22 Metric Spaces

McMaster University

Mathematics and Statistics

$$\int_M d\omega = \int_{\partial M} \omega$$

Mathematics 3A03
Real Analysis I

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Lecture 22
Metric Spaces
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Announcements

- New, exciting topic today. . .

Metric Spaces

The metric structure of $\mathbb{R}$

Definition (Absolute Value function)

For any $x \in \mathbb{R}$,

$$|x| \overset{\text{def}}{=} \begin{cases} x & \text{if } x \geq 0, \\ -x & \text{if } x < 0. \end{cases}$$

Theorem (Properties of the Absolute Value function)

For all $x, y \in \mathbb{R}$:

1. $-|x| \leq x \leq |x|$.
2. $|xy| = |x|\,|y|$.
3. $|x + y| \leq |x| + |y|$.
4. $|x| - |y| \leq |x - y|$.

The metric structure of $\mathbb{R}$

Definition (Distance function or metric)

The distance between two real numbers x and y is

$$d(x,y) = |x-y| \ .$$

Theorem (Properties of distance function or metric)

1. $d(x,y) \geq 0$ — *distances are positive or zero*
2. $d(x,y) = 0 \iff x = y$ — *distinct points have distance* > 0
3. $d(x,y) = d(y,x)$ — *distance is symmetric*
4. $d(x,y) \leq d(x,z) + d(z,y)$ — *the triangle inequality*

Note: Any function satisfying these properties can be considered a "distance" or "metric".

The metric structure of $\mathbb{R}$

Given $d(x, y) = |x - y|$, the properties of the distance function are equivalent to:

Theorem (Metric properties of the absolute value function)

For all $x, y \in \mathbb{R}$*:*

1. $|x| \geq 0$
2. $|x| = 0 \iff x = 0$
3. $|x| = |-x|$
4. $|x + y| \leq |x| + |y|$ *(the triangle inequality)*

Slick proof of the triangle inequality

Theorem (The Triangle Inequality for the standard metric on $\mathbb{R}$)

$|x+y| \leq |x| + |y|$ *for all* $x, y \in \mathbb{R}$.

Proof.

Let $s = \text{sign}(x+y)$. Then

$$|x+y| \ = \ s(x+y) \ = \ sx + sy \ \leq \ |x| + |y| \ ,$$

as required. □

A non-standard metric on $\mathbb{R}$

Example (finite distance between every pair of real numbers)

Let $f(x) = \frac{x}{1+x}$, and define $d(x,y) = f(|x-y|)$. Prove that $d(x,y)$ can be interpreted as a distance between x and y because it satisfies all the properties of a metric.

Proof: The only metric property that is non-trivial to prove is the triangle inequality. Note that $f(x)$ is an increasing function on $[0,\infty)$, so the usual triangle inequality, $|x-y| \le |x-z| + |z-y|$, implies

$$\begin{aligned} f(|x-y|) &\le f(|x-z| + |z-y|) = \frac{|x-z| + |z-y|}{1+|x-z| + |z-y|} \\ &= \frac{|x-z|}{1+|x-z| + |z-y|} + \frac{|z-y|}{1+|x-z| + |z-y|} \\ &\le \frac{|x-z|}{1+|x-z|} + \frac{|z-y|}{1+|z-y|} = f(|x-z|) + f(|z-y|) \end{aligned}$$

i.e., $d(x,y) \le d(x,z) + d(z,y)$. □

Poll

- Go to
  https://www.childsmath.ca/childsa/forms/main_login.php
- Click on Math 3A03
- Click on Take Class Poll
- Fill in poll **Metric spaces: Is "$=$ vs $\neq$" a metric?**
- Submit.

Discrete metric

Example (Discrete metric on $\mathbb{R}$)

Let $d(x,y) = \begin{cases} 0 & x = y, \\ 1 & x \neq y. \end{cases}$ Is d a metric on $\mathbb{R}$?

By definition, $d(x,y)$ is non-negative, zero iff $x = y$, and symmetric. For the triangle inequality, if $x = y$ then $d(x,y) = 0$ so the inequality holds for any z. If $x \neq y$ then $d(x,y) = 1$, and at least one of x and y must not equal z, so the inequality says either $1 \leq 1$ or $1 \leq 2$.

Example (Discrete metric on any set X)

The argument that $d(x,y)$ is a metric on $\mathbb{R}$ has nothing to do with $\mathbb{R}$ specifically. $d(x,y)$ is a metric on <u>any</u> set X.

General metric space (X, d)

Definition (Metric space)

A ***metric space*** (X, d) is a non-empty set X together with a distance function (or ***metric***) $d : X \times X \to \mathbb{R}$ satisfying

1. $d(x, y) \geq 0$ — *distances are positive or zero*
2. $d(x, y) = 0 \iff x = y$ — *distinct points have distance* > 0
3. $d(x, y) = d(y, x)$ — *distance is symmetric*
4. $d(x, y) \leq d(x, z) + d(z, y)$ — *the triangle inequality*

Much of our analysis of sequences of real numbers and topology of $\mathbb{R}$ generalizes to any metric space. Very often, definitions and proofs depend only on the the existence of a metric, not on $|x|$ specifically. Many useful inferences can be made by identifying a metric on a space of interest.

