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$$\int_M d\omega = \int_{\partial M} \omega$$

Mathematics 3A03
Real Analysis I

Instructor: David Earn

Lecture 19
Sequences and Series of Functions
Friday 28 February 2025

Announcements

- New, exciting topic today...

Sequences and Series of Functions

Limits of Functions

We know that it can be useful to represent functions as limits of other functions.

Example

The power series expansion

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

expresses the exponential e^x as a certain limit of the functions

$$1, \qquad 1+\frac{x}{1!}, \qquad 1+\frac{x}{1!}+\frac{x^2}{2!}, \qquad 1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}, \qquad \ldots$$

Our goal is to give meaning to the phrase "*limit of functions*", and discuss how functions behave under limits.

Pointwise Convergence

- There are multiple inequivalent ways to define the limit of a sequence of functions.
- Consequently, there are multiple different notions of what it means for a sequence of functions to converge.
- Some convergence notions are better behaved than others.

We will begin with the simplest notion of convergence.

Definition (Pointwise Convergence)

Suppose $\{f_n\}$ is a sequence of functions defined on a domain $D \subseteq \mathbb{R}$, and let f be another function defined on D. Then $\{f_n\}$ ***converges pointwise on*** D ***to*** f if, for every $x \in D$, the sequence $\{f_n(x)\}$ of real numbers converges to $f(x)$.

What useful properties of functions does *pointwise convergence* preserve?

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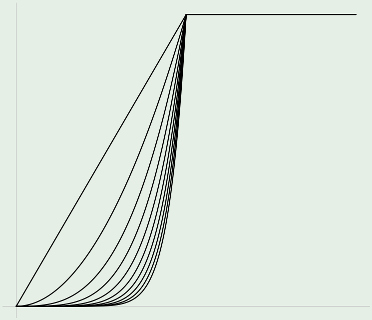
Pointwise Convergence

Example

$$f_n(x) = \begin{cases} x^n & 0 \le x \le 1, \\ 1 & x \ge 1. \end{cases}$$

$$\lim_{n\to\infty} f_n(x) = \begin{cases} 0 & 0 \le x < 1 \\ 1 & x \ge 1 \end{cases}$$



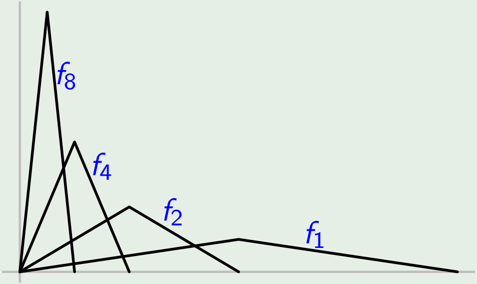
- The limit of this sequence (of continuous functions) is not continuous.
- If we smooth the corner of $f_n(x)$ at $x = 1$, we get a sequence of differentiable functions that converge to a function that is not even continuous.

Pointwise Convergence

Example

Define $f_n(x)$ on $[0,1]$ as follows:

$$f_n(x) = \begin{cases} 2n^2x, & 0 \le x \le \frac{1}{2n} \\ 2n - 2n^2x, & \frac{1}{2n} \le x \le \frac{1}{n} \\ 0, & x \ge \frac{1}{n}. \end{cases}$$



$$\lim_{n\to\infty} f_n(x) = 0 \quad \forall\, x$$

$$\int_0^1 f_n = \frac{1}{2} \quad \forall\, n \in \mathbb{N}$$

$$\int_0^1 \lim_{n\to\infty} f_n = 0$$

Pointwise Convergence

In the previous example, each f_n is integrable and the limit function (the zero function) is also integrable. The example shows that, nevertheless, the sequence of integrals $\{\int f_n\}$ need not converge to the integral of the limit function $\int f$.

Is pointwise convergence sufficient for integrability to be passed on to the limit function?

If so, how do we prove it?
If not, what is a counter-example?

Pointwise Convergence

Example

Let's try to construct a sequence of functions that converges to a non-integrable function. The one such function that we have discussed is

$$f(x) = \begin{cases} 1 & x \in \mathbb{Q}, \\ 0 & x \notin \mathbb{Q}. \end{cases}$$

Let's construct a sequence of integrable functions that converges to f.

