ then $B_\varepsilon(x) \subset W$. Thus all $x \in W$ are interior points. So W is an open set.

Example (Any set in $(X, \text{discrete})$)

$(X, \text{discrete})$ means X is a non-empty set and d is the discrete metric d. So, $\forall x \in X$,

$$B_\varepsilon(x) = \begin{cases} \{x\}, & \text{if } 0 < \varepsilon \leq 1, \\ X, & \text{if } \varepsilon > 1. \end{cases}$$

Therefore, for any subset $E \subseteq X$ and for any point $x \in E$, $B_1(x) = \{x\} \subseteq E$, so every point of any set E is an interior point, so **all** sets in the discrete metric are open sets!

Topology of metric spaces

Definition (Closed set)

A set $F \subseteq X$ is ***closed*** if F^c is open.

Example (Lower half-plane in $(\mathbb{R}^2, \text{Euclidean})$)

Let $Z = \{x = (x_1, x_2) : x_2 \leq 0\}$. Then $Z = W^c$, where W is the (open) upper half-plane. So Z is a closed set.

Note: In $(\mathbb{R}^2, \text{Euclidean})$, a half-plane is open or closed depending on whether none or all of the x-axis is included. If neither none nor all of the x-axis is included then the half-plane is neither open nor closed.

Example (Any set in $(X, \text{discrete})$)

Any set $E \subseteq X$ is open. So given any $F \subseteq X$, $F^c \subseteq X$ so F^c is open. Hence F is closed. So **all** sets are *both open and closed* with respect to the discrete metric.

Topology of metric spaces

When studying $X = \mathbb{R}$, we defined closed sets differently.
Could we have used the same definition for a general metric space?

Definition (Accumulation Point or Limit Point or Cluster Point)

If $E \subseteq X$ in a metric space (X, d) then x is an ***accumulation point*** of E if every neighbourhood of x contains infinitely many points of E,

$$i.e., \qquad \forall \varepsilon > 0 \qquad B_\varepsilon(x) \cap (E \setminus \{x\}) \neq \varnothing .$$

Equivalently, $x \in X$ is an accumulation point of the set E if and only if there exists a sequence $(x_n)_{n \in \mathbb{N}}$ with $x_n \in E$ $\forall n \in \mathbb{N}$, such that $x_n \neq x$ $\forall n$ and $x_n \xrightarrow{n \to \infty} x$.

Topology of metric spaces

Theorem (Closed sets in a metric space)

A set $F \subseteq X$ in a metric space (X,d) is closed if and only if F contains all its accumulation points.

Proof.

First, suppose F is closed, and that $x \in X$ is an accumulation point of F. If $x \notin F$, then $x \in F^c$, which is open. Therefore, $\exists \varepsilon > 0$ such that $B_\varepsilon(x) \subset F^c$, *i.e.*, $B_\varepsilon(x) \cap F = \varnothing$. But this contradicts x being an accumulation point of F. So we must have $x \in F$.

Conversely, suppose F contains all its accumulation points. Then, if $z \in F^c$, z cannot be an accumulation point of F. But for any $\varepsilon > 0$, $z \in B_\varepsilon(z)$, so $\exists \varepsilon > 0$ for which $B_\varepsilon(z) \cap F = \varnothing$, *i.e.*, $B_\varepsilon(z) \subset F^c$. Hence F^c is open, so F is closed. □

Topology of metric spaces

Theorem (Properties of open sets in a metric space (X, d))

1. *The sets X and $\varnothing$ are open.*
2. *Any intersection of a finite number of open sets is open.*
3. *Any union of an arbitrary collection of open sets is open.*
4. *The complement of an open set is closed.*

Theorem (Properties of closed sets in a metric space (X, d))

1. *The sets X and $\varnothing$ are closed.*
2. *Any union of a finite number of closed sets is closed.*
3. *Any intersection of an arbitrary collection of closed sets is closed.*
4. *The complement of a closed set is open.*

Poll

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- Submit.

