24 Differentiation

Differentiation

McMaster University

Mathematics and Statistics

$$\int_M d\omega = \int_{\partial M} \omega$$

Mathematics 3A03
Real Analysis I

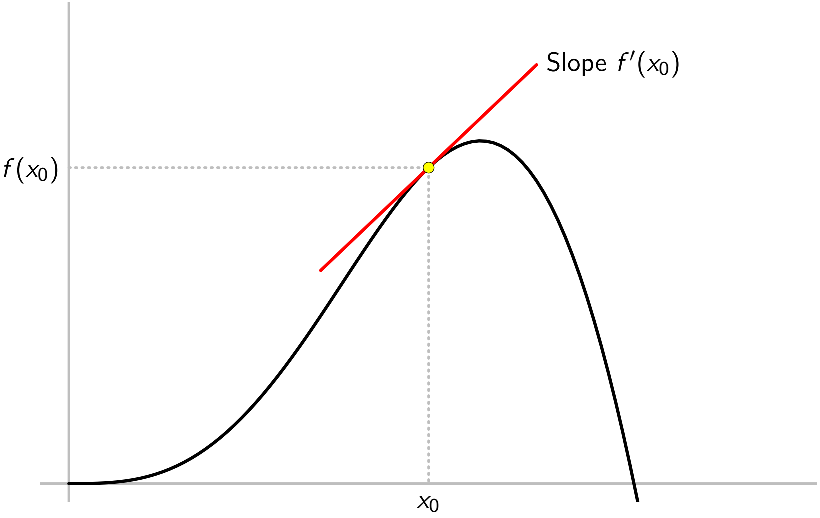
Instructor: David Earn

Lecture 24
Differentiation
Tuesday 5 November 2019

Announcements

- Assignment 4 is posted and is due on Tuesday 12 Nov 2019, 2:25pm, via crowdmark.

The Derivative



The Derivative

Definition (Derivative)

Let f be defined on an interval I and let $x_0 \in I$. The ***derivative*** of f at x_0, denoted by $f'(x_0)$, is defined as

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0},$$

provided either that this limit exists or is infinite. If $f'(x_0)$ is finite we say that f is ***differentiable*** at x_0. If f is differentiable at every point of a set $E \subseteq I$, we say that f is differentiable on E. If E is all of I, we simply say that f is a ***differentiable function***.

Note: "Differentiable" and "a derivative exists" always mean that the derivative is finite.

The Derivative

Example

$f(x) = x^2$. Find $f'(2)$.

$$f'(2) = \lim_{x\to 2} \frac{x^2-4}{x-2} = \lim_{x\to 2} \frac{(x+2)(x-2)}{x-2} = \lim_{x\to 2} x+2 = 4$$

Note:

- In the first two limits, we must have $x \neq 2$.
- But in the third limit, we just plug in $x = 2$.
- Two things are equal, but in one $x \neq 2$ and in the other $x = 2$.
- Good illustration of why it is important to define the meaning of limits rigorously.

Poll

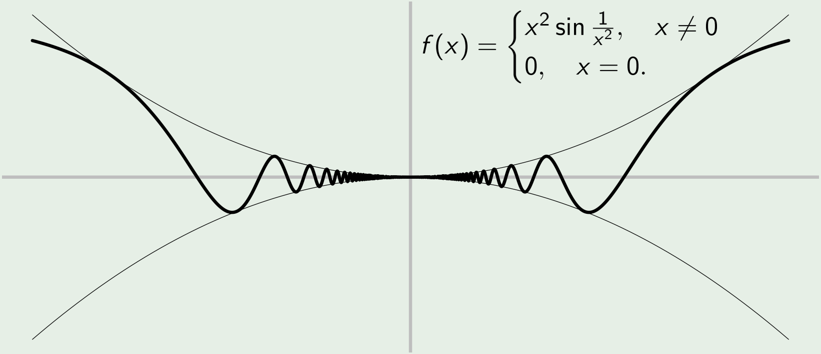
- Go to https://www.childsmath.ca/childsa/forms/main_login.php
- Click on Math 3A03
- Click on Take Class Poll
- Fill in poll **Lecture 24: Differentiable at 0**
- Submit.

