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$$\int_M d\omega = \int_{\partial M} \omega$$

Mathematics 3A03
Real Analysis I

Instructor: David Earn

Lecture 19
Sequences and Series of Functions
Friday 27 February 2026

Announcements

- New, exciting topic today...
- Assignment 4 is posted on the course web site.

Sequences and Series of Functions

Limits of Functions

We know that it can be useful to represent functions as limits of other functions.

Example

The power series expansion

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

expresses the exponential e^x as a certain limit of the functions

$$1, \qquad 1+\frac{x}{1!}, \qquad 1+\frac{x}{1!}+\frac{x^2}{2!}, \qquad 1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}, \qquad \cdots$$

Our goal is to give meaning to the phrase "*limit of functions*", and discuss how functions behave under limits.

Pointwise Convergence

- There are multiple inequivalent ways to define the limit of a sequence of functions.
- Consequently, there are multiple different notions of what it means for a sequence of functions to converge.
- Some convergence notions are better behaved than others.

We will begin with the simplest notion of convergence.

Definition (Pointwise Convergence)

Suppose $\{f_n\}$ is a sequence of functions defined on a domain $D \subseteq \mathbb{R}$, and let f be another function defined on D. Then $\{f_n\}$ ***converges pointwise on*** D ***to*** f if, for every $x \in D$, the sequence $\{f_n(x)\}$ of real numbers converges to $f(x)$.

What useful properties of functions does *pointwise convergence* preserve?

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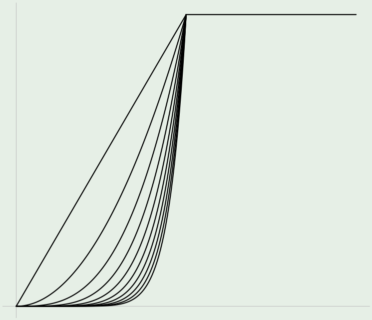
Pointwise Convergence

Example

$$f_n(x) = \begin{cases} x^n & 0 \le x \le 1, \\ 1 & x \ge 1. \end{cases}$$

$$\lim_{n\to\infty} f_n(x) = \begin{cases} 0 & 0 \le x < 1 \\ 1 & x \ge 1 \end{cases}$$



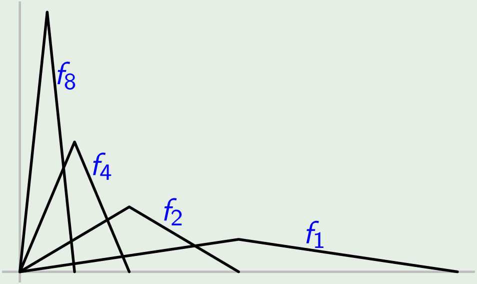
- The limit of this sequence (of continuous functions) is not continuous.
- If we smooth the corner of $f_n(x)$ at $x = 1$, we get a sequence of differentiable functions that converge to a function that is not even continuous.

Pointwise Convergence

Example

Define $f_n(x)$ on $[0,1]$ as follows:

$$f_n(x) = \begin{cases} 2n^2x, & 0 \le x \le \frac{1}{2n} \\ 2n - 2n^2x, & \frac{1}{2n} \le x \le \frac{1}{n} \\ 0, & x \ge \frac{1}{n}. \end{cases}$$



$$\lim_{n\to\infty} f_n(x) = 0 \quad \forall\, x$$

$$\int_0^1 f_n = \frac{1}{2} \quad \forall\, n \in \mathbb{N}$$

$$\int_0^1 \lim_{n\to\infty} f_n = 0$$

Pointwise Convergence

In the previous example, each f_n is integrable and the limit function (the zero function) is also integrable. The example shows that, nevertheless, the sequence of integrals $\{\int f_n\}$ need not converge to the integral of the limit function $\int f$.

Is pointwise convergence sufficient for integrability to be passed on to the limit function?

If so, how do we prove it?
If not, what is a counter-example?

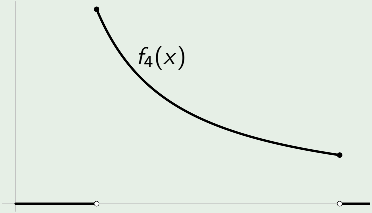
Pointwise Convergence

Example

Let's try to construct a sequence of functions that converges to a non-integrable function. One approach is for each f_n to be bounded, but to converge to an unbounded function.

$$f_n(x) = \begin{cases} 0 & x < \frac{1}{n}, \\ \frac{1}{x} & \frac{1}{n} < x \leq 1, \\ 0 & 1 < x. \end{cases}$$

$$\lim_{n\to\infty} f_n(x) = \begin{cases} 0 & x \leq 0 \\ \frac{1}{x} & 0 < x \geq 1 \\ 0 & 1 < x \end{cases}$$



- The limit of this sequence (of integrable functions) is not bounded on $[0, 1]$, hence not integrable on $[0, 1]$.
- What if $|f_n(x)| \leq M \; \forall x \forall n$, *i.e.*, if $\{f_n\}$ is ***uniformly bounded***?

