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McMaster University

Mathematics and Statistics

$$\int_M d\omega = \int_{\partial M} \omega$$

Mathematics 3A03
Real Analysis I

Instructor: David Earn

Lecture 22
Metric Spaces
Monday 10 March 2025

Announcements

- New, exciting topic today...

Metric Spaces

The metric structure of $\mathbb{R}$

Definition (Absolute Value function)

For any $x \in \mathbb{R}$,

$$|x| \stackrel{\text{def}}{=} \begin{cases} x & \text{if } x \geq 0, \\ -x & \text{if } x < 0. \end{cases}$$

Theorem (Properties of the Absolute Value function)

For all $x, y \in \mathbb{R}$:

1. $-|x| \leq x \leq |x|$.
2. $|xy| = |x|\,|y|$.
3. $|x + y| \leq |x| + |y|$.
4. $|x| - |y| \leq |x - y|$.

The metric structure of $\mathbb{R}$

Definition (Distance function or metric)

The distance between two real numbers x and y is

$$d(x,y) = |x-y| \ .$$

Theorem (Properties of distance function or metric)

1. $d(x,y) \geq 0$ *distances are positive or zero*
2. $d(x,y) = 0 \iff x = y$ *distinct points have distance > 0*
3. $d(x,y) = d(y,x)$ *distance is symmetric*
4. $d(x,y) \leq d(x,z) + d(z,y)$ *the triangle inequality*

Note: Any function satisfying these properties can be considered a "distance" or "metric".

The metric structure of $\mathbb{R}$

Given $d(x, y) = |x - y|$, the properties of the distance function are equivalent to:

Theorem (Metric properties of the absolute value function)

For all $x, y \in \mathbb{R}$:

1. $|x| \geq 0$
2. $|x| = 0 \iff x = 0$
3. $|x| = |-x|$
4. $|x + y| \leq |x| + |y|$ (*the triangle inequality*)

Slick proof of the triangle inequality

Theorem (The Triangle Inequality for the standard metric on $\mathbb{R}$)

$|x + y| \leq |x| + |y|$ *for all* $x, y \in \mathbb{R}$.

Proof.

Let $s = \text{sign}(x + y)$. Then

$$|x + y| \; = \; s(x + y) \; = \; sx + sy \; \leq \; |x| + |y| \; ,$$

as required. □

A non-standard metric on $\mathbb{R}$

Example (finite distance between every pair of real numbers)

Let $f(x) = \frac{x}{1+x}$, and define $d(x,y) = f(|x-y|)$. Prove that $d(x,y)$ can be interpreted as a distance between x and y because it satisfies all the properties of a metric.

Proof: The only metric property that is non-trivial to prove is the triangle inequality. Note that $f(x)$ is an increasing function on $[0,\infty)$, so the usual triangle inequality, $|x-y| \le |x-z| + |z-y|$, implies

$$\begin{aligned} f(|x-y|) &\le f(|x-z| + |z-y|) = \frac{|x-z| + |z-y|}{1+|x-z| + |z-y|} \\ &= \frac{|x-z|}{1+|x-z| + |z-y|} + \frac{|z-y|}{1+|x-z| + |z-y|} \\ &\le \frac{|x-z|}{1+|x-z|} + \frac{|z-y|}{1+|z-y|} = f(|x-z|) + f(|z-y|) \end{aligned}$$

i.e., $d(x,y) \le d(x,z) + d(z,y)$. □

Poll

- Go to
  https://www.childsmath.ca/childsa/forms/main_login.php
- Click on Math 3A03
- Click on Take Class Poll
- Fill in poll **Metric spaces: Is "$=$ vs $\neq$" a metric?**
- Submit.

Discrete metric

Example (Discrete metric on $\mathbb{R}$)

Let $d(x,y) = \begin{cases} 0 & x = y, \\ 1 & x \neq y. \end{cases}$ Is d a metric on $\mathbb{R}$?

By definition, $d(x,y)$ is non-negative, zero iff $x = y$, and symmetric. For the triangle inequality, if $x = y$ then $d(x,y) = 0$ so the inequality holds for any z. If $x \neq y$ then $d(x,y) = 1$, and at least one of x and y must not equal z, so the inequality says either $1 \leq 1$ or $1 \leq 2$.

Example (Discrete metric on any set X)

The argument that $d(x,y)$ is a metric on $\mathbb{R}$ has nothing to do with $\mathbb{R}$ specifically. $d(x,y)$ is a metric on <u>any</u> set X.

