Announcements

- Assignment 3 is posted (and complete).
  Due Tuesday 22 October 2019 at 2:25pm via crowdmark.
- **Math 3A03 Test #1**
  Tuesday 29 October 2019, 5:30–7:00pm, in JHE 264
  (room is booked for 90 minutes; you should not feel rushed)
- Math 3A03 **Final Exam**: Fri 6 Dec 2019, 9:00am–11:30am
  Location: `MDCL 1105`

Continuous Functions

Mathematics and Statistics

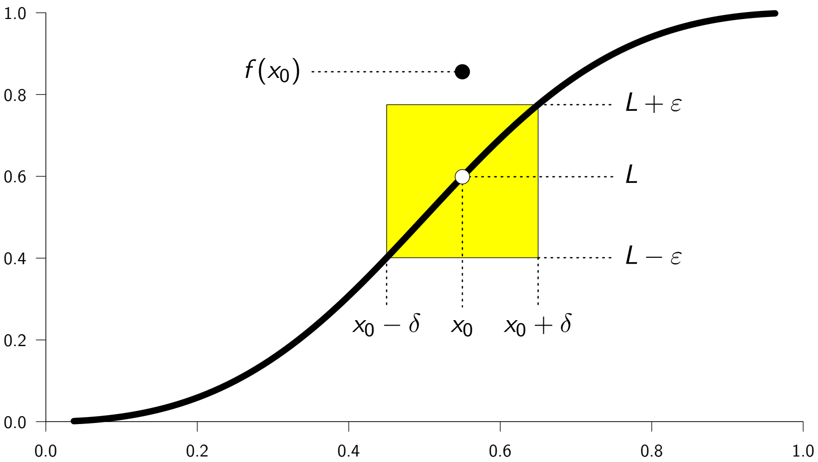
$$\int_M d\omega = \int_{\partial M} \omega$$

Mathematics 3A03
Real Analysis I

Instructor: David Earn

Lecture 18
Continuity
Friday 11 October 2019

Limits of functions



Limits of functions

Definition (Limit of a function on an interval (a, b))

Let $a < x_0 < b$ and $f : (a, b) \to \mathbb{R}$. Then f is said to ***approach the limit*** L ***as*** x ***approaches*** x_0, often written "$f(x) \to L$ as $x \to x_0$" or

$$\lim_{x \to x_0} f(x) = L\,,$$

iff for all $\varepsilon > 0$ there exists $\delta > 0$ such that if $0 < |x - x_0| < \delta$ then $|f(x) - L| < \varepsilon$.

Shorthand version:

$\forall \varepsilon > 0\ \exists \delta > 0 \ni 0 < |x - x_0| < \delta \implies |f(x) - L| < \varepsilon$.

limitdefinterval

Limits of functions

The function f need not be defined on an entire interval.
It is enough for f to be defined on a set with at least one accumulation point.

Definition (Limit of a function with domain $E \subseteq \mathbb{R}$)

Let $E \subseteq \mathbb{R}$ and $f : E \to \mathbb{R}$. Suppose x_0 is a point of accumulation of E. Then f is said to ***approach the limit*** L ***as*** x ***approaches*** x_0, *i.e.,*

$$\lim_{x \to x_0} f(x) = L\,,$$

iff for all $\varepsilon > 0$ there exists $\delta > 0$ such that if $x \in E$, $x \neq x_0$, and $|x - x_0| < \delta$ then $|f(x) - L| < \varepsilon$.

Shorthand version:

$\forall \varepsilon > 0\ \exists \delta > 0 \ni (x \in E \ \wedge\ 0 < |x - x_0| < \delta) \implies |f(x) - L| < \varepsilon$.

Limits of functions

Example

Prove directly from the definition of a limit that

$$\lim_{x \to 3} (2x + 1) = 7 \,.$$

Proof that $2x + 1 \to 7$ as $x \to 3$.

We must show that $\forall \varepsilon > 0 \; \exists \delta > 0$ such that $0 < |x - 3| < \delta \implies |(2x + 1) - 7| < \varepsilon$. Given ε, to determine how to choose δ, note that

$$|(2x+1) - 7| < \varepsilon \iff |2x - 6| < \varepsilon \iff 2\,|x - 3| < \varepsilon \iff |x - 3| < \frac{\varepsilon}{2}$$

Therefore, given $\varepsilon > 0$, let $\delta = \frac{\varepsilon}{2}$. Then $|x - 3| < \delta \implies |(2x + 1) - 7| = |2x - 6| = 2\,|x - 3| < 2\frac{\varepsilon}{2} = \varepsilon$, as required. □

Limits of functions

Example

Prove directly from the definition of a limit that

$$\lim_{x \to 2} x^2 = 4 \,.$$

(Solution on next slide)

Limits of functions

Proof that $x^2 \to 4$ as $x \to 2$.