Pointwise Convergence

Example

Let's try to construct a *uniformly bounded* sequence of functions that converges to a non-integrable, bounded function. Recall the non-integrable function

$$f(x) = \begin{cases} 1 & x \in \mathbb{Q}, \\ 0 & x \notin \mathbb{Q}. \end{cases}$$

Let's construct a sequence of integrable functions that converges to f.

Since $\mathbb{Q}$ is countable, we can list all of its elements in a sequence $\{q_k : k = 1, 2, \ldots\}$. Now define f_n on $[0, 1]$ via

$$f_n(x) = \begin{cases} 1 & x \in \{q_1, \ldots, q_n\}, \\ 0 & \text{otherwise.} \end{cases}$$

Then $f_n(x) \leq 1 \; \forall x$ and on any closed interval (*e.g.*, $[0, 1]$) each f_n is integrable, since it is piecewise continuous, but $f_n \to f$, which is not integrable on <u>any</u> interval. □

Uniform Convergence

A much better behaved notion of convergence is the following.

Definition ($f_n \to f$ uniformly)

Suppose $\{f_n\}$ is a sequence of functions defined on a domain $D \subseteq \mathbb{R}$, and let f be another function defined on D. Then $\{f_n\}$ ***converges uniformly on*** D ***to*** f if, for every $\varepsilon > 0$, there is some $N \in \mathbb{N}$ so that, for all $x \in D$,

$$n \geq N \quad \Longrightarrow \quad |f_n(x) - f(x)| < \varepsilon.$$

Note that $\{f_n\}$ converges uniformly to f if and only if $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ such that

$$n \geq N \quad \Longrightarrow \quad \sup_{x \in D} |f_n(x) - f(x)| < \varepsilon.$$

uniform convergence $\underset{\not\Longleftarrow}{\Longrightarrow}$ pointwise convergence

Uniform Convergence

The sense in which uniform convergence is better behaved than pointwise convergence is that it does preserve at least some properties of the sequence of functions.

Which properties?

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Uniform Convergence

Theorem (Continuity and Uniform Convergence)

Suppose $\{f_n\}$ is a sequence of functions that converges uniformly on $[a,b]$ to f. If each f_n is continuous on $[a,b]$, then f is continuous on $[a,b]$.

What should our proof strategy be?

Our goal is to show that the limit function f is continuous for all $x \in [a,b]$. So given $x \in [a,b]$, we must show that for any $\varepsilon > 0$ we can find a small enough neighbourhood of x, say $(x-\delta, x+\delta)$ for some small δ, such that $|f(x) - f(y)| < \varepsilon$ if $y \in (x-\delta, x+\delta)$, *i.e.*, if $|x-y| < \delta$.

Somehow we have to manage this using the facts that (i) each f_n is continuous and (ii) $f_n \to f$ uniformly.

The key is that (for any n) if x and y are close then $f_n(x)$ and $f_n(y)$ are close, and, if n is large enough, f_n is (uniformly) close to f throughout $[a,b]$, so continuity is "passed through" to the limit.

Let's make this precise...

Uniform Convergence

Proof: f_n continuous $\forall n$ and $f_n \to f$ uniformly $\implies$ f continuous.

Fix $x \in [a, b]$ and $\varepsilon > 0$. We must show $\exists \delta > 0$ such that if $y \in [a, b]$ and $|x - y| < \delta$ then $|f(x) - f(y)| < \varepsilon$.

Since $f_n \to f$ uniformly, $\exists N \in \mathbb{N} \ni |f_N(y) - f(y)| < \frac{\varepsilon}{3}\ \forall y \in [a, b]$ (in particular, $x \in [a, b]$, so we have $|f_N(x) - f(x)| < \frac{\varepsilon}{3}$).
Fix such an integer N.