General metric space (X, d)

Definition (Metric space)

A ***metric space*** (X, d) is a non-empty set X together with a distance function (or ***metric***) $d : X \times X \to \mathbb{R}$ satisfying

1. $d(x, y) \geq 0$ *distances are positive or zero*
2. $d(x, y) = 0 \iff x = y$ *distinct points have distance* > 0
3. $d(x, y) = d(y, x)$ *distance is symmetric*
4. $d(x, y) \leq d(x, z) + d(z, y)$ *the triangle inequality*

Much of our analysis of sequences of real numbers and topology of $\mathbb{R}$ generalizes to any metric space. Very often, definitions and proofs depend only on the the existence of a metric, not on $|x|$ specifically. Many useful inferences can be made by identifying a metric on a space of interest.

Examples of metric spaces

Example (Metric spaces (X, d))

- $X = \mathbb{Q}$, with the standard metric $d(x, y) = |x - y|$.
  As $\mathbb{Q} \subset \mathbb{R}$, each condition for d is satisfied in $\mathbb{Q}$.

  How different is $(\mathbb{Q}, d)$ from $(\mathbb{R}, d)$?

- $X = \mathbb{N}$, with the standard metric $d(x, y) = |x - y|$.
  As $\mathbb{N} \subset \mathbb{R}$, each condition for d is satisfied in $\mathbb{N}$.

- $X = \mathbb{R}^2$ with $d(x, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$, where we write the vectors $x = (x_1, x_2)$, $y = (y_1, y_2) \in \mathbb{R}^2$.

- $X = \mathbb{R}^n$ with $d(x, y) = \sqrt{\sum_{i=1}^{n}(x_i - y_i)^2}$, where $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n) \in \mathbb{R}^n$.

  This metric on $\mathbb{R}^n$ is called the ***Euclidean distance***.

Metrics from norms

The Euclidean metric on $\mathbb{R}^n$ is the (Euclidean) length of the difference of two vectors. This connection between length and distance generalizes to any vector space in which *length* is defined.

Definition (Norm)

A ***norm*** on a vector space X is a real-valued function on X such that if $x, y \in X$ and $\alpha \in \mathbb{R}$ then

1. $\|x\| \geq 0$ and $\|x\| = 0$ iff x is the zero element in X;
2. $\|\alpha x\| = |\alpha| \, \|x\|$;
3. $\|x + y\| \leq \|x\| + \|y\|$.

A vector space X equipped with a norm $\|\cdot\|$ is said to be a ***normed vector space***. Any norm $\|\cdot\|$ ***induces*** a metric d via

$$d(x, y) = \|x - y\| \, .$$

Proving that a function is a norm is not necessarily easy. Let's try for the Euclidean norm... To that end, recall the notion of inner product ...

Norms from inner products

Definition (Inner product)

An ***inner product*** on a vector space V over $\mathbb{R}$ is a function

$$\langle \cdot, \cdot \rangle : V \times V \to \mathbb{R}$$

such that for all $u, v, w \in V$ and all scalars $\alpha \in \mathbb{R}$:

1. $\langle u, v \rangle = \overline{\langle v, u \rangle}$ *conjugate symmetry*
2. $\langle \alpha u + v, w \rangle = \alpha \langle u, w \rangle + \langle v, w \rangle$ *linearity in 1st argument*
3. $\langle v, v \rangle \geq 0$ with equality iff $v = 0$ *positive definiteness*

A vector space equipped with an inner product is called an *inner product space*.

Definition (Inner Product Norm)

The norm induced by an inner product $\langle \cdot, \cdot \rangle$ is $\|u\| = \sqrt{\langle u, u \rangle}$.

Norms from inner products

Theorem (Cauchy-Schwarz inequality)

Let V be a (real) inner product space with inner product $\langle\cdot,\cdot\rangle$. For all vectors $u, v \in V$, we have

$$|\langle u, v\rangle| \ \le \ \|u\| \, \|v\|$$

where $\|u\| = \sqrt{\langle u, u\rangle}$ *is the norm induced by the inner product.*

Proof.

The standard proof begins with an idea that probably took someone a long time to think of: Since $\langle v, v\rangle \ge 0$ for any $v \in V$, for any $t \in \mathbb{R}$ we have

$$\begin{aligned} 0 \ \le \ \langle u + tv, u + tv\rangle \ &= \ \langle u, u\rangle + t\,\langle u, v\rangle + t\,\langle v, u\rangle + t^2\,\langle v, v\rangle \\ &= \ \langle u, u\rangle + 2t\,\langle u, v\rangle + t^2\,\langle v, v\rangle \end{aligned}$$

This is a quadratic polynomial in t, which is non-negative for all $t \in \mathbb{R}$. Hence, this quadratic has at most one real root. Consequently, its discriminant is non-positive, *i.e.,* $(2\,\langle u, v\rangle)^2 - 4\,\langle u, u\rangle\,\langle v, v\rangle \ \le \ 0$.

continued...

Norms from inner products

Proof of Cauchy-Schwarz inequality (continued).