We must show that $\forall \varepsilon > 0\ \exists \delta > 0$ such that $0 < |x-2| < \delta \implies |x^2-4| < \varepsilon$. Given ε, to determine how to choose δ, note that

$$\left|x^2-4\right| < \varepsilon \iff |(x-2)(x+2)| < \varepsilon \iff |x-2|\,|x+2| < \varepsilon.$$

We can make $|x-2|$ as small as we like by choosing δ sufficiently small. Moreover, if x is close to 2 then $x+2$ will be close to 4, so we should be able to ensure that $|x+2| < 5$. To see how, note that

$$\begin{aligned} |x+2| < 5 \iff -5 < x+2 < 5 \iff -9 < x-2 < 1 \\ \Longleftarrow -1 < x-2 < 1 \iff |x-2| < 1\,. \end{aligned}$$

Therefore, given $\varepsilon > 0$, let $\delta = \min(1, \frac{\varepsilon}{5})$. Then $|x^2-4| = |(x-2)(x+2)| = |x-2|\,|x+2| < \frac{\varepsilon}{5} 5 = \varepsilon$. □

Poll

- Go to https://www.childsmath.ca/childsa/forms/main_login.php
- Click on Math 3A03
- Click on Take Class Poll
- Fill in poll **Lecture 18: ε-δ definition of limit**
- Submit.

Limits of functions

Rather than the ε-δ definition, we can exploit our experience with sequences to define "$f(x) \to L$ as $x \to x_0$".

Definition (Limit of a function via sequences)

Let $E \subseteq \mathbb{R}$ and $f : E \to \mathbb{R}$. Suppose x_0 is a point of accumulation of E. Then

$$\lim_{x \to x_0} f(x) = L$$

iff for every sequence $\{e_n\}$ of points in $E \setminus \{x_0\}$,

$$\lim_{n \to \infty} e_n = x_0 \quad \Longrightarrow \quad \lim_{n \to \infty} f(e_n) = L\,.$$

McMaster University

Mathematics and Statistics

$$\int_M d\omega = \int_{\partial M} \omega$$

Mathematics 3A03
Real Analysis I

Instructor: David Earn

Lecture 19
Continuity II
Tuesday 22 October 2019

Announcements

- Assignment 3 was due today at 2:25pm via crowdmark. Solutions will be posted today.
- **Math 3A03 Test #1**
  Tuesday 29 October 2019, 5:30–7:00pm, in JHE 264
  (room is booked for 90 minutes; you should not feel rushed)
- An incomplete version of Assignment 4 is posted on the course web site. Due 5 November 2019 at 2:25pm via crowdmark. BUT you should do the posted questions before Test #1 (check again later in the week and over the weekend for additional posted questions).
- Math 3A03 **Final Exam**: Fri 6 Dec 2019, 9:00am–11:30am
  Location: `MDCL 1105`

Last time...

- ε-δ definition of limit of a function
- Sequence definition of limit of a function

Poll

- Go to https://www.childsmath.ca/childsa/forms/main_login.php
- Click on Math 3A03
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- Fill in poll **Lecture 19: ε-δ vs sequence definition of a limit**
- Submit.

Limits of functions

Lemma (Equivalence of limit definitions)

The ε-δ definition of limits and the sequence definition of limits are equivalent.

(proof on next two slides)

Note: The definition of a limit via sequences is sometimes easier to use than the ε-δ definition.

Proof of Equivalence of ε-δ definition and sequence definition of limit.

Proof (ε-$\delta \implies$ seq).

Suppose the ε-δ definition holds and $\{e_n\}$ is a sequence in $E \setminus \{x_0\}$ that converges to x_0. Given $\varepsilon > 0$, there exists $\delta > 0$ such that if $0 < |x - x_0| < \delta$ then $|f(x) - L| < \varepsilon$. But since $e_n \to x_0$, given $\delta > 0$, there exists $N \in \mathbb{N}$ such that, for all $n \geq N$, $|e_n - x_0| < \delta$. This means that if $n \geq N$ then $x = e_n$ satisfies $0 < |x - x_0| < \delta$, implying that we can put $x = e_n$ in the statement $|f(x) - L| < \varepsilon$. Hence, for all $n \geq N$, $|f(e_n) - L| < \varepsilon$. Thus,

