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**Mathematics
and Statistics**

$$\int_M d\omega = \int_{\partial M} \omega$$

Mathematics 3A03
Real Analysis I

Instructor: David Earn

Lecture 19
Sequences and Series of Functions
Friday 27 February 2026

Announcements

- New, exciting topic today…
- Assignment 4 is posted on the course web site.

Sequences and Series of Functions

Limits of Functions

We know that it can be useful to represent functions as limits of
other functions.

Example

The power series expansion

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

expresses the exponential e^x as a certain limit of the functions

$$1, \qquad 1 + \frac{x}{1!}, \qquad 1 + \frac{x}{1!} + \frac{x^2}{2!}, \qquad 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!}, \qquad \cdots$$

Our goal is to give meaning to the phrase "*limit of functions*", and
discuss how functions behave under limits.

Pointwise Convergence

- There are multiple <u>inequivalent</u> ways to define the <u>limit</u> of a sequence of functions.
- Consequently, there are multiple different notions of what it means for a sequence of functions to <u>converge</u>.
- Some convergence notions are <u>better behaved</u> than others.

We will begin with the simplest notion of convergence.

Definition (Pointwise Convergence)

Suppose $\{f_n\}$ is a sequence of functions defined on a domain $D \subseteq \mathbb{R}$, and let f be another function defined on D. Then $\{f_n\}$ **converges pointwise on D to f** if, for every $x \in D$, the sequence $\{f_n(x)\}$ of real numbers converges to $f(x)$.

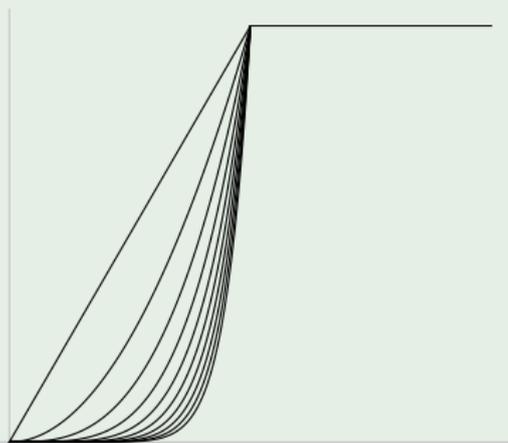What useful properties of functions does *pointwise convergence* preserve?

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Pointwise Convergence

Example

$$f_n(x) = \begin{cases} x^n & 0 \le x \le 1, \\ 1 & x \ge 1. \end{cases}$$

$$\lim_{n \to \infty} f_n(x) = \begin{cases} 0 & 0 \le x < 1 \\ 1 & x \ge 1 \end{cases}$$
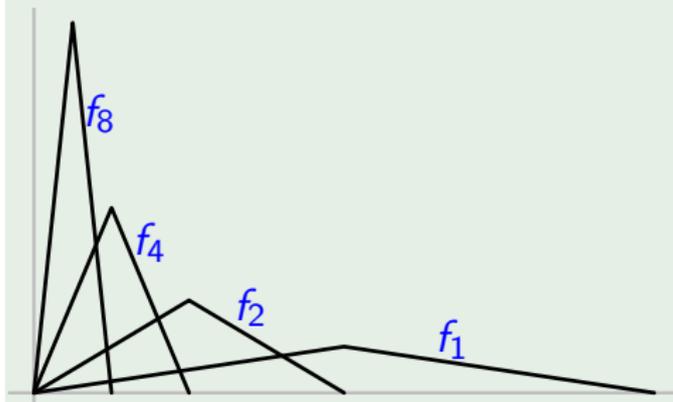

- The limit of this sequence (of continuous functions) is not continuous.

- If we smooth the corner of $f_n(x)$ at $x = 1$, we get a sequence of <u>differentiable</u> functions that converge to a function that is <u>not</u> even <u>continuous</u>.