Topology of metric spaces

Example (Set differences of balls)

Suppose $0 < r_1 < r_2$ and (X, d) is a metric space. If $x \in X$, is it necessarily true that $B_{r_2}(x) \setminus B_{r_1}(x)$ is open or closed?

No, it depends on the metric d. If $(X, d) = (\mathbb{R}^n, \text{Euclidean})$ then $B_{r_2}(x) \setminus B_{r_1}(x)$ is neither open nor closed, *e.g.*, in $\mathbb{R}^1$,

$$\begin{aligned} B_{r_2}(x) \setminus B_{r_1}(x) &= (x - r_2, x + r_2) \setminus (x - r_1, x_+ r_1) \\ &= (x - r_2, x - r_1] \cup [x + r_1, x + r_2). \end{aligned}$$

In contrast, if $(X, d) = (\mathbb{R}^n, \text{discrete})$ then $B_{r_2}(x) \setminus B_{r_1}(x)$ is open since any set in (X, d) is both open and closed.

In general, if A and B are sets then

$$A \setminus B = A \cap B^c,$$

so if A and B are both open, and B is also closed, then $A \setminus B$ is open.

Topology of metric spaces

Definition (Isolated point)

If $x \in E \subseteq X$ in a metric space (X, d) then x is an ***isolated point*** of E if there is a neighbourhood of x for which the only point in E is x itself, *i.e.*,

$$\exists \varepsilon > 0 \quad \ni \quad B_\varepsilon(x) \cap E = \{x\}\,.$$

Example

Consider the metric space $(X, d) = ([0, 1], \text{standard})$.
What are the isolated points of (X, d)?

There are no isolated points!

Now suppose $(X, d) = ([0, 1], \text{discrete})$.
What are the isolated points of (X, d)?

All points are isolated!

McMaster University

Mathematics and Statistics

$$\int_M d\omega = \int_{\partial M} \omega$$

Mathematics 3A03
Real Analysis I

Instructor: David Earn

Lecture 26
Metric Spaces IV
Wednesday 19 March 2025

Announcements

Last time. . .

- More definitions and examples related to metric topology

Topology of metric spaces

Definition (Closure of a set)

If (X, d) is a metric space and $E \subseteq X$ then the ***closure*** of E, denoted $\overline{E}$, is the smallest closed set that contains E.

$\therefore$ $\overline{E}$ is closed, $E \subseteq \overline{E}$, and if F is closed and $E \subseteq F$, then $\overline{E} \subseteq F$.

Theorem

$x \in \overline{E} \iff x$ is either an element of E or an accumulation point of E.
i.e., $x \in \overline{E} \iff \forall \varepsilon > 0,\ B_\varepsilon(x) \cap E \neq \varnothing$.

Proof.

($\Longleftarrow$) Suppose $\forall \varepsilon > 0$ $B_\varepsilon(x) \cap E \neq \varnothing$. In order to derive a contradiction, assume $x \notin \overline{E}$. Then $x \in (\overline{E})^c$, which is open, so $\exists \varepsilon > 0$ such that $B_\varepsilon(x) \subseteq (\overline{E})^c$, *i.e.*, $B_\varepsilon(x) \cap \overline{E} = \varnothing$. But $E \subset \overline{E}$, so $B_\varepsilon(x) \cap E = \varnothing$. $\Rightarrow\Leftarrow$

($\Longrightarrow$) Suppose $x \in \overline{E}$ and, in order to derive a contradiction, suppose $\exists \varepsilon > 0$ $\ni$ $B_\varepsilon(x) \cap E = \varnothing$. Then $E \subseteq (B_\varepsilon(x))^c$. Hence $\overline{E} \subseteq \overline{(B_\varepsilon(x))^c} = (B_\varepsilon(x))^c$. But $x \in \overline{E}$ and $x \notin (B_\varepsilon(x))^c$. $\Rightarrow\Leftarrow$. $\therefore \forall \varepsilon > 0,\ B_\varepsilon(x) \cap E \neq \varnothing$. $\square$