Since f_N is continuous, there is some $\delta > 0$ such that if $y \in [a, b]$ satisfies $|x - y| < \delta$, then $|f_N(x) - f_N(y)| < \frac{\varepsilon}{3}$. For such y, we then have

$$\begin{aligned} |f(x) - f(y)| &= |f(x) - f_N(x) + f_N(x) - f_N(y) + f_N(y) - f(y)| \\ &\le |f(x) - f_N(x)| + |f_N(x) - f_N(y)| + |f_N(y) - f(y)| \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon, \end{aligned}$$

as required. $\square$

Uniform Convergence

Theorem (Integrability and Uniform Convergence)

Suppose $\{f_n\}$ *is a sequence of functions that converges uniformly on* $[a, b]$ *to* f. *If each* f_n *is integrable on* $[a, b]$, *then* f *is integrable and*

$$\int_a^b f = \lim_{n \to \infty} \int_a^b f_n .$$

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Mathematics and Statistics

$$\int_M d\omega = \int_{\partial M} \omega$$

Mathematics 3A03
Real Analysis I

Instructor: David Earn

Lecture 20
Sequences and Series of Functions II
Tuesday 3 March 2026

Announcements

- I have posted Test 1 and my solutions on the course web site.
- I have added another (simpler) example showing pointwise convergence is insufficient to preserve integrability.

Last time...

Convergence of sequences of functions:

- Pointwise convergence
- Uniform convergence
- Theorem about continuity and uniform convergence

Uniform Convergence

Theorem (Integrability and Uniform Convergence)

Suppose $\{f_n\}$ is a sequence of functions that converges uniformly on $[a,b]$ to f. If each f_n is integrable on $[a,b]$, then f is integrable and

$$\int_a^b f = \lim_{n\to\infty} \int_a^b f_n \,. \qquad (*)$$

(TBB §9.5.2, p. 571ff)

Note: To prove $(*)$, we will need the fact that if f is integrable then so is $|f|$, and $\left|\int_a^b f\right| \le \int_a^b |f|$. *This "triangle inequality" is an excellent exercise.*

Uniform Convergence

Lemma (A useful estimate for upper minus lower sums)

Let g, h be bounded on $[a, b]$. If $\sup_{x\in[a,b]} |h(x) - g(x)| \le \delta$, then for every partition P of $[a, b]$,

$$U(h, P) - L(h, P) \le \big(U(g, P) - L(g, P)\big) + 2\delta(b - a).$$

Proof.

Fix a partition $P = \{a = t_0 < t_1 < \cdots < t_m = b\}$. For each subinterval $I_i = [t_{i-1}, t_i]$ we have

$$\sup_{x\in I_i} h(x) \le \sup_{x\in I_i} \big(g(x) + \delta\big) = \Big(\sup_{x\in I_i} g(x)\Big) + \delta,$$

and similarly

$$\inf_{x\in I_i} h(x) \ge \inf_{x\in I_i} \big(g(x) - \delta\big) = \Big(\inf_{x\in I_i} g(x)\Big) - \delta.$$

Multiplying by $\Delta t_i = t_i - t_{i-1}$ and summing over i,

$$U(h, P) \le U(g, P) + \delta(b - a) \qquad \text{and} \qquad L(h, P) \ge L(g, P) - \delta(b - a).$$

Subtract: $U(h, P) - L(h, P) \le U(g, P) - L(g, P)) + 2\delta(b - a)$. □

Uniform Convergence

Proof that f is integrable (using the ε–P criterion).

First, note that each f_n is bounded (integrable $\Rightarrow$ bounded). Since $f_n \to f$ uniformly, the limit function f is bounded too,so $U(f,P)$ and $L(f,P)$ make sense. We must show: $\forall \varepsilon > 0, \quad \exists$ a partition $P \ni U(f,P) - L(f,P) < \varepsilon$.

Given $\varepsilon > 0$, choose $\delta = \frac{\varepsilon}{4(b-a)}$. Since $f_n \to f$ uniformly, $\exists\, N \in \mathbb{N}$ such that

$$\sup_{x\in[a,b]} |f(x) - f_N(x)| \ < \ \delta.$$

Because f_N is integrable , the ε–P criterion gives a partition P such that

$$U(f_N,P) - L(f_N,P) \ < \ \frac{\varepsilon}{2}.$$

Apply the estimate from the lemma (previous slide) with $g = f_N$ and $h = f$:

$$\begin{aligned} U(f,P) - L(f,P) \ &\le \ \big(U(f_N,P) - L(f_N,P)\big) + 2\delta(b-a) \\ &< \ \frac{\varepsilon}{2} + 2\cdot\frac{\varepsilon}{4(b-a)}(b-a) \ = \ \varepsilon. \end{aligned}$$

This is exactly the ε–P criterion for f to be integrable. $\square$

Now that we know f is integrable , we can prove $(*)$ ...

Uniform Convergence

Proof that $\int_a^b f = \lim_{n\to\infty} \int_a^b f_n$ given that f is integrable.

