18 Examples and Q&A

McMaster University

Mathematics and Statistics

$$\int_M d\omega = \int_{\partial M} \omega$$

Mathematics 3A03
Real Analysis I

Instructor: David Earn

Lecture 18
Examples and Q&A
Tuesday 24 February 2026

Poll

- Go to https://www.childsmath.ca/childsa/forms/main_login.php
- Click on Math 3A03
- Click on Take Class Poll
- Fill in poll **Poll on polls**
- `Submit`.

Announcements

- The participation deadline for Assignment 3 was TODAY @ 2:25pm.
- Solutions to assignment 3 were posted a few days ago.
- The midterm TEST is on THURSDAY @ 7:00pm in T13 123.
- The room is booked for 7:00–10:00 pm, but the intention is that a reasonable amount of time for the test is 90 minutes. You will be given double time.
- Last year's (winter 2025) midterm test and solutions are posted on the course web site.
- No class on Thursday this week.
  Office hour instead (in HH-317).

Midterm Test

What you need to know:

- Everything discussed in class, including all definitions/concepts and theorems/lemmas/corollaries.
- Everything in assignments and solutions to assignments. *Make sure you fully understand all the solutions to all the problems in all the assignments.*
- Most—but not all—of the material that you are responsible for is covered in the chapters of the textbooks indicated on the course web page. You are not responsible for material in the textbooks that was not mentioned in lectures, tutorials or assignments.
- It is essential that you understand how to use the definitions and theorems to construct proofs.

Poll

- Go to https://www.childsmath.ca/childsa/forms/main_login.php
- Click on Math 3A03
- Click on Take Class Poll
- Fill in poll **Example: sup of sum**
- Submit.

Example (sup of sum)

Suppose $E \subseteq \mathbb{R}$, and f and g are defined and bounded on E. Prove that

$$\sup_{x\in E}\{f(x)+g(x)\} \le \sup_{x\in E}\{f(x)\} + \sup_{x\in E}\{g(x)\}.$$

Proof

$\sup_{x\in E}\{f(x)\}$ is an upper bound for $f(x)$ on E.

Consequently, $f(x) \le \sup_{x\in E}\{f(x)\}$ for all $x \in E$.

Similarly, $g(x) \le \sup_{x\in E}\{g(x)\}$ for all $x \in E$.

$$\therefore \quad f(x)+g(x) \le \sup_{x\in E}\{f(x)\} + \sup_{x\in E}\{g(x)\} \qquad \text{for all} \quad x \in E$$

$$\implies \quad \sup_{x\in E}\,[f(x)+g(x)] \le \sup_{x\in E}\{f(x)\} + \sup_{x\in E}\{g(x)\} \qquad \square$$

Example (Integral of sum)

Prove that if f and g are integrable on $[a, b]$ then $f + g$ is integrable on $[a, b]$ and

$$\int_a^b (f + g) = \int_a^b f + \int_a^b g.$$

Proof

To establish that $f + g$ is integrable, we need to show that for any $\varepsilon > 0$ there is a partition P of $[a, b]$ such that

$$U(f + g, P) - L(f + g, P) < \varepsilon.$$

We will use the fact that f and g are both integrable.

...continued...

Proof of sum of integrals theorem (continued)

Since f is integrable, for any $\varepsilon > 0$ there is a partition of $[a, b]$, say P_f, such that

$$U(f, P_f) - L(f, P_f) < \frac{\varepsilon}{2}.$$

Similarly, since g is integrable, for any $\varepsilon > 0$ there is a partition of $[a, b]$, say P_g, such that

$$U(g, P_g) - L(g, P_g) < \frac{\varepsilon}{2}.$$

Create a finer partition $P = P_f \cup P_g$. Then

$$L(f, P_f) \ \le \ L(f, P) \ \le \ \int_a^b f \ \le \ U(f, P) \ \le \ U(f, P_f),$$

and similarly

$$L(g, P_g) \ \le \ L(g, P) \ \le \ \int_a^b g \ \le \ U(g, P) \ \le \ U(g, P_g).$$

...continued...