Topology of metric spaces

Example (Cubic balls in $\mathbb{R}^n$)

In $\mathbb{R}^n$ with the max norm $\|\cdot\|_\infty$, the distance between $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ is

$$d(x,y) \;=\; \|x-y\|_\infty \;=\; \max\{|x_i - y_i| : 1 \leq i \leq n\}.$$

Consider the set $E \subset \mathbb{R}^n$, where

$$E \;=\; \left\{x \in \mathbb{R}^n \,:\, 0 < x_i < 1,\ \forall i \in \{1, \ldots, n\}\right\}.$$

What are the interior E° and closure $\overline{E}$ of the set E?

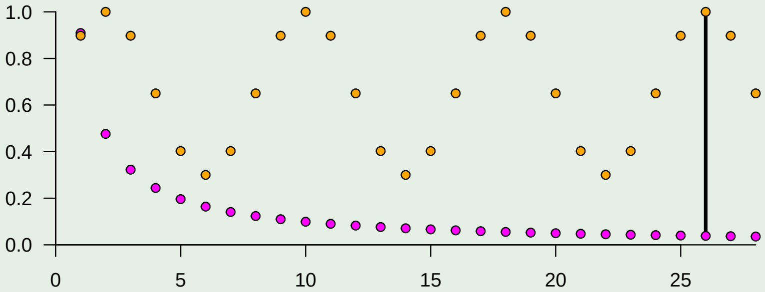
- $\boldsymbol{n = 1}$: $E = (0,1)$, an open interval. $E^\circ = (0,1) = E$. $\overline{E} = [0,1]$.
- $\boldsymbol{n = 2}$: $E = (0,1) \times (0,1)$, an open square (an open ball in this metric). $E^\circ = (0,1)^2 = E$. $\overline{E} = [0,1]^2$.
- $\boldsymbol{n > 2}$: $E = (0,1)^n$, an open n-cube (an open ball in this metric). $E^\circ = (0,1)^n = E$. $\overline{E} = [0,1]^n$.

  When we imagine an n-cube for $n > 3$, we are probably thinking about $n = 3$ in our minds. But we can easily represent individual points in $\mathbb{R}^n$ in the plane. *How?*

Topology of metric spaces

Example (Cubic balls in $\mathbb{R}^n$)

Suppose $x, y \in E \subset \mathbb{R}^{28}$, $\quad d(x,y) = \|x - y\|_\infty$.



The vertical **black line** indicates the distance between x and y.

Note: The same picture works for $\ell^\infty(\mathbb{R})$, the space of sequences that are bounded, and in which the norm is defined by sup rather than max.

Topology of metric spaces

For the space of continuous functions on a closed interval $[a, b]$, we defined the Euclidean norm via the standard inner product. We can also define a p-norm on this space for any $p \geq 1$,

$$\|f\|_p = \left(\int_a^b |f(x)|^p \, dx \right)^{1/p} .$$

As in finite dimensions, $\lim_{p\to\infty} \|f\|_p = \|f\|_\infty$, where

$$\|f\|_\infty \equiv \sup\{|f(x)| : a \leq x \leq b\} = \max\{|f(x)| : a \leq x \leq b\}.$$

Note: In the metric space $(C[a, b], d)$ with $d(f, g) = \|f - g\|_\infty$, convergence of sequences of "points", $f_n \xrightarrow{n\to\infty} f$, implies $f_n \xrightarrow[\text{unif}]{n\to\infty} f \in C[a, b]$.