Simplifying the non-positive discriminant condition, we have

$$(\langle u, v\rangle)^2 \;\leq\; \langle u, u\rangle \langle v, v\rangle \;.$$

Taking square roots, we have

$$|\langle u, v\rangle| \;\leq\; \|u\| \, \|v\| \;,$$

as required. □

How might you come up with such a proof?

Perhaps by guessing the result (based on knowing it in $\mathbb{R}^2$) and then working backwards.

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Mathematics 3A03
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Lecture 23
Metric Spaces II
Wednesday 12 March 2025

Announcements

- Participation deadline for Assignment 4 was at 11:25am today.

Last time. . .

- Introduction to metric spaces
  - Critical ingredient: the triangle inequality
- Cauchy-Schwarz inequality
  (proved for real inner product spaces)

Norms from inner products

If X is an inner product space, then Cauchy-Schwarz allows us to prove that the induced norm really is a norm (*i.e.*, satisfies the triangle inequality). For $x, y \in X$, we have

$$\begin{aligned} \|x+y\|^2 &= \langle x+y, x+y \rangle \\ &= \langle x, x\rangle + \langle x, y\rangle + \langle y, x\rangle + \langle y, y\rangle \\ &= \|x\|^2 + \|y\|^2 + 2\langle x, y\rangle \\ &\le \|x\|^2 + \|y\|^2 + 2\|x\|\,\|y\| \\ &= (\|x\| + \|y\|)^2 \end{aligned}$$

$$\Longrightarrow \quad \|x+y\| \le \|x\| + \|y\| \,.$$

In particular, $\mathbb{R}^n$ with the usual "dot product" $\langle x, y\rangle = \sum_{i=1}^n x_i y_i$ induces the Euclidean norm $\|\cdot\|_2$, which therefore really is a norm, and $d(x, y) = \|x - y\|_2$ really is a metric (the Euclidean distance).

What about other norms induced by inner products?

Other metric spaces induced by inner products

We are accustomed to finite vectors:

$$x = (x_1, x_2) \in X = \mathbb{R}^2$$
$$x = (x_1, x_2, x_3) \in X = \mathbb{R}^3$$
$$x = (x_1, x_2, \ldots, x_n) \in X = \mathbb{R}^n$$

We can think of a sequence as an infinite vector:

$$x = (x_1, x_2, \ldots) \in X = \big\{\{x_n\} : n \in \mathbb{N}\big\}$$

The points in this space (X) are infinite-dimensional vectors.

We can think of an infinite vector as a function:

$$x_n = f(n) \quad \Longrightarrow \quad (x_1, x_2, \ldots) = \big(f(1), f(2), \ldots\big)$$

The points in this space are functions: $X = \big\{f \,\big|\, f : \mathbb{N} \to \mathbb{R}\big\}$.

So we can generalize to other spaces via functions, *e.g.*,

$$C[0,1] = \big\{f : [0,1] \to \mathbb{R} \,\big|\, f \text{ continuous}\big\}$$

All of the above spaces have a natural inner product, and hence a natural norm and metric.

Other metric spaces induced by inner products

Inner products that convert the spaces on the previous slide into (Euclidean) metric spaces:

$$\mathbb{R}^n: \quad \langle x, y \rangle = \sum_{i=1}^{n} x_i y_i$$

$$\ell^2(\mathbb{R}): \quad \langle x, y \rangle = \sum_{i=1}^{\infty} x_i y_i$$

$$C[a,b]: \quad \langle f, g \rangle = \int_a^b f(x) g(x)\, dx$$

Note: ℓ^2 includes only ***square-summable*** sequences: $\displaystyle\sum_{n=1}^{\infty} x_n^2 < \infty$

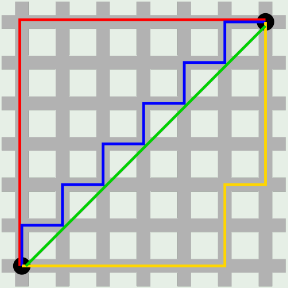
Do we need to specify that $C[a,b]$ contains only ***square-integrable*** *functions?*

Metrics from norms

Example (Taxicab distance)

Taxicab norm on $\mathbb{R}^n$

$$\|x\|_1 = \sum_{i=1}^{n} |x_i|$$



In taxicab geometry, the lengths of the **red**, **blue**, **green**, and **yellow** paths all equal 12, the taxicab distance between the opposite corners, and all four paths are shortest paths. Instead, in Euclidean geometry, the **red**, **blue**, and **yellow** paths still have length 12 but the **green** path is the unique shortest path, with length equal to the Euclidean distance between the opposite corners, $6\sqrt{2} \approx 8.49$.

Image and caption from Wikipedia article on "Taxicab geometry".