Pointwise Convergence

Example

Define $f_n(x)$ on $[0, 1]$ as follows:

$$f_n(x) = \begin{cases} 2n^2 x, & 0 \leq x \leq \frac{1}{2n} \\ 2n - 2n^2 x, & \frac{1}{2n} \leq x \leq \frac{1}{n} \\ 0, & x \geq \frac{1}{n}. \end{cases}$$

$$\lim_{n \to \infty} f_n(x) = 0 \quad \forall x$$

$$\int_0^1 f_n = \frac{1}{2} \quad \forall n \in \mathbb{N}$$

$$\int_0^1 \lim_{n \to \infty} f_n = 0$$

Pointwise Convergence

In the previous example, each f_n is integrable and the limit function (the zero function) is also integrable. The example shows that, nevertheless, the sequence of integrals $\{\int f_n\}$ need not converge to the integral of the limit function $\int f$.

Is pointwise convergence sufficient for integrability to be passed on to the limit function?

If so, how do we prove it?
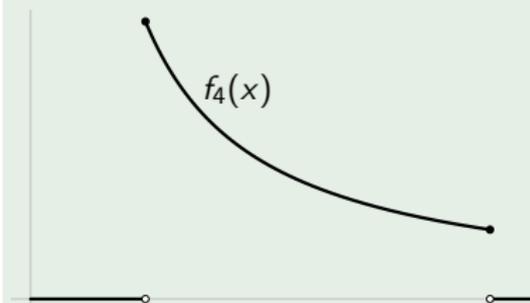If not, what is a counter-example?

Pointwise Convergence

Example

Let's try to construct a sequence of functions that converges to a non-integrable function. One approach is for each f_n to be bounded, but to converge to an unbounded function.

$$f_n(x) = \begin{cases} 0 & x < \frac{1}{n}, \\ \frac{1}{x} & \frac{1}{n} < x \leq 1, \\ 0 & 1 < x. \end{cases} \qquad \lim_{n \to \infty} f_n(x) = \begin{cases} 0 & x \leq 0 \\ \frac{1}{x} & 0 < x \geq 1 \\ 0 & 1 < x \end{cases}$$

$f_4(x)$

- The limit of this sequence (of integrable functions) is not bounded on $[0, 1]$, hence not integrable on $[0, 1]$.

- What if $|f_n(x)| \leq M \; \forall x \forall n$, *i.e.*, if $\{f_n\}$ is ***uniformly bounded***?

Pointwise Convergence

Example

Let's try to construct a *uniformly bounded* sequence of functions that converges to a non-integrable, bounded function. Recall the non-integrable function

$$f(x) = \begin{cases} 1 & x \in \mathbb{Q}, \\ 0 & x \notin \mathbb{Q}. \end{cases}$$

Let's construct a sequence of integrable functions that converges to f.

Since $\mathbb{Q}$ is countable, we can list all of its elements in a sequence $\{q_k : k = 1, 2, \ldots\}$. Now define f_n on $[0, 1]$ via

$$f_n(x) = \begin{cases} 1 & x \in \{q_1, \ldots, q_n\}, \\ 0 & \text{otherwise.} \end{cases}$$

Then $f_n(x) \le 1 \ \forall x$ and on any closed interval (*e.g.*, $[0, 1]$) each f_n is integrable, since it is piecewise continuous, but $f_n \to f$, which is not integrable on <u>any</u> interval. $\square$

Uniform Convergence

A much better behaved notion of convergence is the following.

Definition ($f_n \to f$ uniformly)

Suppose $\{f_n\}$ is a sequence of functions defined on a domain $D \subseteq \mathbb{R}$, and let f be another function defined on D. Then $\{f_n\}$ **converges uniformly on D to f** if, for every $\varepsilon > 0$, there is some $N \in \mathbb{N}$ so that, <u>for all $x \in D$</u>,
$$n \geq N \implies |f_n(x) - f(x)| < \varepsilon.$$

Note that $\{f_n\}$ converges uniformly to f if and only if $\quad \forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ such that
$$n \geq N \implies \sup_{x \in D} |f_n(x) - f(x)| < \varepsilon.$$

uniform convergence $\quad \overset{\Longrightarrow}{\nLeftarrow} \quad$ pointwise convergence

Uniform Convergence

The sense in which uniform convergence is <u>better behaved</u>
than pointwise convergence is that it <u>does</u> preserve at least
some properties of the sequence of functions.