Given that f is integrable, to prove the equality, we will show that

$$\forall \varepsilon > 0, \quad \exists N \in \mathbb{N} \quad \text{such that} \quad \left| \int_a^b f - \int_a^b f_n \right| < \varepsilon \qquad \forall n \geq N.$$

For any $n \in \mathbb{N}$, we have

$$\left| \int_a^b f - \int_a^b f_n \right| = \left| \int_a^b (f - f_n) \right| \leq \int_a^b |f - f_n|$$
$$\leq U\Big(|f - f_n|, \{a, b\}\Big) = \Big(\sup_{x\in[a,b]} \big| f(x) - f_n(x) \big| \Big)(b - a) .$$

But f_n converges uniformly to f, which means that

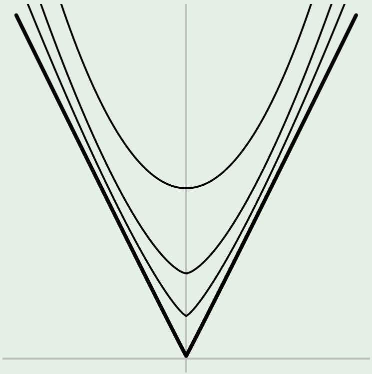
$$\exists N \in \mathbb{N} \quad \text{such that} \quad \sup_{x\in[a,b]} |f(x) - f_n(x)| < \frac{\varepsilon}{b-a} \qquad \forall n \geq N.$$

For such n, we have $\left| \int_a^b f - \int_a^b f_n \right| < \varepsilon$, as required. □

Uniform Convergence

The interaction between uniform convergence and differentiability is more subtle.

Example (f_n diff'ble $\forall n$ and $f_n \to f$ uniformly $\not\Rightarrow$ f diff'ble)



$$f_n(x) = \frac{1}{2n} + (x^2)^{\left(1+\frac{1}{n}\right)/2}$$

Each f_n is differentiable

$f_n(x) \to f(x) = |x|$ *uniformly*

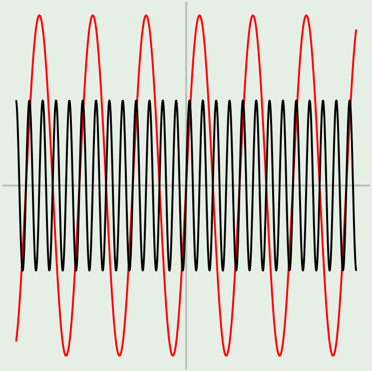
Limit function f is not differentiable.

Note: Graph shows $n = 1, 2, 4, 64$ for $x \in [-1, 1]$.

Uniform Convergence

Even if $f_n \to f$ uniformly, and all the f_n and f are differentiable, it is not necessarily true that $f_n' \to f'$.

Example ($f_n \to f$ uniformly and f_n', f' exist $\not\Rightarrow$ $f_n' \to f'$)



$$f_n(x) = \frac{1}{n}\sin(n^2 x)$$

$$f_n(x) \to f(x) \equiv 0$$

uniformly

$$f_n'(x) = n\cos(n^2 x)$$

$\lim_{n\to\infty} f_n'(x)$ does not exist (*e.g.*, $f_n(0) = n$, which diverges as $n \to \infty$)

Note: Graph shows $n = 1, 2$ on interval $[-20, 20]$.

Uniform Convergence

The theorem on integrability and uniform convergence, together with the fundamental theorem of calculus, must yield some result on uniformly convergent sequences of differentiable functions.

But the result must make hypotheses that avoid the failures in the examples on the two previous slides.

Theorem (Differentiability and Uniform Convergence)

Suppose $\{f_n\}$ *is a sequence of differentiable functions on* $[a, b]$ *such that*

1. f_n' *is integrable for each* n*,*
2. *the sequence* $\{f_n'\}$ *converges uniformly on* $[a, b]$ *to a continuous function* g*,*
3. *the sequence* $\{f_n\}$ *converges pointwise to a function* f*.*

Then f *is differentiable and* $\{f_n'\}$ *converges uniformly to* f'*.*

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Mathematics 3A03
Real Analysis I

Instructor: David Earn

Lecture 21
Sequences and Series of Functions III
Thursday 5 March 2026

Announcements

- Assignment 4 is posted on the course web site.

Last time: Convergence of sequences of functions:

- Pointwise convergence
- Uniform convergence
- Theorem about continuity and uniform convergence
- Theorem about integrability and uniform convergence
- Theorem about differentiability and uniform convergence

Uniform convergence

Proof of theorem on Differentiability and Uniform Convergence.