Since $\mathbb{Q}$ is countable, we can list all of its elements in a sequence $\{q_k : k = 1, 2, \dots\}$. Now define f_n on $[0, 1]$ via

$$f_n(x) = \begin{cases} 1 & x \in \{q_1, \dots, q_n\}, \\ 0 & \text{otherwise.} \end{cases}$$

Then on any closed interval (*e.g.*, $[0, 1]$) each f_n is integrable, since it is piecewise continuous, but $f_n \to f$, which is not integrable on <u>any</u> interval. □

Uniform Convergence

A much better behaved notion of convergence is the following.

Definition ($f_n \to f$ uniformly)

Suppose $\{f_n\}$ is a sequence of functions defined on a domain $D \subseteq \mathbb{R}$, and let f be another function defined on D. Then $\{f_n\}$ ***converges uniformly on*** D ***to*** f if, for every $\varepsilon > 0$, there is some $N \in \mathbb{N}$ so that, for all $x \in D$,

$$n \geq N \quad \Longrightarrow \quad |f_n(x) - f(x)| < \varepsilon.$$

Note that $\{f_n\}$ converges uniformly to f if and only if $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ such that

$$n \geq N \quad \Longrightarrow \quad \sup_{x \in D} |f_n(x) - f(x)| < \varepsilon.$$

$$\text{uniform convergence} \quad \overset{\Longrightarrow}{\not\Longleftarrow} \quad \text{pointwise convergence}$$

Uniform Convergence

The sense in which uniform convergence is better behaved than pointwise convergence is that it does preserve at least some properties of the sequence of functions.

Which properties?

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Uniform Convergence

Theorem (Continuity and Uniform Convergence)

Suppose $\{f_n\}$ is a sequence of functions that converges uniformly on $[a,b]$ to f. If each f_n is continuous on $[a,b]$, then f is continuous on $[a,b]$.

What should our proof strategy be?

Our goal is to show that the limit function f is continuous for all $x \in [a,b]$. So given $x \in [a,b]$, we must show that for any $\varepsilon > 0$ we can find a small enough neighbourhood of x, say $(x-\delta, x+\delta)$ for some small δ, such that $|f(x) - f(y)| < \varepsilon$ if $y \in (x-\delta, x+\delta)$, *i.e.*, if $|x-y| < \delta$.

Somehow we have to manage this using the facts that (i) each f_n is continuous and (ii) $f_n \to f$ uniformly.

The key is that (for any n) if x and y are close then $f_n(x)$ and $f_n(y)$ are close, and, if n is large enough, f_n is (uniformly) close to f throughout $[a,b]$, so continuity is "passed through" to the limit.

Let's make this precise...

Uniform Convergence

Proof: f_n continuous $\forall n$ and $f_n \to f$ uniformly $\implies$ f continuous.

Fix $x \in [a,b]$ and $\varepsilon > 0$. We must show $\exists \delta > 0$ such that if $y \in [a,b]$ and $|x - y| < \delta$ then $|f(x) - f(y)| < \varepsilon$.

Since $f_n \to f$ uniformly, $\exists N \in \mathbb{N} \ni |f_N(y) - f(y)| < \frac{\varepsilon}{3} \ \forall y \in [a,b]$ (in particular, $x \in [a,b]$, so we have $|f_N(x) - f(x)| < \frac{\varepsilon}{3}$).
Fix such an integer N.

Since f_N is continuous, there is some $\delta > 0$ such that if $y \in [a,b]$ satisfies $|x - y| < \delta$, then $|f_N(x) - f_N(y)| < \frac{\varepsilon}{3}$. For such y, we then have

$$\begin{aligned} |f(x) - f(y)| &= |f(x) - f_N(x) + f_N(x) - f_N(y) + f_N(y) - f(y)| \\ &\le |f(x) - f_N(x)| + |f_N(x) - f_N(y)| + |f_N(y) - f(y)| \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon, \end{aligned}$$

as required. □

Uniform Convergence

Theorem (Integrability and Uniform Convergence)

Suppose $\{f_n\}$ is a sequence of functions that converges uniformly on $[a, b]$ to f. If each f_n is integrable on $[a, b]$, then f is integrable and

$$\int_a^b f = \lim_{n\to\infty} \int_a^b f_n \,.$$

McMaster University

Mathematics and Statistics

$$\int_M d\omega = \int_{\partial M} \omega$$

Mathematics 3A03
Real Analysis I

Instructor: David Earn

Lecture 20
Sequences and Series of Functions
Monday 3 March 2025

Announcements

- I have posted the test and solutions on the course web site.

Last time. . .

Convergence of sequences of functions:

- Pointwise convergence
- Uniform convergence
- I have added another example to the slides for the previous lecture: there is now a new example on integrability and pointwise convergence.