Which properties?

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Uniform Convergence

Theorem (Continuity and Uniform Convergence)

Suppose $\{f_n\}$ is a sequence of functions that converges uniformly on $[a, b]$ to f. If each f_n is continuous on $[a, b]$, then f is continuous on $[a, b]$.

What should our proof strategy be?

Our goal is to show that the limit function f is continuous for all $x \in [a, b]$. So given $x \in [a, b]$, we must show that for any $\varepsilon > 0$ we can find a small enough neighbourhood of x, say $(x - \delta, x + \delta)$ for some small δ, such that $|f(x) - f(y)| < \varepsilon$ if $y \in (x - \delta, x + \delta)$, *i.e.*, if $|x - y| < \delta$.

Somehow we have to manage this using the facts that (i) each f_n is continuous and (ii) $f_n \to f$ <u>uniformly</u>.

The key is that (for any n) if x and y are close then $f_n(x)$ and $f_n(y)$ are close, and, if n is large enough, f_n is (uniformly) close to f throughout $[a, b]$, so continuity is "passed through" to the limit.

Let's make this precise…

Uniform Convergence

Proof: f_n continuous $\forall n$ and $f_n \to f$ uniformly $\implies$ f continuous.

Fix $x \in [a, b]$ and $\varepsilon > 0$. We must show $\exists \delta > 0$ such that if $y \in [a, b]$ and $|x - y| < \delta$ then $|f(x) - f(y)| < \varepsilon$.

Since $f_n \to f$ uniformly, $\exists N \in \mathbb{N} \ni |f_N(y) - f(y)| < \frac{\varepsilon}{3} \ \forall y \in [a, b]$ (in particular, $x \in [a, b]$, so we have $|f_N(x) - f(x)| < \frac{\varepsilon}{3}$). Fix such an integer N.

Since f_N is continuous, there is some $\delta > 0$ such that if $y \in [a, b]$ satisfies $|x - y| < \delta$, then $|f_N(x) - f_N(y)| < \frac{\varepsilon}{3}$. For such y, we then have

$$
\begin{aligned}
|f(x) - f(y)| &= |f(x) - f_N(x) + f_N(x) - f_N(y) + f_N(y) - f(y)| \\
&\leq |f(x) - f_N(x)| + |f_N(x) - f_N(y)| + |f_N(y) - f(y)| \\
&< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon,
\end{aligned}
$$

as required. $\qquad\square$

Uniform Convergence

Theorem (Integrability and Uniform Convergence)

Suppose $\{f_n\}$ is a sequence of functions that converges uniformly on $[a, b]$ to f. If each f_n is integrable on $[a, b]$, then f is integrable and

$$\int_a^b f = \lim_{n \to \infty} \int_a^b f_n \,.$$

**Mathematics
and Statistics**

$$\int_M d\omega = \int_{\partial M} \omega$$

Mathematics 3A03
Real Analysis I

Instructor: David Earn

Lecture 20
Sequences and Series of Functions II
Tuesday 3 March 2026

Announcements

- I have posted Test 1 and my solutions on the course web site.

- I have added another (simpler) example showing pointwise convergence is insufficient to preserve integrability.

Last time...

Convergence of sequences of functions:

- Pointwise convergence

- Uniform convergence

- Theorem about continuity and uniform convergence

Uniform Convergence

Theorem (Integrability and Uniform Convergence)

Suppose $\{f_n\}$ is a sequence of functions that converges uniformly on $[a, b]$ to f. If each f_n is integrable on $[a, b]$, then f is integrable and

$$\int_a^b f = \lim_{n \to \infty} \int_a^b f_n. \qquad (*)$$

(TBB §9.5.2, p. 571ff)

Note: To prove $(*)$, we will need the fact that if f is integrable then so is $|f|$, and $\left| \int_a^b f \right| \leq \int_a^b |f|$. *This "triangle inequality" is an excellent exercise.*

Uniform Convergence

Lemma (A useful estimate for upper minus lower sums)

Let g, h be bounded on $[a, b]$. If $\sup_{x \in [a,b]} |h(x) - g(x)| \leq \delta$, *then for every partition P of $[a, b]$,*

$$U(h, P) - L(h, P) \leq \big(U(g, P) - L(g, P)\big) + 2\delta(b - a).$$

Proof.