Since the function g to which f_n' converges is continuous, it is certainly integrable. So we can apply the theorem on integrability and uniform convergence on the interval $[a, x]$ to infer that

$$\begin{aligned} \int_a^x g &= \lim_{n\to\infty} \int_a^x f_n' && f_n' \to g \text{ uniformly} \\ &= \lim_{n\to\infty} \Big(f_n(x) - f_n(a) \Big) && \text{SFTC} \\ &= f(x) - f(a) && f_n \to f \text{ pointwise} \end{aligned}$$

Since g is continuous, FFTC implies that

$$g(x) = \lim_{n\to\infty} f_n'(x) = f'(x)$$

for all $x \in [a, b]$. □

Series of Functions

Series of Real Numbers

Suppose $\{x_n\}$ is a sequence of real numbers. Recall that the ***sequence of partial sums*** is the sequence $\{s_n\}$ defined by

$$s_n = \sum_{k=1}^{n} x_n \,.$$

If the sequence of partial sums converges, then we write the limit as

$$\sum_{k=1}^{\infty} x_k = \lim_{n\to\infty} \sum_{k=1}^{n} x_n = \lim_{n\to\infty} s_n \,.$$

In this case, we call $\sum_{k=1}^{\infty} x_k$ a ***convergent series***. A ***divergent series*** is a sequence of partial sums that diverges; we sometimes abuse notation and write $\sum_{k=1}^{\infty} x_k$ for divergent series as well.

A ***series*** is either a convergent series or a divergent series.

Our goal now is to extend this notion of series to sequences of functions.

Series of Functions

Suppose $\{f_n\}$ is a sequence of functions defined on a set $D \subseteq \mathbb{R}$. The ***sequence of partial sums*** is the sequence $\{S_n\}$ where S_n is the function defined on D by

$$S_n(x) = \sum_{k=1}^{n} f_k(x)\,.$$

When talking about limits of the S_n, we will write $\sum_{k=1}^{\infty} f_k$ and refer to this as a ***series***.

Keep in mind that this is very informal, since the terminology does not specify any sense in which the S_n converge, nor does it assume that the S_n converge at all!

We will now make this more formal.

Series of Functions

Suppose $\{f_n\}$ is a sequence of functions defined on a domain D, and $\{S_n\}$ is its sequence of partial sums.

Definition (Convergence of Series)

If the sequence of partial sums $\{S_n\}$ converges pointwise on D to a function f, then we say that the series $\sum_{k=1}^{\infty} f_k$ ***converges pointwise on*** D ***to*** f.

If the $\{S_n\}$ converge uniformly on D to a function f, then we say that the series $\sum_{k=1}^{\infty} f_k$ ***converges uniformly on*** D ***to*** f.

In both cases, we will write $f = \sum_{k=1}^{\infty} f_k$ to denote that the ***series converges to*** f.

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Series of Functions

The theorems on convergence of sequences of integrable, continuous and differentiable functions have several immediate implications for series of functions.

In the following, we assume that $\{f_n\}$ is a sequence of functions defined on an interval $[a, b]$.

Corollary (Integrals of Series)

Suppose the f_n are integrable and $\sum_{k=1}^{\infty} f_k$ converges uniformly to a function f. Then f is integrable and

$$\int_a^b f = \sum_{k=1}^{\infty} \int_a^b f_k \,.$$

Series of Functions

Corollary (Continuity of Series)

Suppose the f_n are continuous and $\sum_{k=1}^{\infty} f_k$ converges uniformly to a function f. Then f is continuous.

Corollary (Differentiability of Series)

Suppose $\{f_n\}$ is a sequence of differentiable functions on $[a, b]$ such that

- *f_n' is integrable for each n,*
- *the series $\sum_{k=1}^{\infty} f_k'$ converges uniformly on $[a, b]$ to a continuous function g,*
- *the series $\sum_{k=1}^{\infty} f_k$ converges pointwise to a function f.*

Then f is differentiable and $f' = \sum_{k=1}^{\infty} f_k'$.

Proving Uniform Convergence of Series of Functions

We have seen that several useful conclusions can be drawn when a series converges uniformly. The following gives a practical way of proving uniform convergence for a series of functions.

Theorem (Weierstrass M-test)

Let $\{f_n\}$ be a sequence of functions defined on $D \subseteq \mathbb{R}$, and suppose $\{M_n\}$ is a sequence of real numbers such that

$$|f_n(x)| \;\le\; M_n, \qquad \forall x \in D,\ \forall n \in \mathbb{N}.$$

If $\displaystyle\sum_{n=1}^{\infty} M_n$ converges, then $\displaystyle\sum_{k=1}^{\infty} f_k$ converges uniformly.

Proving Uniform Convergence of Series of Functions

Approach to proving the Weierstrass M-test:

- Let $S_n = \sum_{k=1}^{n} f_k$ be the n^{th} partial sum.
- Show that for every $\varepsilon > 0$, there is some $N \in \mathbb{N}$ so that

$$\sup_{x \in D} |S_n(x) - S_m(x)| < \varepsilon \,, \qquad \forall n, m \geq N.$$

  This condition is called the ***uniform Cauchy criterion***.