Note: The **green** path can be followed. All the points of $\mathbb{R}^2$ are still present when we measure distance with the taxicab metric. Any monotonic curve path that joins the two points can be approximated by an arbitrarily fine grid, and will have the same length. The metric does not impose a particular grid.

Metrics from norms

Example (p-metric)

p-norm on $\mathbb{R}^n$ (for $p \geq 1$) $$\|x\|_p = \Big(\sum_{i=1}^{n} |x_i|^p\Big)^{\frac{1}{p}}.$$

- $p = 1$ is the taxicab norm.
- $p = 2$ is the Euclidean norm.

What happens as $p \to \infty$? For any $x \in \mathbb{R}^n$, we have

$$\|x\|_p = \Big(\sum_{i=1}^{n} |x_i|^p\Big)^{\frac{1}{p}} = |x_k| \Big(\sum_{i=1}^{n} \Big|\frac{x_i}{x_k}\Big|^p\Big)^{\frac{1}{p}} \qquad (|x_k| > |x_i| \;\; \forall i \neq k)$$

$$= |x_k| \Big(1 + \sum_{i \neq k} \Big|\frac{x_i}{x_k}\Big|^p\Big)^{\frac{1}{p}} \xrightarrow{p \to \infty} |x_k|$$

What further work is required if $\nexists k \ni |x_k| > |x_i| \;\; \forall i \neq k$*?*

Therefore, we define $\|\cdot\|_\infty$ to be

Max norm on $\mathbb{R}^n$ $$\|x\|_\infty = \max_{1 \leq i \leq n} |x_i|$$

Poll

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Metrics from norms

Example (p-metric)

Proving that $p = 1$ and $p = \infty$ yield norms is a *good exercise*.

Only $p = 2$ is induced by an inner product.

That the p-norms for $p \neq 1, 2, \infty$ are norms is harder to prove (but true), so

$$d_p(x, y) = \|x - y\|_p$$

is a metric on $\mathbb{R}^n$ for any $p \geq 1$.

Topology of metric spaces

We can generalize the notion of "neighbourhood of a point" to any metric space:

Definition (Open ball)

Let (X, d) be a metric space. If $x_0 \in X$ and $r > 0$ then the ***open ball of radius r about x_0*** is

$$B_r(x_0) = \{x \in X : d(x, x_0) < r\}\,.$$

x_0 is said to be the ***centre*** of $B_r(x_0)$.

Note: The notation $B(x_0, r)$ is also common (and used in TBB).

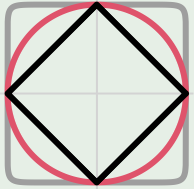
Definition (Neighbourhood)

A ***neighborhood*** of x is any set that contains an open ball $B_r(x)$ for some $r > 0$.

Topology of metric spaces

Example (Open balls in metric spaces)

- In the metric space $(\mathbb{R}, \mathit{standard})$, *i.e.*, $\mathbb{R}$ with $d(x, y) = |x - y|$, $B_r(x) = (x - r, x + r)$, an open interval of length $2r$ centred at x.
- In $\mathbb{R}^n$ with Euclidean metric $d(x, y) = \|x - y\|_2$, $B_r(x)$ has a spherical boundary (circular boundary if $n = 2$).
- In $\mathbb{R}^n$ with a p-norm $\|\cdot\|_p$, the ball is not spherical. For $\mathbb{R}^2$ with the Taxicab metric $d(x, y) = \|x - y\|_1$, $B_r(x)$ is diamond shaped, and for $\mathbb{R}^2$ with the Max norm $\|\cdot\|_\infty$, $B_r(x)$ is a square. We write $B_r^p(x)$ for balls in the p-norm ("***p-balls***").



$B_r^p(x)$

- $p = 1$
- $p = 2$
- $p = 16$

Topology of metric spaces

Example (*p*-norm inequalities and containments)

The following inequalities relate the various *p*-norms on $\mathbb{R}^n$,

$$\|x\|_\infty \le \|x\|_2 \le \|x\|_1 \le n\,\|x\|_\infty\,, \quad \text{and} \quad \|x\|_1 \le \sqrt{n}\,\|x\|_2\,.$$

A good exercise.

Balls in the norm $\|\cdot\|_p$ are often written $B_r^p(x)$. The inequalities above imply that the following sets are *nested:*

$$B^2_{r/n}(x) \subset B^\infty_{r/n}(x) \subset B^1_r(x) \subset B^2_r(x) \subset B^\infty_r(x).$$

Another good exercise.

Example (Balls in the discrete metric)

For any set X, in the *discrete metric* the balls are simple, but strange. If $0 < r \le 1$ then $B_r(x) = \{x\}$, a single point! If $r > 1$ then $B_r(x) = X$, the whole space! You can't be "close" to a point x unless you are at x itself!