McMaster University

Mathematics and Statistics

$$\int_M d\omega = \int_{\partial M} \omega$$

Mathematics 3A03
Real Analysis I

Instructor: David Earn

Lecture 14
Topology of $\mathbb{R}$
Monday 10 February 2025

Announcements

- Solutions to Assignment 2 have been reposted after correcting some errors (thanks to Kieran for spotting these).
  - There were typos in Q2(b) and Q4.
  - Q3 was incomplete because I assumed $f(x)$ was positive.
- Assignment 3 is posted on the course web site. Participation deadline is Monday 24 Feb 2025 @ 11:25 am.
- I reposted the slides for Lecture 13. Slide 79 now contains a sequence of hints for proving π is irrational.
- The midterm TEST is on Thursday 27 February 2025 @ 7:00pm in Hamilton Hall 302.
- The room is booked for 7:00–10:00 pm, but the intention is that a reasonable amount of time for the test is one hour. You will be given double time.

Topology of $\mathbb{R}$

Intervals

Open interval:

$$(a, b) = \{x : a < x < b\}$$

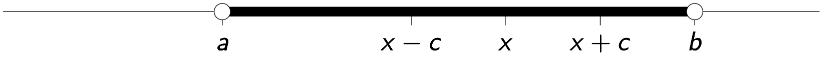
Closed interval:

$$[c, d] = \{x : c \leq x \leq d\}$$

Half-open interval:

$$(e, f] = \{x : e < x \leq f\}$$

Interior point



Definition (Interior point)

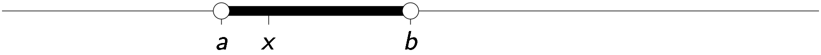
If $E \subseteq \mathbb{R}$ then x is an ***interior point*** of E if x lies in an open interval that is contained in E, *i.e.*,

$$\exists c > 0 \quad \ni \quad (x - c, x + c) \subset E.$$

Interior point examples

Set E	Interior points?
$(-1,1)$	
$[0,1]$	
$\mathbb{N}$	
$\mathbb{R}$	
$\mathbb{Q}$	
$(-1,1)\cup[0,1]$	
$(-1,1)\setminus\{\frac{1}{2}\}$	

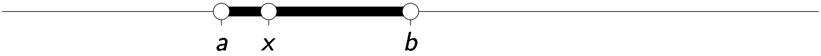
Neighbourhood



Definition (Neighbourhood)

A ***neighbourhood*** of a point $x \in \mathbb{R}$ is an open interval containing x.

Deleted neighbourhood

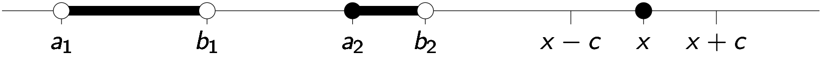


$$(a, b) \setminus \{x\}$$

Definition (Deleted neighbourhood)

A ***deleted neighbourhood*** of a point $x \in \mathbb{R}$ is a set formed by removing x from a neighbourhood of x.

Isolated point



$$E = (a_1, b_1) \cup [a_2, b_2) \cup \{x\}$$

Definition (Isolated point)

If $x \in E \subseteq \mathbb{R}$ then x is an ***isolated point*** of E if there is a neighbourhood of x for which the only point in E is x itself, *i.e.*,

$$\exists c > 0 \quad \ni \quad (x-c, x+c) \cap E \; = \; \{x\} \,.$$

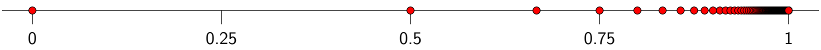
Poll

- Go to
  https://www.childsmath.ca/childsa/forms/main_login.php
- Click on Math 3A03
- Click on Take Class Poll
- Fill in poll **Topology: Isolated points**
- `Submit`.