- Prove that the uniform Cauchy criterion implies uniform convergence.
  - This part is an excellent exercise for you.

    Note: The proof is similar to the proof of the Cauchy criterion for real numbers.

Proving Uniform Convergence of Series of Functions

Proof of the Weierstrass M-test.

Let $\varepsilon > 0$. Suppose the series $\sum M_n$ converges. By the Cauchy criterion for real numbers, there is some integer N so that

$$\left|\sum_{k=1}^{n} M_k - \sum_{k=1}^{m} M_k\right| \;<\; \varepsilon\,, \qquad \forall n, m \geq N.$$

Without loss of generality, we can assume $m < n$, so the above can be written

$$M_{m+1} + M_{m+2} + \cdots + M_n \;<\; \varepsilon.$$

Note that we have $\quad S_n - S_m \;=\; f_{m+1} + f_{m+2} + \cdots + f_n, \quad$ so the assumption that $|f_k(x)| \leq M_k$ implies that

$$\begin{aligned} |S_n(x) - S_m(x)| \;&=\; |f_{m+1}(x) + f_{m+2}(x) + \cdots + f_n(x)| \\ &\leq\; M_{m+1} + M_{m+2} + \cdots + M_n \;<\; \varepsilon. \end{aligned}$$

This is true $\forall x \in D$, hence $\quad \sup_{x \in D} |S_n(x) - S_m(x)| < \varepsilon\,, \quad$ *i.e.*, the uniform Cauchy criterion is satisfied. □

Proving Uniform Convergence of Series of Functions

In order to use the Weierstrass M-test, we need to know whether an associated series of real numbers converges. The most useful standard results are:

- The geometric series $\displaystyle\sum_{n=0}^{\infty} a^n$ converges if and only if $|a| < 1$, in which case the sum of the series is $\dfrac{1}{1-a}$.

- The ***Ratio Test***: (TBB Theorem 3.28)

  If $a_n > 0$ for all $n \in \mathbb{N}$ and $\displaystyle\lim_{n\to\infty} \frac{a_{n+1}}{a_n} < 1$ then $\displaystyle\sum_{n=1}^{\infty} a_n$ converges.

- The ***Root Test***: (TBB Theorem 3.30)

  If $a_n \geq 0$ for all $n \in \mathbb{N}$ and $\displaystyle\lim_{n\to\infty} \sqrt[n]{a_n} < 1$ then $\displaystyle\sum_{n=1}^{\infty} a_n$ converges.

Proving Uniform Convergence of Series of Functions

- The ***Integral Test***: (TBB Theorem 3.35)

Let f be a nonnegative decreasing function on $[1,\infty)$ such that

$$\int_1^K f \quad \text{exists} \quad \text{for all } K > 1.$$

Then

$$\sum_{k=1}^{\infty} f(k) \quad \text{converges} \qquad \Longleftrightarrow \qquad \lim_{K\to\infty} \int_1^K f(x)\,dx \quad \text{exists.}$$

(*There are many other known results concerning convergence series of real numbers.* See TBB Chapter 3.)

Proving Uniform Convergence of Series of Functions

Example

Let $p > 1$, and consider the series $$\sum_{k=1}^{\infty} \frac{\sin(kx)}{k^p}\,.$$

This series satisfies $$\left|\frac{\sin(kx)}{k^p}\right| \le \frac{1}{k^p} \quad \text{for all } x \in \mathbb{R}.$$

Since the series $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^p}$ converges (by the integral test), it follows from the Weierstrass M-test that the series $\sum_{k=1}^{\infty} \frac{\sin(kx)}{k^p}$ converges uniformly.

Hence it is a continuous function.

In fact, if $p > 2$ then the series $\sum_{k=1}^{\infty} \frac{\sin(kx)}{k^p}$ is differentiable:

Let $f_k(x) = \frac{\sin(kx)}{k^p}$. The f_k' are continuous and another application of the Weierstrass M-test shows that $\sum_{k=1}^{\infty} f_k'$ converges uniformly. Hence the series is differentiable and the derivative is $\sum_{k=1}^{\infty} f_k'$.

Mathematics and Statistics

$$\int_M d\omega = \int_{\partial M} \omega$$

Mathematics 3A03
Real Analysis I

Instructor: David Earn

Lecture 22
Sequences and Series of Functions IV
Friday 6 March 2026

Announcements

Last time:

- Differentiability and uniform convergence
- Convergence of series
- Theorems about uniform convergence of series of functions
- Weierstrass M-test
  - Example

Today:

- Power series
- Assignment 4 partcipation poll will open after class

Power Series

Power Series

Suppose $\{a_n\}$ is a sequence of real numbers.

