24 π is irrational

Mathematics
and Statistics

$$\int_M d\omega = \int_{\partial M} \omega$$

Mathematics 3A03
Real Analysis I

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Lecture 24
π is irrational
Friday 14 March 2025

Announcements

Happy π day!

What is π ?

- ☐ π is a Greek letter;
- ☐ $\pi = \frac{22}{7}$;
- ☐ $\pi = 3.14$;
- ☐ $\pi = 3.1415926$;
- ☐ $\pi = 3.141592653589793238462643383279502884197$
- ☐ π is the ratio of the circumference to the diameter of a circle;
- ☐ π is the area of the unit circle;
- ☐ None of the above.

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What is π ?

What else do we know about π?

- π is the first positive root of $\sin(x)$
- $e^{i\pi} = -1$
- π is irrational
  - How do you know?
  - Can you prove it?
  - Let's try...

Warmup: $\sqrt{2}$ is irrational

Theorem

$\sqrt{2} \notin \mathbb{Q}$.

Proof.

Suppose $\sqrt{2} \in \mathbb{Q}$. Then there exist two positive integers m and n with $\gcd(m, n) = 1$ such that $m/n = \sqrt{2}$.

$$\therefore \left(\frac{m}{n}\right)^2 = (\sqrt{2})^2 \implies \frac{m^2}{n^2} = 2 \implies m^2 = 2n^2.$$

$\therefore m^2$ is even $\implies$ m is even ($\because$ odd numbers have odd squares).

$\therefore m = 2k$ for some $k \in \mathbb{N}$.

$$\therefore 4k^2 = m^2 = 2n^2 \implies 2k^2 = n^2 \implies n \text{ is even}.$$

$\therefore$ 2 is a factor of both m and n. **Contradiction!** $\therefore \sqrt{2} \notin \mathbb{Q}$. $\square$

Warmup 2: e is irrational

The power series expansion for e^x implies that

$$e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{3!} + \frac{1}{4!} + \cdots \qquad (*)$$

Suppose e is rational, *i.e.*, $e = \frac{m}{n}$ for $m, n \in \mathbb{N}$. Then, multiplying $(*)$ by $n!$, we have

$$n!e \;=\; \underbrace{n!\frac{m}{n}}_{A} \;=\; \underbrace{n! + \frac{n!}{2!} + \cdots + \frac{n!}{n!}}_{B} + \underbrace{\frac{n!}{(n+1)!} + \frac{n!}{(n+2)!}}_{C} + \cdots$$

$A \in \mathbb{N}$ and $B \in \mathbb{N}$, so $C = A - B \in \mathbb{N}$. But

$$0 \;<\; C \;<\; \frac{1}{n+1} + \frac{1}{(n+2)(n+1)} + \cdots \;<\; \frac{1}{n+1} + \frac{1}{(n+1)^2} + \cdots \;=\; \frac{1}{n} \;\leq\; 1,$$

so $0 < C < 1$, which is impossible since $C \in \mathbb{N}$. $\square$

What about π?

To get started, let's review some facts about the sine function:

$\sin(x)$ is the unique solution to the differential equation:

$$y'' + y = 0, \qquad y(0) = 0, \quad y'(0) = 1$$

Thus:
$$\begin{aligned} \sin''(x) &= -\sin(x) \quad \text{for all } x \in \mathbb{R}, \\ \sin(0) &= 0, \\ \sin'(0) &= 1 \quad [= \cos(0)] \end{aligned}$$

In addition: $\mathbf{\sin(\pi) = 0}$

i.e., $x = \pi$ *is the first positive root of* $\sin(x)$.

Motivating ideas

$\sin(x)$ can be expanded in a power series:

$$\sin x \;=\; x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots \;=\; \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^{2n-1}}{(2n-1)!}$$