Isolated point examples

Set E	**Isolated points?**
$(-1,1)$	
$[0,1]$	
$\mathbb{N}$	
$\mathbb{R}$	
$\mathbb{Q}$	
$(-1,1)\cup[0,1]$	
$(-1,1)\setminus\{\frac{1}{2}\}$	

Accumulation point



$$E = \left\{ 1 - \frac{1}{n} : n \in \mathbb{N} \right\}$$

Definition (Accumulation Point or Limit Point or Cluster Point)

If $E \subseteq \mathbb{R}$ then x is an ***accumulation point*** of E if every neighbourhood of x contains infinitely many points of E,

$$i.e., \qquad \forall c > 0 \qquad (x - c, x + c) \cap (E \setminus \{x\}) \neq \varnothing\,.$$

Note:

- It is possible but not necessary that $x \in E$.
- The shorthand condition is equivalent to saying that every deleted neighbourhood of x contains at least one point of E.

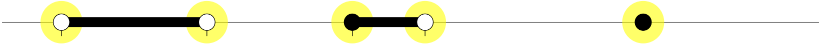
Poll

- Go to
  https://www.childsmath.ca/childsa/forms/main_login.php
- Click on Math 3A03
- Click on Take Class Poll
- Fill in poll **Topology: Accumulation points**
- Submit.

Accumulation point examples

Set E	**Accumulation points?**
$(-1,1)$	
$[0,1]$	
$\mathbb{N}$	
$\mathbb{R}$	
$\mathbb{Q}$	
$(-1,1)\cup[0,1]$	
$(-1,1)\setminus\{\frac{1}{2}\}$	
$\left\{1-\frac{1}{n} : n\in\mathbb{N}\right\}$	

Boundary point



Definition (Boundary Point)

If $E \subseteq \mathbb{R}$ then x is a ***boundary point*** of E if every neighbourhood of x contains at least one point of E and at least one point not in E, *i.e.*,

$$\forall c > 0 \qquad (x-c, x+c) \cap E \neq \varnothing \quad \wedge \quad (x-c, x+c) \cap (\mathbb{R} \setminus E) \neq \varnothing \,.$$

Note: It is possible but not necessary that $x \in E$.

Definition (Boundary)

If $E \subseteq \mathbb{R}$ then the ***boundary*** of E, denoted ∂E, is the set of all boundary points of E.

Boundary point examples

Set E	Boundary points?
$(-1, 1)$	
$[0, 1]$	
$\mathbb{N}$	
$\mathbb{R}$	
$\mathbb{Q}$	
$(-1, 1) \cup [0, 1]$	
$(-1, 1) \setminus \{\frac{1}{2}\}$	
$\left\{1 - \frac{1}{n} : n \in \mathbb{N}\right\}$	

Closed set

Definition (Closed set)

A set $E \subseteq \mathbb{R}$ is ***closed*** if it contains all of its accumulation points.

Definition (Closure of a set)

If $E \subseteq \mathbb{R}$ and E' is the set of accumulation points of E then the ***closure*** of E is

$$\overline{E} = E \cup E'.$$

Note: If the set E has no accumulation points, then E is closed because there are no accumulation points to check.

Open set

Definition (Open set)

A set $E \subseteq \mathbb{R}$ is ***open*** if every point of E is an interior point.

Definition (Interior of a set)

If $E \subseteq \mathbb{R}$ then the ***interior*** of E, denoted $\text{int}(E)$ or E°, is the set of all interior points of E.

Poll

- Go to
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- Click on Math 3A03
- Click on Take Class Poll
- Fill in poll **Topology: Open or Closed**
- Submit.