Fix a partition $P = \{a = t_0 < t_1 < \cdots < t_m = b\}$. For each subinterval $I_i = [t_{i-1}, t_i]$ we have

$$\sup_{x \in I_i} h(x) \leq \sup_{x \in I_i} \big(g(x) + \delta\big) = \Big(\sup_{x \in I_i} g(x)\Big) + \delta,$$

and similarly

$$\inf_{x \in I_i} h(x) \geq \inf_{x \in I_i} \big(g(x) - \delta\big) = \Big(\inf_{x \in I_i} g(x)\Big) - \delta.$$

Multiplying by $\Delta t_i = t_i - t_{i-1}$ and summing over i,

$$U(h, P) \leq U(g, P) + \delta(b - a) \qquad \text{and} \qquad L(h, P) \geq L(g, P) - \delta(b - a).$$

Subtract: $U(h, P) - L(h, P) \leq U(g, P) - L(g, P)) + 2\delta(b - a)$. $\qquad \square$

Uniform Convergence

Proof that f is integrable (using the $\varepsilon-P$ criterion).

First, note that each f_n is bounded (integrable $\Rightarrow$ bounded). Since $f_n \to f$ uniformly, the limit function f is bounded too, so $U(f,P)$ and $L(f,P)$ make sense. We must show: $\forall \varepsilon > 0, \quad \exists$ a partition $P \ni U(f,P) - L(f,P) < \varepsilon$.

Given $\varepsilon > 0$, choose $\delta = \frac{\varepsilon}{4(b-a)}$. Since $f_n \to f$ uniformly, $\exists N \in \mathbb{N}$ such that

$$\sup_{x \in [a,b]} |f(x) - f_N(x)| < \delta.$$

Because f_N is integrable, the $\varepsilon-P$ criterion gives a partition P such that

$$U(f_N, P) - L(f_N, P) < \frac{\varepsilon}{2}.$$

Apply the estimate from the lemma (previous slide) with $g = f_N$ and $h = f$:

$$U(f,P) - L(f,P) \leq \big(U(f_N,P) - L(f_N,P)\big) + 2\delta(b-a)$$
$$< \frac{\varepsilon}{2} + 2 \cdot \frac{\varepsilon}{4(b-a)}(b-a) = \varepsilon.$$

This is exactly the $\varepsilon-P$ criterion for f to be integrable. $\square$

Now that we know f is integrable, we can prove $(*)$ ...

Uniform Convergence

Proof that $\quad \int_a^b f = \lim_{n \to \infty} \int_a^b f_n \quad$ <u>given</u> that f is integrable.

Given that f is integrable, to prove the equality, we will show that

$$\forall \varepsilon > 0, \quad \exists N \in \mathbb{N} \quad \text{such that} \quad \left| \int_a^b f - \int_a^b f_n \right| < \varepsilon \qquad \forall n \geq N.$$

For any $n \in \mathbb{N}$, we have

$$\left| \int_a^b f - \int_a^b f_n \right| = \left| \int_a^b (f - f_n) \right| \leq \int_a^b |f - f_n|$$

$$\leq U\big(|f - f_n|, \{a, b\} \big) = \left(\sup_{x \in [a,b]} \big| f(x) - f_n(x) \big| \right)(b - a).$$