The Derivative

Example

Let f be defined in a neighbourhood I of 0, and suppose $|f(x)| \leq x^2$ for all $x \in I$. Is f necessarily differentiable at 0? *e.g.*,

$$f(x) = \begin{cases} x^2 \sin \frac{1}{x^2}, & x \neq 0 \\ 0, & x = 0. \end{cases}$$



The Derivative

Example (Trapping principle)

Suppose $f(x) = \begin{cases} x^2 \sin \frac{1}{x^2}, & x \neq 0 \\ 0, & x = 0. \end{cases}$ Then:

$$\forall x \neq 0 : \quad \left|\frac{f(x) - f(0)}{x - 0}\right| = \left|\frac{f(x)}{x}\right| = \left|\frac{x^2 \sin \frac{1}{x^2}}{x}\right| = \left|x \sin \frac{1}{x^2}\right| \leq |x|$$

Therefore:

$$|f'(0)| = \left|\lim_{x\to 0} \frac{f(x) - f(0)}{x - 0}\right| = \lim_{x\to 0} \left|\frac{f(x) - f(0)}{x - 0}\right| \leq \lim_{x\to 0} |x| = 0.$$

$\therefore$ f is differentiable at 0 and $f'(0) = 0$. □

The Derivative

Definition (One-sided derivatives)

Let f be defined on an interval I and let $x_0 \in I$. The ***right-hand derivative*** of f at x_0, denoted by $f'_+(x_0)$, is the limit

$$f'_+(x_0) = \lim_{x \to x_0^+} \frac{f(x) - f(x_0)}{x - x_0},$$

provided either that this one-sided limit exists or is infinite.

Similarly, the ***left-hand derivative*** of f at x_0, denoted by $f'_-(x_0)$, is the limit

$$f'_-(x_0) = \lim_{x \to x_0^-} \frac{f(x) - f(x_0)}{x - x_0}.$$

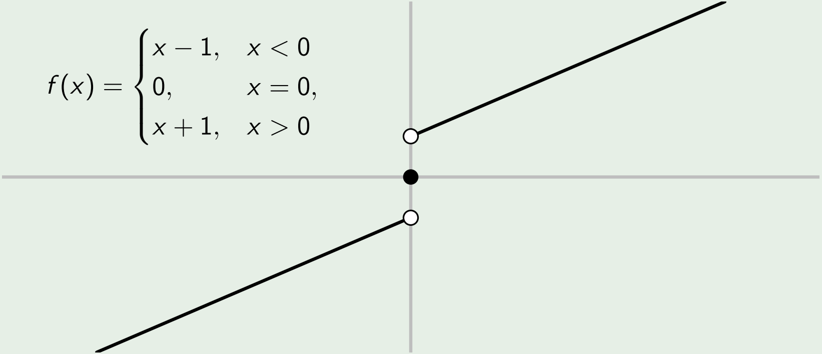
Note:

If $x_0 \in I^\circ$ then f is differentiable at x_0 iff $f'_+(x_0) = f'_-(x_0) \neq \pm\infty$.

The Derivative

Example

$$f(x) = \begin{cases} x-1, & x<0 \\ 0, & x=0, \\ x+1, & x>0 \end{cases}$$



- Same slope from left and right. Why isn't f differentiable???
- $\lim_{x\to 0^-} f'(x) = \lim_{x\to 0^+} f'(x) = \lim_{x\to 0} f'(x) = 1.$
- $f'_-(0) = f'_+(0) = f'(0) = \lim_{x\to 0} \frac{f(x)-f(0)}{x-0} = \infty.$

The Derivative

- Higher derivatives: we write
  - $f'' = (f')'$ if f' is differentiable;
  - $f^{(n+1)} = (f^{(n)})'$ if $f^{(n)}$ is differentiable.

- Other standard notation for derivatives:

$$\frac{df}{dx} = f'(x)$$

$$D = \frac{d}{dx}$$

$$D^n f(x) = \frac{d^n f}{dx} = f^{(n)}(x)$$

The Derivative

Theorem (Differentiable $\implies$ continuous)

If f is defined in a neighbourhood I of x_0 and f is differentiable at x_0 then f is continuous at x_0.