Example

In the metric space $C([a, b])$, with distance given by the sup-norm,

$$d(f, g) = \|f - g\|_\infty = \sup\{|f(x) - g(x)| : a \leq x \leq b\},$$

let

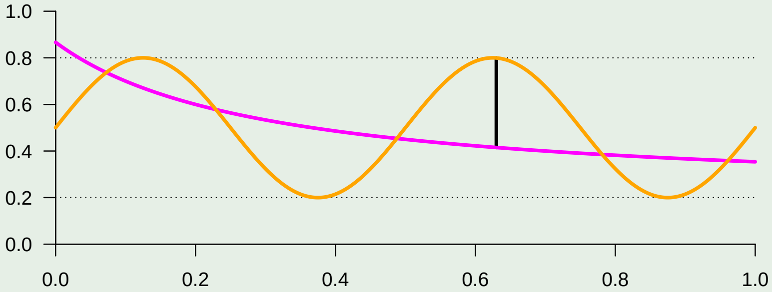
$$E = \Big\{ f \in C([a, b]) \, : \, 0 < f(x) < 1, \ \forall x \in [a, b] \Big\}.$$

What are the interior E° and closure $\overline{E}$ of the set E?

Topology of metric spaces

Example (Cubic balls in $C[0,1]$)

Suppose $f, g \in E \subset C[0,1]$, $\quad d(f,g) = \|f-g\|_\infty$.

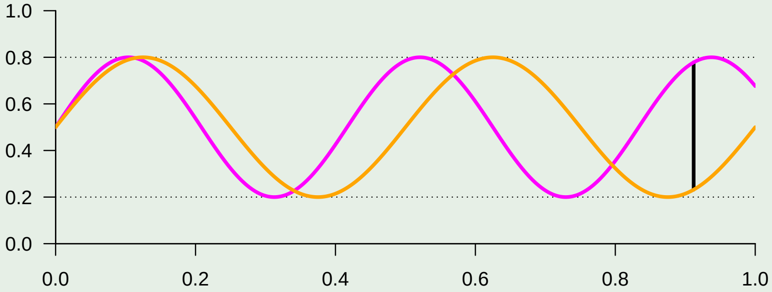


The vertical **black line** indicates the distance between f and g.

Topology of metric spaces

Example (Cubic balls in $C[0,1]$)

Suppose $f, g \in E \subset C[0,1]$, $\quad d(f,g) = \|f-g\|_\infty$.

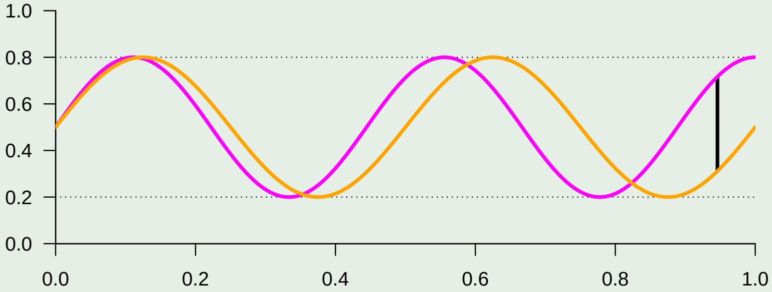


The vertical **black line** indicates the distance between f and g.

Topology of metric spaces

Example (Cubic balls in $C[0,1]$)

Suppose $f, g \in E \subset C[0,1]$, $\quad d(f,g) = \|f-g\|_\infty$.



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Topology of metric spaces

Example (Cubic balls in $C[0,1]$)

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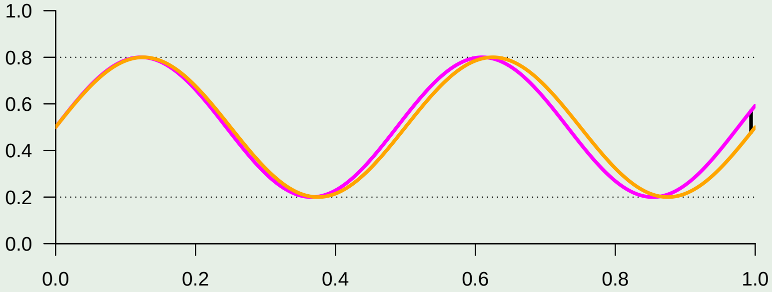

The vertical **black line** indicates the distance between f and g.