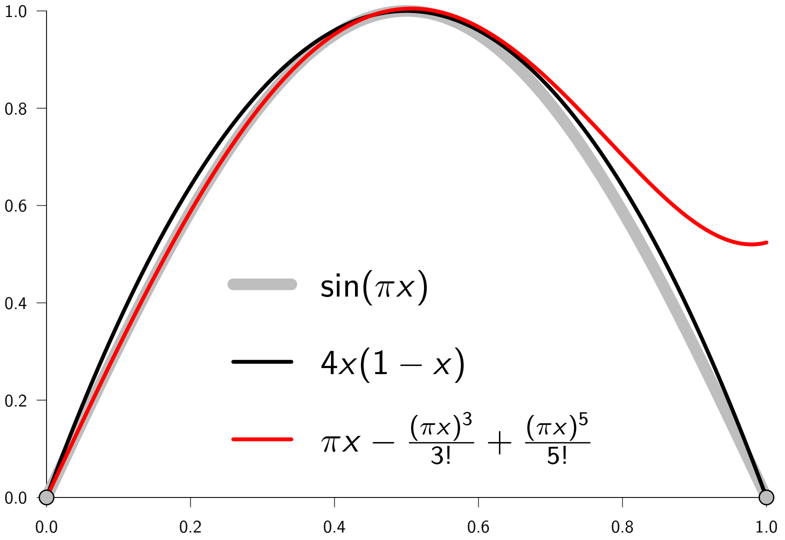
The sequence of powers $\{x^{2n-1}\}$ is just one example of a sequence of polynomials in which we can expand nice functions like $\sin x$.
If we want to approximate $\sin x$ on the interval $[0, \pi]$ it might be better to expand in a sequence of polynomials that all have the same zeros as $\sin(x)$, *i.e.*, at 0 and π, and the same maximum point, *i.e.*, at $x = \frac{\pi}{2}$, for example:

$$\big(x(\pi - x)\big)^n .$$

Equivalently, we could approximate $\sin(\pi x)$ on the interval $[0, 1]$ using the sequence

$$x^n (1 - x)^n .$$

Motivating ideas



Motivating ideas

The graph on the previous slide makes it plausible that, on the interval $[0, 1]$, expressing $\sin(\pi x)$ as a series in powers of $x(1-x)$ is more efficient than expanding in powers of x.

So, it might be useful to consider the series

$$\sum_{n=1}^{\infty} C_n \, f_n(x),$$

where

$$f_n(x) \; = \; \frac{x^n(1-x)^n}{n!}.$$

How can we find the coefficients C_n?

Motivating ideas

We can think of $\{f_n\}$ as a basis for the space of functions that can be represented by power series. To find the coefficients of each basis function, we need to consider inner products of sin with each basis function, *i.e.*, the projection of sin onto each f_n. Thus, we will need to calculate the integrals

$$\int_0^1 f_n(x)\sin(\pi x)\,dx \qquad (\heartsuit)$$

If the series converges, then its terms must vanish as $n \to \infty$, so it is plausible that $(\heartsuit)$ might get arbitrarily small for large enough n.

Strategy for proving π is irrational:

- Show that $0 < (\heartsuit) < 1$ for sufficiently large n;
- Show that if π is rational then $(\heartsuit)$ is a positive integer.

This won't quite work as stated, but the idea will work...

Step 1: Properties of $f_n(x)$

1. $f_n(x) = \dfrac{x^n(1-x)^n}{n!}$
2. $f_n(0) = f_n(1) = 0$
3. $0 < f_n(x) < \dfrac{1}{n!}$ for $0 < x < 1$
4. $f_n(x) = \dfrac{1}{n!}\displaystyle\sum_{i=n}^{2n} c_i x^i$ for integers c_i
5. $f_n^{(k)}(0)$ is an integer for all k
6. $f_n(1-x) = f_n(x)$ for $0 < x < 1$
7. $f_n^{(k)}(1-x) = (-1)^k f_n^{(k)}(x)$
8. $f_n^{(k)}(1)$ is an integer for all k

Step 2: Compute $\int_0^1 f_n(x)\sin(\pi x)\,dx$ by parts

Let $u = f_n(x)$, $dv = \sin(\pi x)\,dx \implies du = f_n'(x)\,dx$, $v = -\frac{\cos(\pi x)}{\pi}$