$$e_n \to x_0 \implies f(e_n) \to L\,,$$

as required. □

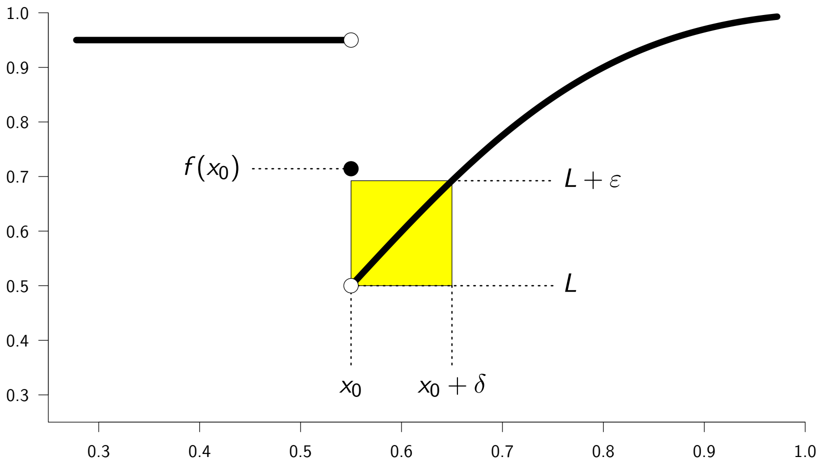
Proof of Equivalence of ε-δ definition and sequence definition of limit.

Proof (seq $\implies$ ε-δ) via contrapositive.

Suppose that as $x \to x_0$, $f(x) \not\to L$ according to the ε-δ definition. We must show that $f(x) \not\to L$ according to the sequence definition.

Since the ε-δ criterion does not hold, $\exists \varepsilon > 0$ such that $\forall \delta > 0$ there is some $x_\delta \in E$ for which $0 < |x_\delta - x_0| < \delta$ and yet $|f(x_\delta) - L| \geq \varepsilon$. This is true, in particular, for $\delta = 1/n$, where n is any natural number. Thus, $\exists \varepsilon > 0$ such that: $\forall n \in \mathbb{N}$, there exists $x_n \in E$ such that $0 < |x_n - x_0| < 1/n$ and yet $|f(x_n) - L| \geq \varepsilon$. This demonstrates that there is a sequence $\{x_n\}$ in $E \setminus \{x_0\}$ for which $x_n \to x_0$ and yet $f(x_n) \not\to L$. Hence, $f(x) \not\to L$ as $x \to x_0$ according to the sequence criterion, as required. □

One-sided limits



One-sided limits

Definition (Right-Hand Limit)

Let $f : E \to \mathbb{R}$ be a function with domain E and suppose that x_0 is a point of accumulation of $E \cap (x_0, \infty)$. Then we write

$$\lim_{x \to x_0^+} f(x) = L$$

if for every $\varepsilon > 0$ there is a $\delta > 0$ so that

$$|f(x) - L| < \varepsilon$$

whenever $x_0 < x < x_0 + \delta$ and $x \in E$.

One-sided limits

One-sided limits can also be expressed in terms of sequence convergence.

Definition (Right-Hand Limit – sequence version)

Let $f : E \to \mathbb{R}$ be a function with domain E and suppose that x_0 is a point of accumulation of $E \cap (x_0, \infty)$. Then we write

$$\lim_{x \to x_0^+} f(x) = L$$

if for every decreasing sequence $\{e_n\}$ of points of E with $e_n > x_0$ and $e_n \to x_0$ as $n \to \infty$,

$$\lim_{n \to \infty} f(e_n) = L\,.$$

Infinite limits

Definition (Right-Hand Infinite Limit)

Let $f : E \to \mathbb{R}$ be a function with domain E and suppose that x_0 is a point of accumulation of $E \cap (x_0, \infty)$. Then we write

$$\lim_{x \to x_0^+} f(x) = \infty$$

if for every $M > 0$ there is a $\delta > 0$ such that $f(x) \geq M$ whenever $x_0 < x < x_0 + \delta$ and $x \in E$.

Properties of limits

There are theorems for limits of functions of a real variable that correspond (and have similar proofs) to the various results we proved for limits of sequences:

- Uniqueness of limits
- Algebra of limits
- Order properties of limits
- Limits of absolute values
- Limits of Max/Min

See Chapter 5 of textbook for details.

Limits of compositions of functions

When is $\displaystyle \lim_{x\to x_0} g(f(x)) = g\Big(\lim_{x\to x_0} f(x)\Big)$?