Topology of metric spaces

Example (E° and $\overline{E}$ for $E = \{f \in C([a,b]) : 0 < f(x) < 1 \quad \forall x \in [a,b]\}$)

If $f \in C([a,b])$ then f is continuous on $[a,b]$ so, by the Extreme Value Theorem, f attains its minimum and maximum values on $[a,b]$, say at $u, v \in [a,b]$. Since $f(x) \in (0,1)$ for all $x \in [a,b]$, the extreme values, $f(u)$ and $f(v)$, must also be in the open interval $(0,1)$. Therefore,

$$0 < f(u) \le f(x) \le f(v) < 1, \quad \forall x \in [a,b]. \qquad (\heartsuit)$$

Thus, the range of f is $[f(u), f(v)] \subset (0,1)$. There is a gap (a finite interval) that "insulates" f from the extreme values (0 and 1).

Let $\varepsilon = \min\{f(u), 1 - f(v)\}$. Then $\varepsilon > 0$. Now consider $g \in B_\varepsilon(f) \subset C([a,b])$.

Then:

$$\begin{aligned}
&\|g - f\|_\infty < \varepsilon \\
\Longrightarrow\ & \max\{|g(x) - f(x)| : 0 \le x \le 1\} < \varepsilon \\
\Longrightarrow\ & |g(x) - f(x)| < \varepsilon \quad \forall x \in [0,1] \\
\Longrightarrow\ & -\varepsilon < g(x) - f(x) < \varepsilon \quad \forall x \in [0,1] \\
\Longrightarrow\ & f(x) - \varepsilon < g(x) < f(x) + \varepsilon \quad \forall x \in [0,1]
\end{aligned}$$

Topology of metric spaces

Example (E° and $\overline{E}$ for $E = \{f \in C([a,b]) : 0 < f(x) < 1 \quad \forall x \in [a,b]\}$)

Now using ($\heartsuit$), we have

$$0 \le f(u) - \varepsilon \le f(x) - \varepsilon < g(x) < f(x) + \varepsilon \le f(v) + \varepsilon \le 1, \quad \forall x \in [a,b],$$

from which we conclude that $g \in E$.

Since g was an arbitrary "point" in $B_\varepsilon(f)$, it follows that $B_\varepsilon(f) \subseteq E$, so any $f \in E$ is an interior point, so E is open and $E^\circ = E$.

What about $\overline{E}$, the closure of E?

We will show that the closure of E is the set

$$F = \{f \in C([a,b]) : 0 \le f(x) \le 1, \ \forall x \in [a,b]\}. \qquad (\spadesuit)$$

We will show $\overline{E} \subseteq F$ and then $F \subseteq \overline{E}$.

Topology of metric spaces

Example (E° and $\overline{E}$ for $E = \{f \in C([a,b]) : 0 < f(x) < 1 \quad \forall x \in [a,b]\}$)

($\overline{E} \subseteq F$) First, consider any sequence $(f_n)_{n\in\mathbb{N}}$ with $f_n \in E$, $\forall n \in \mathbb{N}$, and suppose $f_n \xrightarrow{n\to\infty} f$ in the sup-norm, *i.e.*, $f_n \xrightarrow[\text{unif}]{n\to\infty} f$. Since uniform convergence implies pointwise convergence, and $f_n(x) \in (0,1)$, we must have $0 \le f(x) \le 1$, $\forall x \in \mathbb{N}$, and since $f_n \xrightarrow[\text{unif}]{n\to\infty} f$, we know $f \in C([a,b])$, so $f \in F$. Since f is a limit point of E and any limit point of E must lie in $\overline{E}$, we must have $\overline{E} \subseteq F$.