Examples of metric spaces

Example (Metric spaces (X, d))

- $X = \mathbb{Q}$, with the standard metric $d(x, y) = |x - y|$.
  As $\mathbb{Q} \subset \mathbb{R}$, each condition for d is satisfied in $\mathbb{Q}$.

  How different is $(\mathbb{Q}, d)$ from $(\mathbb{R}, d)$?

- $X = \mathbb{N}$, with the standard metric $d(x, y) = |x - y|$.
  As $\mathbb{N} \subset \mathbb{R}$, each condition for d is satisfied in $\mathbb{N}$.

- $X = \mathbb{R}^2$ with $d(x, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$, where we write the vectors $x = (x_1, x_2)$, $y = (y_1, y_2) \in \mathbb{R}^2$.

- $X = \mathbb{R}^n$ with $d(x, y) = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}$, where $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n) \in \mathbb{R}^n$.

  This metric on $\mathbb{R}^n$ is called the ***Euclidean distance***.

Metrics from norms

The Euclidean metric on $\mathbb{R}^n$ is the (Euclidean) length of the difference of two vectors. This connection between length and distance generalizes to any vector space in which *length* is defined.

Definition (Norm)

A ***norm*** on a vector space X is a real-valued function on X such that if $x, y \in X$ and $\alpha \in \mathbb{R}$ then

1. $\|x\| \geq 0$ and $\|x\| = 0$ iff x is the zero element in X;
2. $\|\alpha x\| = |\alpha| \, \|x\|$;
3. $\|x + y\| \leq \|x\| + \|y\|$.

A vector space X equipped with a norm $\|\cdot\|$ is said to be a ***normed vector space***. Any norm $\|\cdot\|$ ***induces*** a metric d via

$$d(x, y) = \|x - y\| \,.$$

Proving that a function is a norm is not necessarily easy. Let's try for the Euclidean norm... To that end, recall the notion of inner product ...

Norms from inner products

Definition (Inner product)

An ***inner product*** on a vector space V over $\mathbb{R}$ is a function

$$\langle \cdot, \cdot \rangle : V \times V \to \mathbb{R}$$

such that for all $u, v, w \in V$ and all scalars $\alpha \in \mathbb{R}$:

1. $\langle u, v \rangle = \overline{\langle v, u \rangle}$ *conjugate symmetry*
2. $\langle \alpha u + v, w \rangle = \alpha \langle u, w \rangle + \langle v, w \rangle$ *linearity in 1st argument*
3. $\langle v, v \rangle \geq 0$ with equality iff $v = 0$ *positive definiteness*

A vector space equipped with an inner product is called an *inner product space*.

Definition (Inner Product Norm)

The norm induced by an inner product $\langle \cdot, \cdot \rangle$ is $\|u\| = \sqrt{\langle u, u \rangle}$.

Norms from inner products

Theorem (Cauchy-Schwarz inequality)

Let V be a (real) inner product space with inner product $\langle \cdot, \cdot \rangle$. For all vectors $u, v \in V$, we have

$$|\langle u, v \rangle| \leq \|u\| \, \|v\|$$

where $\|u\| = \sqrt{\langle u, u \rangle}$ *is the norm induced by the inner product.*

Proof.

The standard proof begins with an idea that probably took someone a long time to think of: Since $\langle v, v \rangle \geq 0$ for any $v \in V$, for any $t \in \mathbb{R}$ we have

$$\begin{aligned} 0 \leq \langle u + tv, u + tv \rangle &= \langle u, u \rangle + t \langle u, v \rangle + t \langle v, u \rangle + t^2 \langle v, v \rangle \\ &= \langle u, u \rangle + 2t \langle u, v \rangle + t^2 \langle v, v \rangle \end{aligned}$$

This is a quadratic polynomial in t, which is non-negative for all $t \in \mathbb{R}$. Hence, this quadratic has at most one real root. Consequently, its discriminant is non-positive, *i.e.,* $(2 \langle u, v \rangle)^2 - 4 \langle u, u \rangle \langle v, v \rangle \leq 0$.

continued. . .

Norms from inner products

Proof of Cauchy-Schwarz inequality (continued).

Simplifying the non-positive discriminant condition, we have

$$(\langle u, v\rangle)^2 \;\le\; \langle u, u\rangle \,\langle v, v\rangle \;.$$

Taking square roots, we have

$$|\langle u, v\rangle| \;\le\; \|u\|\,\|v\| \;,$$

as required. □

How might you come up with such a proof?

Perhaps by guessing the result (based on knowing it in $\mathbb{R}^2$)
and then working backwards.