Proof.

Must show $\lim_{x \to x_0} f(x) = f(x_0)$, *i.e.*, $\lim_{x \to x_0} (f(x) - f(x_0)) = 0$.

$$\begin{aligned}
\lim_{x \to x_0} (f(x) - f(x_0)) &= \lim_{x \to x_0} \left(\frac{f(x) - f(x_0)}{x - x_0} \times (x - x_0) \right) \\
&= \lim_{x \to x_0} \left(\frac{f(x) - f(x_0)}{x - x_0} \right) \times \lim_{x \to x_0} (x - x_0) \\
&= f'(x_0) \times 0 = 0\,,
\end{aligned}$$

where we have used the theorem on the algebra of limits. □

The Derivative

Theorem (Algebra of derivatives)

Supppose f and g are defined on an interval I and $x_0 \in I$. If f and g are differentiable at x_0 then $f + g$ and fg are differentiable at x_0. If, in addition, $g(x_0) \neq 0$ then f/g is differentiable at x_0. Under these conditions:

1. $(cf)'(x_0) = cf'(x_0)$ *for all* $c \in \mathbb{R}$;
2. $(f + g)'(x_0) = (f' + g')(x_0)$;
3. $(fg)'(x_0) = (f'g + fg')(x_0)$;
4. $\left(\frac{f}{g}\right)'(x_0) = \left(\frac{gf' - fg'}{g^2}\right)(x_0) \qquad (g(x_0) \neq 0)$.

(Textbook (TBB) Theorem 7.7, p. 408)

The Derivative

Theorem (Chain rule)

Suppose f is defined in a neighbourhood U of x_0 and g is defined in a neighbourhood V of $f(x_0)$ such that $f(U) \subseteq V$. If f is differentiable at x_0 and g is differentiable at $f(x_0)$ then the composite function $h = g \circ f$ is differentiable at x_0 and

$$h'(x_0) = (g \circ f)'(x_0) = g'(f(x_0))f'(x_0)\,.$$

(Textbook (TBB) §7.3.2, p. 411)

TBB provide a very good motivating discussion of this proof, which is quite technical.

The Derivative

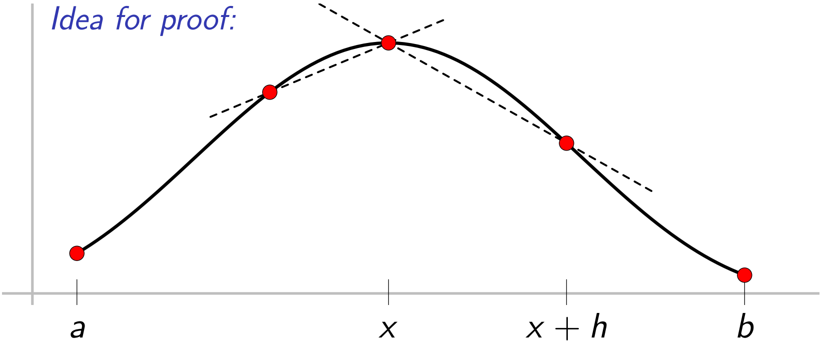
Theorem (Derivative at local extrema)

Let $f : (a,b) \to \mathbb{R}$. *If* x *is a maximum or minimum point of* f *in* (a,b), *and* f *is differentiable at* x, *then* $f'(x) = 0$.

(Textbook (TBB) Theorem 7.18, p. 424)

Note: f need not be differentiable or even continuous at other points.

Idea for proof:



Last time. . .

- Definition of the derivative.
- Proved differentiable $\Longrightarrow$ continuous.
- Discussed algebra of derivatives and chain rule.
- Pictorial argument that derivative is zero at extrema.
- Defined one-sided derivatives
  - Example

The Mean Value Theorem

Theorem (Rolle's theorem)

If f is continuous on $[a, b]$ and differentiable on (a, b), and $f(a) = f(b)$, then there exists $x \in (a, b)$ such that $f'(x) = 0$.