Theorem (Limit of composition)

Suppose

$$\lim_{x\to x_0} f(x) = L\,.$$

If g is a function defined in a neighborhood of the point L and

$$\lim_{z\to L} g(z) = g(L)$$

then

$$\lim_{x\to x_0} g(f(x)) = g\Big(\lim_{x\to x_0} f(x)\Big) = g(L)\,.$$

(Textbook (TBB) §5.2.5)

Limits of compositions of functions – more generally

Note: It is a little more complicated to generalize the statement of this theorem so as to minimize the set on which g must be defined but the proof is no more difficult.

Theorem (Limit of composition)

Let $A, B \subseteq \mathbb{R}$, $f : A \to \mathbb{R}$, $f(A) \subseteq B$, and $g : B \to \mathbb{R}$. Suppose x_0 is an accumulation point of A and

$$\lim_{x \to x_0} f(x) = L\,.$$

Suppose further that g is defined at L. If L is an accumulation point of B and

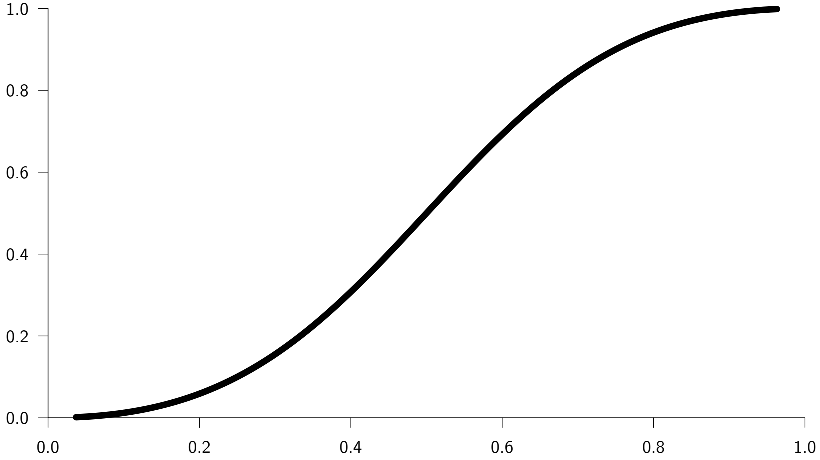
$$\lim_{z \to L} g(z) = g(L)\,,$$

or $\exists \delta > 0$ such that $f(x) = L$ for all $x \in (x_0 - \delta, x_0 + \delta) \cap A$, then

$$\lim_{x \to x_0} g\big(f(x)\big) = g\Big(\lim_{x \to x_0} f(x)\Big) = g(L)\,.$$

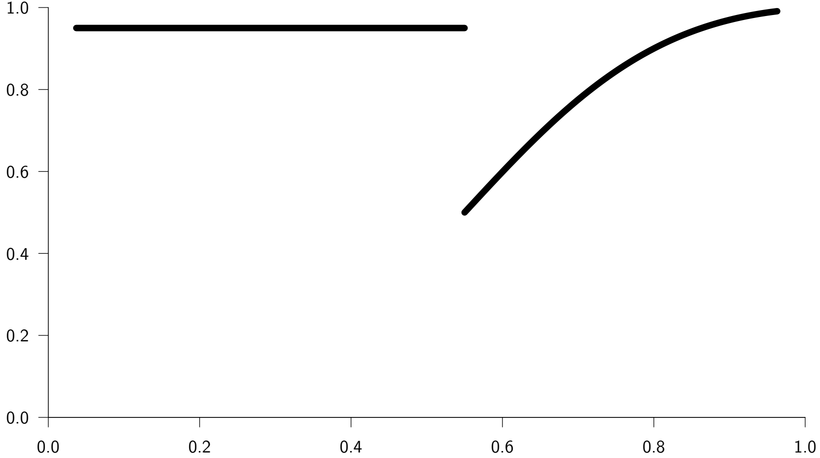
Continuity

Intuitively, a function f is ***continuous*** if you can draw its graph without lifting your pencil from the paper. . .



Continuity

and ***discontinuous*** otherwise. . .



Continuity

In order to develop a rigorous foundation for the theory of functions, we need to be more precise about what we mean by “continuous”.

The main challenge is to define “continuity” in a way that works consistently on sets other than intervals (and generalizes to spaces that are more abstract than $\mathbb{R}$).

We will define:

- continuity at a single point;
- continuity on an open interval;
- continuity on a closed interval;
- continuity on more general sets.

Pointwise continuity

Definition (Continuous at an interior point of the domain of f)

If the function f is defined in a neighbourhood of the point x_0 then we say f is ***continuous at*** x_0 iff

$$\lim_{x \to x_0} f(x) = f(x_0)\,.$$

This definition works more generally provided x_0 is a point of accumulation of the domain of f (notation: $\mathrm{dom}(f)$).

We will also consider a function to be continuous at any isolated point in its domain.