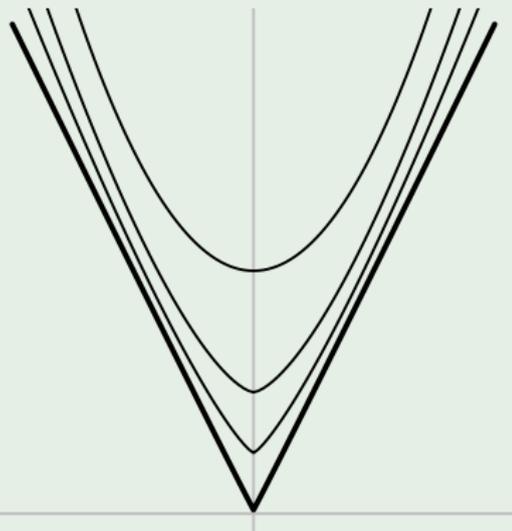But f_n converges uniformly to f, which means that

$$\exists N \in \mathbb{N} \quad \text{such that} \quad \sup_{x \in [a,b]} |f(x) - f_n(x)| < \frac{\varepsilon}{b - a} \qquad \forall n \geq N.$$

For such n, we have $\left| \int_a^b f - \int_a^b f_n \right| < \varepsilon$, as required. $\qquad\qquad\square$

Uniform Convergence

The interaction between uniform convergence and differentiability is more subtle.

Example (f_n diff'ble $\forall n$ and $f_n \to f$ uniformly $\not\Longrightarrow$ f diff'ble)

$f_n(x) = \frac{1}{2n} + (x^2)^{(1+\frac{1}{n})/2}$

Each f_n is differentiable
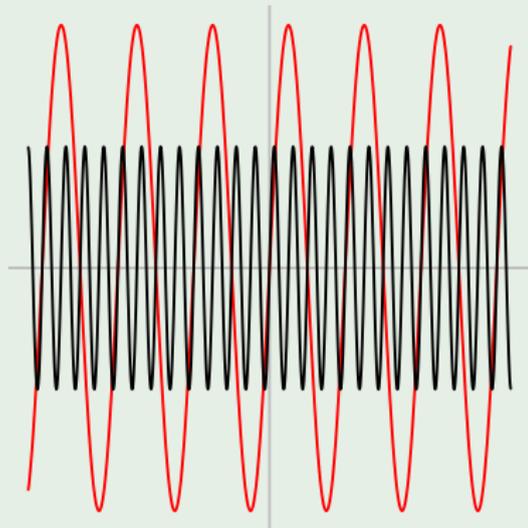
$f_n(x) \to f(x) = |x|$
  uniformly

Limit function f is <u>not</u> differentiable.

Note: Graph shows $n = 1, 2, 4, 64$ for $x \in [-1, 1]$.

Uniform Convergence

Even if $f_n \to f$ uniformly, and all the f_n and f are differentiable, it is <u>not</u> necessarily true that $f_n' \to f'$.

Example ($f_n \to f$ uniformly and f_n', f' exist $\implies\!\!\!\!\!/\ \ f_n' \to f'$)

$f_n(x) \;=\; \frac{1}{n}\sin\left(n^2 x\right)$

$f_n(x) \to f(x) \equiv 0$
  uniformly

$f_n'(x) = n\cos\left(n^2 x\right)$

$\displaystyle \lim_{n\to\infty} f_n'(x)$ does <u>not</u> exist
(*e.g.*, $f_n(0) = n$, which
diverges as $n \to \infty$)

Note: Graph shows $n = 1, 2$ on interval $[-20, 20]$.

Uniform Convergence

The theorem on integrability and uniform convergence, together with the fundamental theorem of calculus, must yield some result on uniformly convergent sequences of differentiable functions.

But the result must make hypotheses that avoid the failures in the examples on the two previous slides.

Theorem (Differentiability and Uniform Convergence)

Suppose $\{f_n\}$ is a sequence of differentiable functions on $[a, b]$ such that

1 *f_n' is integrable for each n,*

2 *the sequence $\{f_n'\}$ converges uniformly on $[a, b]$ to a continuous function g,*

3 *the sequence $\{f_n\}$ converges pointwise to a function f.*

Then f is differentiable and $\{f_n'\}$ converges uniformly to f'.