($F \subseteq \overline{E}$) Suppose $f \in F$, and define the sequence $f_n \in E$ by

$$f_n(x) = \begin{cases} 1 - \frac{1}{n}, & \text{if } f(x) > 1 - \frac{1}{n}, \\ f(x), & \text{if } \frac{1}{n} \le f(x) \le 1 - \frac{1}{n}, \\ \frac{1}{n}, & \text{if } f(x) < \frac{1}{n}. \end{cases}$$

By construction, $f_n \xrightarrow[\text{unif}]{n\to\infty} f$, and so f is a limit point of E so (by the theorem about closures again) $F \subseteq \overline{E}$. *(Note: $f_n \to f$ is illustrated in the next few slides.)*

Therefore, $F = \overline{E}$. □

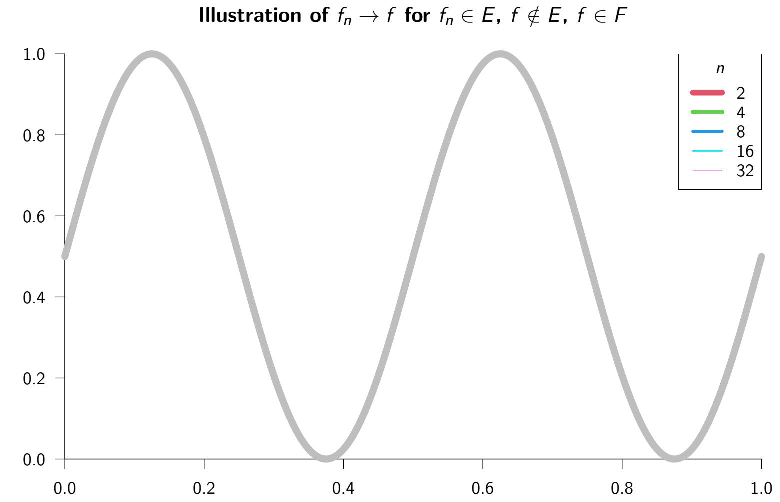
Illustration of $f_n \to f$ for $f_n \in E$, $f \notin E$, $f \in F$
1.0
0.8
0.6
0.4
0.2
0.0
0.0
0.2
0.4
0.6
0.8
1.0
n
2
4
8
16
32

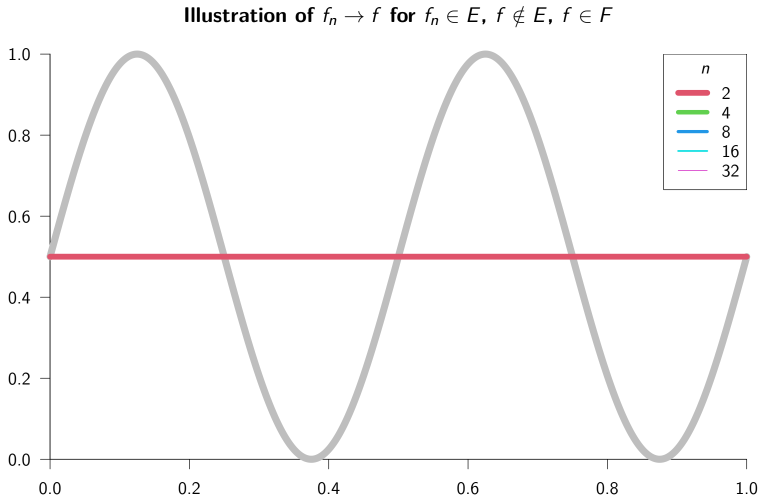
Illustration of $f_n \to f$ for $f_n \in E$, $f \notin E$, $f \in F$
1.0
0.8
0.6
0.4
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0.0
0.2
0.4
0.6
0.8
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n
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16
32