Uniform Convergence

Theorem (Integrability and Uniform Convergence)

Suppose $\{f_n\}$ *is a sequence of functions that converges uniformly on* $[a,b]$ *to* f. *If each* f_n *is integrable on* $[a,b]$, *then* f *is integrable and*

$$\int_a^b f = \lim_{n\to\infty} \int_a^b f_n \,. \qquad (*)$$

(TBB §9.5.2, p. 571ff)

The proof that f is integrable is rather involved. We will skip it, and assume that the limit function is integrable.

To prove $(*)$, we will need the fact that if f is integrable then so is $|f|$, and $\left|\int_a^b f\right| \le \int_a^b |f|$. *This "triangle inequality" is an excellent exercise.*

Uniform Convergence

Proof that $\int_a^b f = \lim_{n\to\infty} \int_a^b f_n$ given that f is integrable.

Given that f is integrable, to prove the equality, we will show that

$$\forall \varepsilon > 0, \quad \exists N \in \mathbb{N} \quad \text{such that} \quad \left| \int_a^b f - \int_a^b f_n \right| < \varepsilon \qquad \forall n \ge N.$$

For any $n \in \mathbb{N}$, we have

$$\left| \int_a^b f - \int_a^b f_n \right| = \left| \int_a^b (f - f_n) \right| \le \int_a^b |f - f_n|$$
$$\le U\big(|f - f_n|, \{a, b\}\big) = \Big(\sup_{x\in[a,b]} \big|f(x) - f_n(x)\big| \Big)(b - a).$$

But f_n converges uniformly to f, which means that

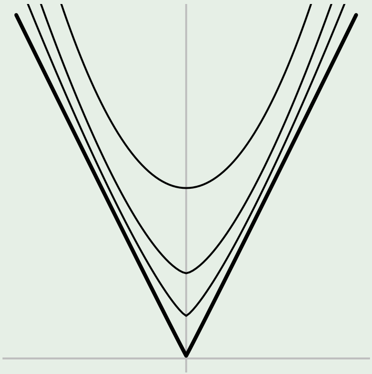
$$\exists N \in \mathbb{N} \quad \text{such that} \quad \sup_{x\in[a,b]} |f(x) - f_n(x)| < \frac{\varepsilon}{b - a} \qquad \forall n \ge N.$$

For such n, we have $\left|\int_a^b f - \int_a^b f_n\right| < \varepsilon$, as required. □

Uniform Convergence

The interaction between uniform convergence and differentiability is more subtle.

Example (f_n diff'ble $\forall n$ and $f_n \to f$ uniformly $\not\Rightarrow$ f diff'ble)



$$f_n(x) = \frac{1}{2n} + (x^2)^{\left(1+\frac{1}{n}\right)/2}$$

Each f_n is differentiable

$f_n(x) \to f(x) = |x|$
uniformly

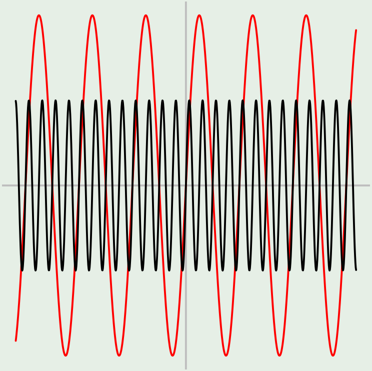
Limit function f is not differentiable.

Note: Graph shows $n = 1, 2, 4, 64$ for $x \in [-1, 1]$.

Uniform Convergence

Even if $f_n \to f$ uniformly, and all the f_n and f are differentiable, it is not necessarily true that $f_n' \to f'$.

Example ($f_n \to f$ uniformly and f_n', f' exist $\not\Longrightarrow$ $f_n' \to f'$)



$$f_n(x) = \frac{1}{n}\sin\left(n^2 x\right)$$

$$f_n(x) \to f(x) \equiv 0$$

uniformly

$$f_n'(x) = n\cos\left(n^2 x\right)$$

$\lim_{n\to\infty} f_n'(x)$ does not exist (*e.g.*, $f_n(0) = n$, which diverges as $n \to \infty$)

Note: Graph shows $n = 1, 2$ on interval $[-20, 20]$.

Uniform Convergence

The theorem on integrability and uniform convergence, together with the fundamental theorem of calculus, must yield some result on uniformly convergent sequences of differentiable functions.

But the result must make hypotheses that avoid the failures in the examples on the two previous slides.