Definition (Power Series)

A ***power series (centred at* 0*)*** is a series of the form

$$\sum_{k=0}^{\infty} a_k x^k \,.$$

More generally, a ***power series centred at*** c has the form

$$\sum_{k=0}^{\infty} a_k (x-c)^k \,.$$

Power Series

Corollary (Convergence of Power Series)

Suppose that the series $f(x_0) = \sum_{k=0}^{\infty} a_k x_0^k$ *converges for some* $x_0 > 0$*, and suppose* $0 < a < x_0$*. Then on* $[-a, a]$*, the series*

$$f(x) = \sum_{k=0}^{\infty} a_k x^k$$

converges uniformly. Moreover, f *is continuous and*

$$\int_c^d f = \sum_{k=0}^{\infty} a_k \int_c^d x^k \qquad \forall c, d \in [-a, a].$$

Finally, f *is differentiable and* $\sum_{k=1}^{\infty} k a_k x^{k-1}$ *converges uniformly on* $[-a, a]$ *to* f'*.*

Power Series

Sketch of proof of convergence of $f(x)=\sum_{k=0}^{\infty} a_k x^k$ *on* $[-a,a]$

- Weierstrass M-test with $M_k = a_k x_0^k$
  $\implies$ uniform convergence to f.
- Uniform convergence to f $\implies$ f is continuous and

$$\int_c^d f = \sum_{k=0}^{\infty} a_k \int_c^d x^k \,.$$

- That the derivative $\sum_{k=1}^{\infty} k\, a_k\, x^{k-1}$ converges uniformly on $[-a,a]$ can be proved via the ratio test (TBB Theorem 3.28) or the root test (TBB Theorem 3.30).
- Uniform convergence of the derivative series
  $\implies$ uniform limit f is differentiable.

Power Series

Example (The simplest power series: $\sum_{k=0}^{\infty} x^k$)

If $0 < x_0 < 1$, then the series $\sum_{k=0}^{\infty} x_0^k$ converges. Consequently, for any $a \in (0,1)$, the series $\sum_{k=0}^{\infty} x^k$ converges uniformly on $[-a,a]$ to a differentiable function, which we know: $\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}$.

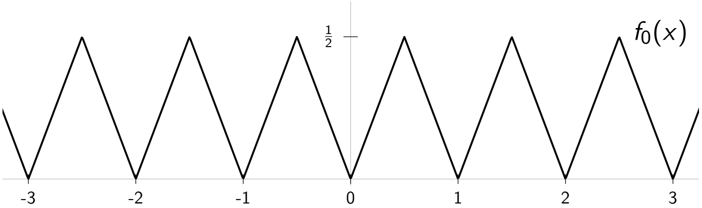
Differentiating we obtain: $\sum_{k=1}^{\infty} k\,x^{k-1} = \frac{1}{(1-x)^2}$.

Integrating (from 0 to x) we obtain: $\sum_{k=0}^{\infty} \frac{x^{k+1}}{k+1} = -\log(1-x)$.

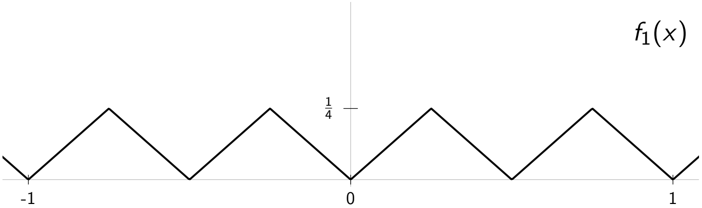
These series are all valid for $x \in (-1,1)$.

How bad can a continuous function be?

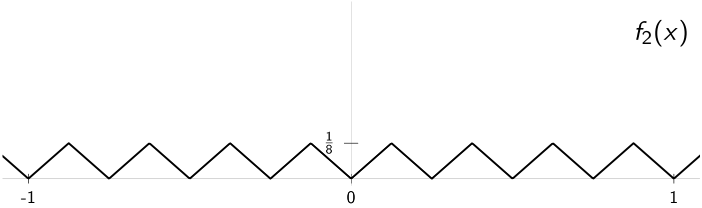
Let $f_0(x) =$ the distance from x to the nearest integer.

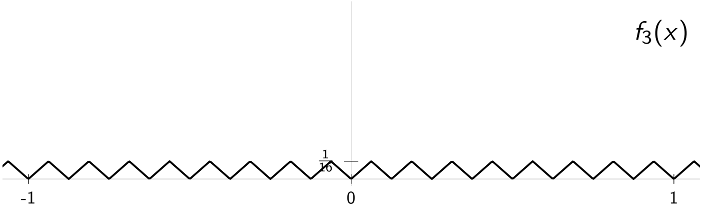


Let $f_n(x) = \dfrac{1}{2^n} f_0(2^n x)$.