Proof of sum of integrals theorem (continued)

Consequently,

$$U(f,P) - L(f,P) \;\le\; U(f,P_f) - L(f,P_f) \;<\; \frac{\varepsilon}{2},$$
$$U(g,P) - L(g,P) \;\le\; U(f,P_g) - L(f,P_g) \;<\; \frac{\varepsilon}{2}.$$

Now recall the definition $U(f,P) = \sum_{i=1}^{n} M_i(t_i - t_{i-1})$, where $M_i = \sup\{(f(x) : t_{i-1} \le x \le t_i\}$, and recall the sup of sum example, which implies

$$U(\,f+g,\;P) \;\le\; U(f,P) + U(g,P). \qquad (*)$$

There is also a corresponding "inf of sum" result, which implies

$$L(f,P) + L(g,P) \;\le\; L(\,f+g,\;P) \qquad (**)$$

From the definition of lower and upper sums, we also know that

$$L(\,f+g,\;P) \le U(\,f+g,\;P). \qquad (***)$$

Putting together $(*)$ $(**)$, and $(***)$, we have

...continued...

Proof of sum of integrals theorem (continued)

$$\begin{aligned} L(f,P)+L(g,P) &\le L(f+g,\,P) \\ &\le U(f+g,\,P) \le U(f,P)+U(g,P) \end{aligned} \qquad (\spadesuit)$$

from which it follows that

$$\begin{aligned} U(f+g,\,P) - L(f+g,\,P) & \\ &\le \Big(U(f,P)+U(g,P)\Big) - \Big(L(f,P)+L(g,P)\Big) \\ &= \Big(U(f,P)-L(f,P)\Big) + \Big(U(g,P)-L(g,P)\Big) \\ &< \frac{\varepsilon}{2}+\frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

$\therefore$ $f+g$ is integrable. Consequently, in ($\spadesuit$), $\int_a^b (f+g)$ is the unique number that lies between $L(f+g,P)$ and $U(f+g,P)$ for any partition P. Similarly, $\int_a^b f + \int_a^b g$ is the unique number that lies between the outermost quantities in ($\spadesuit$) for all P. Therefore, $\int_a^b (f+g) = \int_a^b f + \int_a^b g$, as required. $\square$

Example (Characteristics of open sets)

Prove the fundamental properties of open sets.

Proof that $\mathbb{R}$ is open.

$$x \in \mathbb{R} \quad \Longrightarrow \quad x \in (x-1, x+1) \subset \mathbb{R} \quad \Longrightarrow \quad x \in \mathbb{R}^\circ.$$

$\therefore$ Every $x \in \mathbb{R}$ is an interior point of $\mathbb{R}$, *i.e.*, $\mathbb{R}$ is open. □

Proof that $\varnothing$ is open.

Since there are no points in $\varnothing$, every point in $\varnothing$ is an interior point of $\varnothing$, so $\varnothing$ is open. □

Proof that any union of open sets is open.

Suppose $\mathcal{U}$ is a collection of open sets, and $x \in \bigcup_{U \in \mathcal{U}} U$. Then $x \in U$ for some $U \in \mathcal{U}$, *i.e.*, $x \in U$ for some open set $U \subseteq \bigcup_{U \in \mathcal{U}} U$. So $\bigcup_{U \in \mathcal{U}} U$ is open. □

Proof that any finite intersection of open sets is open.

Suppose U_1 and U_2 are open. If U_1 and U_2 are disjoint, then their intersection is $\varnothing$, which is open. If $U_1 \cap U_2 \neq \varnothing$ then let $x \in U_1 \cap U_2$. Since $x \in U_1$, $\exists \delta_1 > 0 \ni (x - \delta_1, x + \delta_1) \subseteq U_1$. Similarly, since $x \in U_2$, $\exists \delta_2 > 0 \ni (x - \delta_2, x + \delta_2) \subseteq U_2$.

Let $\delta = \min\{\delta_1, \delta_2\}$. Then $(x - \delta, x + \delta) \subseteq U_1 \cap U_2$. So x is an interior point of $U_1 \cap U_2$. Hence $U_1 \cap U_2$ is open.

The result for <u>any</u> finite intersection follows by induction. Since

$$\bigcap_{i=1}^{n} U_i = \left(\bigcap_{i=1}^{n-1} U_i \right) \bigcap U_n \,,$$

we can apply the induction hypothesis together with the result for the intersection of two open sets to infer the result for n open sets. □

Poll

- Go to
  https://www.childsmath.ca/childsa/forms/main_login.php
- Click on Math 3A03
- Click on Take Class Poll
- Fill in poll **Poll on polls**
- `Submit`.