Theorem (Differentiability and Uniform Convergence)

Suppose $\{f_n\}$ is a sequence of differentiable functions on $[a, b]$ such that

1. *f_n' is integrable for each n,*
2. *the sequence $\{f_n'\}$ converges uniformly on $[a, b]$ to a continuous function g,*
3. *the sequence $\{f_n\}$ converges pointwise to a function f.*

Then f is differentiable and $\{f_n'\}$ converges uniformly to f'.

Uniform convergence

Proof of theorem on Differentiability and Uniform Convergence.

Since the function g to which f_n' converges is continuous, it is certainly integrable. So we can apply the theorem on integrability and uniform convergence on the interval $[a, x]$ to infer that

$$\begin{aligned} \int_a^x g &= \lim_{n\to\infty} \int_a^x f_n' && f_n' \to g \text{ uniformly} \\ &= \lim_{n\to\infty} \Big(f_n(x) - f_n(a)\Big) && \text{SFTC} \\ &= f(x) - f(a) && f_n \to f \text{ pointwise} \end{aligned}$$

Since g is continuous, FFTC implies that

$$g(x) = \lim_{n\to\infty} f_n'(x) = f'(x)$$

for all $x \in [a, b]$. □

Series of Functions

Series of Real Numbers

Suppose $\{x_n\}$ is a sequence of real numbers. Recall that the ***sequence of partial sums*** is the sequence $\{s_n\}$ defined by

$$s_n = \sum_{k=1}^{n} x_n \,.$$

If the sequence of partial sums converges, then we write the limit as

$$\sum_{k=1}^{\infty} x_k \;=\; \lim_{n\to\infty} \sum_{k=1}^{n} x_n \;=\; \lim_{n\to\infty} s_n \,.$$

In this case, we call $\sum_{k=1}^{\infty} x_k$ a ***convergent series***. A ***divergent series*** is a sequence of partial sums that diverges; we sometimes abuse notation and write $\sum_{k=1}^{\infty} x_k$ for divergent series as well.

A ***series*** is either a convergent series or a divergent series.

Our goal now is to extend this notion of series to sequences of functions.

Series of Functions

Suppose $\{f_n\}$ is a sequence of functions defined on a set $D \subseteq \mathbb{R}$. The ***sequence of partial sums*** is the sequence $\{S_n\}$ where S_n is the function defined on D by

$$S_n(x) = \sum_{k=1}^{n} f_k(x) .$$

When talking about limits of the S_n, we will write $\sum_{k=1}^{\infty} f_k$ and refer to this as a ***series***.

Keep in mind that this is very informal, since the terminology does

not specify any sense in which the S_n converge, nor does it assume that the S_n converge at all!

We will now make this more formal.

Series of Functions

Suppose $\{f_n\}$ is a sequence of functions defined on a domain D, and $\{S_n\}$ is its sequence of partial sums.

Definition (Convergence of Series)

If the sequence of partial sums $\{S_n\}$ converges pointwise on D to a function f, then we say that the series $\sum_{k=1}^{\infty} f_k$ ***converges pointwise on*** D ***to*** f.

If the $\{S_n\}$ converge uniformly on D to a function f, then we say that the series $\sum_{k=1}^{\infty} f_k$ ***converges uniformly on*** D ***to*** f.

In both cases, we will write $f = \sum_{k=1}^{\infty} f_k$ to denote that the ***series converges to*** f.

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Series of Functions

The theorems on convergence of sequences of integrable, continuous and differentiable functions have several immediate implications for series of functions.

In the following, we assume that $\{f_n\}$ is a sequence of functions defined on an interval $[a, b]$.

Corollary (Integrals of Series)

Suppose the f_n are integrable and $\sum_{k=1}^{\infty} f_k$ converges uniformly to a function f. Then f is integrable and

$$\int_a^b f = \sum_{k=1}^{\infty} \int_a^b f_k \, .$$

Series of Functions

Corollary (Continuity of Series)

Suppose the f_n are continuous and $\sum_{k=1}^{\infty} f_k$ converges uniformly to a function f. Then f is continuous.

Corollary (Differentiability of Series)

Suppose $\{f_n\}$ is a sequence of differentiable functions on $[a, b]$ such that

- *f_n' is integrable for each n,*
- *the series $\sum_{k=1}^{\infty} f_k'$ converges uniformly on $[a, b]$ to a continuous function g,*
- *the series $\sum_{k=1}^{\infty} f_k$ converges pointwise to a function f.*

Then f is differentiable and $f' = \sum_{k=1}^{\infty} f_k'$.