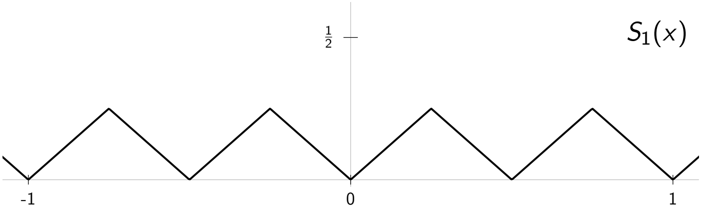
How bad can a continuous function be?

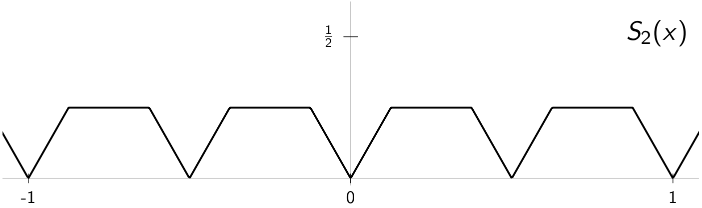




How bad can a continuous function be?

Now define $S_n(x) = \sum_{k=1}^{n} f_n(x)$.





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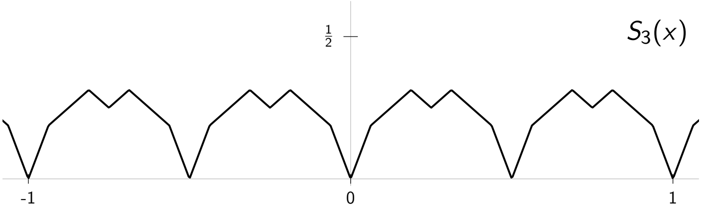
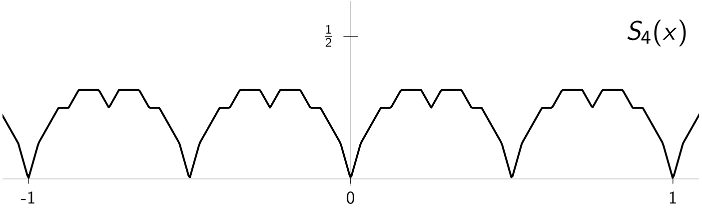
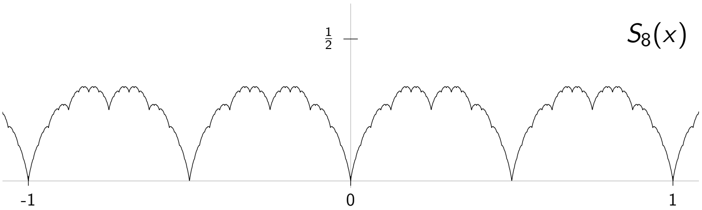
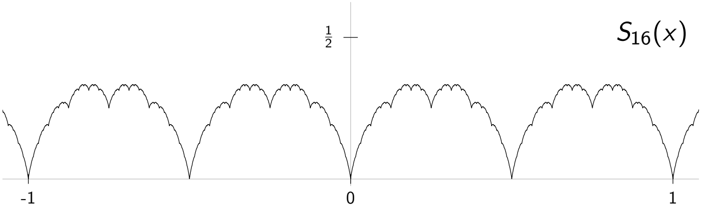




How bad can a continuous function be?





Poll

- Go to
  https://www.childsmath.ca/childsa/forms/main_login.php
- Click on Math 3A03
- Click on Take Class Poll
- Fill in poll **Function sequences: bad function**
- `Submit`.

How bad can a continuous function be?

Now consider:

- Each f_n is continuous, so each $S_n = \sum_{k=1}^{n} f_n$ is continuous.
- $|f_n(x)| \le \dfrac{1}{2^n} \quad \forall x \in \mathbb{R}$.
- $\sum_{n=1}^{\infty} \dfrac{1}{2^n}$ converges.
- $\therefore$ Weierstrass M-test $\implies \sum_{k=1}^{\infty} f_n$ converges uniformly.
- $\therefore$ The uniform limit, say f, is continuous.
- Is f uniformly continuous?
- Is f differentiable?

How bad can a continuous function be?

Extra Challenge Problem:

Prove that the uniform limit function,

$$f = \sum_{k=1}^{\infty} f_n \,,$$

which is continuous on $\mathbb{R}$, is in fact

1. uniformly continuous
2. differentiable **nowhere**

Note: Proving uniform continuity should be really really easy.