$$\begin{aligned}
\int_0^1 f_n(x)\sin(\pi x)\,dx &= -\frac{1}{\pi}\cos(\pi x) f_n(x)\Big|_0^1 - \int_0^1 -\frac{\cos(\pi x)}{\pi} f_n'(x)\,dx \\
&= \frac{1}{\pi}\Big(f_n(1)+f_n(0)\Big) + \frac{1}{\pi}\int_0^1 f_n'(x)\cos(\pi x)\,dx \\
&= \frac{1}{\pi}\Big(f_n(1)+f_n(0)\Big) + \frac{1}{\pi^2}\Big(\sin(\pi) f_n'(1) - \sin(0) f_n'(0)\Big) \\
&\qquad - \frac{1}{\pi^2}\int_0^1 f_n''(x)\sin(\pi x)\,dx \\
&= \frac{1}{\pi}\Big(f_n(1)+f_n(0)\Big) - \frac{1}{\pi^2}\int_0^1 f_n''(x)\sin(\pi x)\,dx \\
&= \frac{1}{\pi}\Big(f_n(1)+f_n(0)\Big) \\
&\qquad - \frac{1}{\pi^2}\left[\frac{1}{\pi}\Big(f_n''(1)+f_n''(0)\Big) - \frac{1}{\pi^2}\int_0^1 f_n^{(4)}(x)\sin(\pi x)\,dx\right]
\end{aligned}$$

Step 3: Induction and simplification

If we now continue integrating by parts again and again, we find

$$\int_0^1 f_n(x)\sin(\pi x)\,dx \;=\; \sum_{k=0}^{n}\frac{(-1)^{2k+1}}{\pi^{2k+1}}\Big(f_n^{(2k)}(0)+f_n^{(2k)}(1)\Big) + \int_0^1 f_n^{(2n+2)}(x)\sin(\pi x)\,dx$$

But f_n is a polynomial of degree $2n$. So $f_n^{(2n+2)}(x)\equiv 0$.
Also, for $k<n$, $f_n^{(k)}(x)$ has a factor x, so $f_n^{(k)}(0)\equiv 0$ if $k<n$. So, if we define

$$g(x) \;=\; f_n(x)-\frac{1}{\pi^2}f_n''(x)+\frac{1}{\pi^4}f_n^{(4)}(x)-\cdots+(-1)^n\frac{1}{\pi^{2n}}f_n^{(2n)}(x)$$

then

$$\pi\int_0^1 f_n(x)\sin(\pi x)\,dx = g(0)+g(1) \qquad (\spadesuit)$$

Step 3: Assume π is rational

Suppose π is rational. Then π^2 is rational, so write

$$\pi^2 = \frac{a}{b}, \qquad a, b \in \mathbb{N}.$$

If we now write $G(x) = b^n \pi^{2n} g(x)$, then we have

$$G(x) \;=\; b^n\big[\pi^{2n} f_n(x) - \pi^{2n-2} f_n''(x) + \pi^{2n-4} f_n^{(4)}(x) - \cdots + (-1)^n f_n^{(2n)}(x)\big].$$

Here, the factors in front of $f_n^{(2k)}$ are

$$b^n \pi^{2n-2k} \;=\; b^n (\pi^2)^{n-k} \;=\; b^n \left(\frac{a}{b}\right)^{n-k} \;=\; a^{n-k} b^k,$$

which is an <u>integer</u>. Looking back at (♠), and multiplying by $b^n \pi^{2n} = a^n$, we have

$$\pi \int_0^1 a^n f_n(x) \sin(\pi x)\, dx \;=\; G(0) + G(1) \quad \text{is an } \underline{\text{integer}} \qquad (\mathbf{I})$$

Step 4: Show $0 < \pi \int_0^1 a^n f_n(x) \sin(\pi x)\, dx < 1$

Since $0 < f_n(x) < \frac{1}{n!}$ for $0 < x < 1$, it follows that

$$0 < \pi \int_0^1 a^n f_n(x) \sin(\pi x)\, dx < \frac{\pi a^n}{n!}\,.$$

This is true *for any n*. But the series $\sum_{n=0}^{\infty} \frac{a^n}{n!}$ converges (to e^a) so

$$\lim_{n\to\infty} \frac{a^n}{n!} = 0.$$

Therefore, for sufficiently large n, $\frac{\pi a^n}{n!} < 1$, hence for sufficiently large n,

$$0 < \pi \int_0^1 a^n f_n(x) \sin(\pi x)\, dx < 1\,.$$

But there is no integer between 0 and 1, which contradicts (**I**).
Therefore, the assumption that π is rational is FALSE! □

Happy π day!