Illustration of $f_n \to f$ for $f_n \in E$, $f \notin E$, $f \in F$

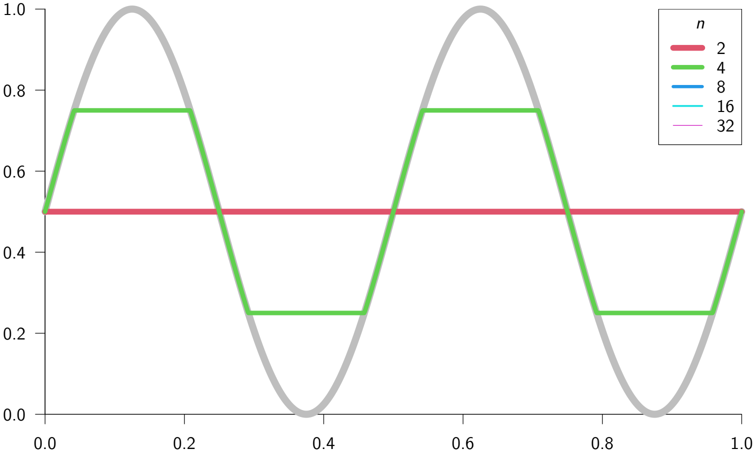
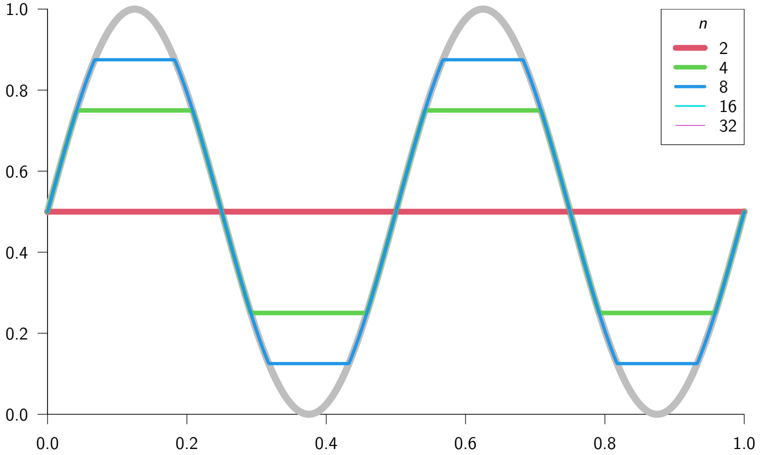


Illustration of $f_n \to f$ for $f_n \in E$, $f \notin E$, $f \in F$



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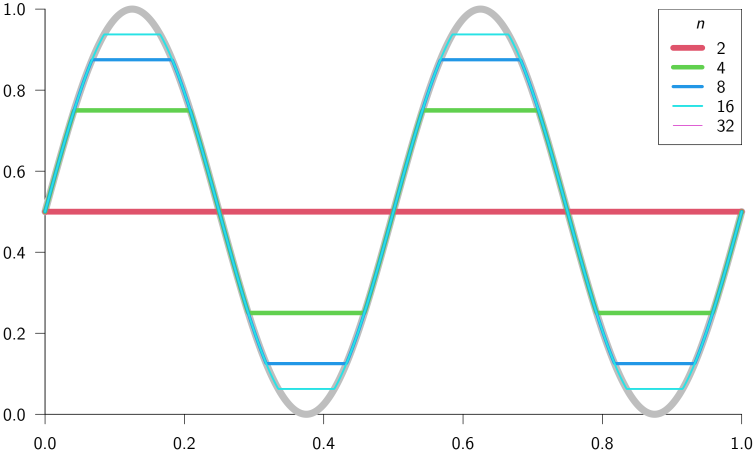

Illustration of $f_n \to f$ for $f_n \in E$, $f \notin E$, $f \in F$