Examples

Set E	Closed?	Open?	$\overline{E}$	E°	∂E
$(-1,1)$					
$[0,1]$					
$\mathbb{N}$					
$\mathbb{R}$					
$\varnothing$					
$\mathbb{Q}$					
$(-1,1)\cup[0,1]$					
$(-1,1)\setminus\{\frac{1}{2}\}$					
$\left\{1-\frac{1}{n} : n\in\mathbb{N}\right\}$					

McMaster University

Mathematics and Statistics

$$\int_M d\omega = \int_{\partial M} \omega$$

Mathematics 3A03
Real Analysis I

Instructor: David Earn

Lecture 15
Topology of $\mathbb{R}$ II
Wednesday 12 February 2025

Announcements (same as on Monday, 10 Feb 2025)

- Solutions to Assignment 2 have been reposted after correcting some errors (thanks to Kieran for spotting these).
  - There were typos in Q2(b) and Q4.
  - Q3 was incomplete because I assumed $f(x)$ was positive.
- Assignment 3 is posted on the course web site. Participation deadline is Monday 24 Feb 2025 @ 11:25 am.
- I reposted the slides for Lecture 13. Slide 79 now contains a sequence of hints for proving π is irrational.
- The midterm TEST is on Thursday 27 February 2025 @ 7:00pm in Hamilton Hall 302.
- The room is booked for 7:00–10:00 pm, but the intention is that a reasonable amount of time for the test is one hour. You will be given double time.

Topological concepts covered so far

- Interval
- Neighbourhood
- Deleted neighbourhood
- Interior point
- Isolated point
- Accumulation point
- Boundary point
- Boundary
- Closed set
- Closure
- Open set
- Interior

Equivalent definitions

Example (Closure)

For $E \subseteq \mathbb{R}$, prove $E \cup E' = E \cup \partial E$, so $\overline{E}$ can be defined either way.

Proof: We must show $E \cup E' \subseteq E \cup \partial E$ and $E \cup E' \supseteq E \cup \partial E$.

- $\subseteq$ Suppose $x \in E \cup E'$. If $x \in E$ then $x \in E \cup A$ for any set A. In particular, $x \in E \cup \partial E$. Alternatively, suppose $x \notin E$, *i.e.*, $x \in E^c$. Then, since $x \in E \cup E'$, it must be that $x \in E'$, which means that any neighbourhood of x contains a point of E. But $x \in E^c$, so any such neighbourhood also contains a point of E^c (namely x). Therefore, $x \in \partial E \subseteq E \cup \partial E$.
- $\supseteq$ Suppose $x \in E \cup \partial E$. If $x \in E$ then $x \in E \cup A$ for any set A. In particular, $x \in E \cup E'$. Alternatively, suppose $x \notin E$, *i.e.*, $x \in E^c$. Then, since $x \in E \cup \partial E$, it must be that $x \in \partial E$, which means that any neighbourhood of x contains a point of E. But that point is not x, since $x \notin E$. Thus, any *deleted* neighbourhood of x contains a point of E, *i.e.*, $x \in E' \subseteq E \cup E'$. □

Question: In the proof above, did we use any properties of $\mathbb{R}$?

Poll

- Go to
  https://www.childsmath.ca/childsa/forms/main_login.php
- Click on Math 3A03
- Click on Take Class Poll
- Fill in poll **Topology: The most general type of open set**
- Submit.

Component intervals of open sets

What does the most general open set look like?

Theorem (Component intervals)

If G is an open subset of $\mathbb{R}$ and $G \neq \varnothing$ then there is a unique (possibly finite) sequence of disjoint open intervals $\{(a_n, b_n)\}$ such that

$$G = (a_1, b_1) \cup (a_2, b_2) \cup \cdots \cup (a_n, b_n) \cup \cdots ,$$

$$\text{i.e.,} \qquad G = \bigcup_{n=1}^{\infty} (a_n, b_n) .$$

*The open intervals (a_n, b_n) are said to be the **component intervals** of G.*

(TBB Theorem 4.15, p. 231)

Component intervals of open sets

Main ideas of proof of component intervals theorem:

- $x \in G \implies x$ is an interior point of $G \implies$
  - some neighbourhood of x is contained in G,
    i.e., $\exists c > 0$ such that $(x - c, x + c) \subseteq G$
  - $\exists$ a <u>largest</u> neighbourhood of x that is contained in G: this "***component of G***" is $I_x = (\alpha, \beta)$, where
    $$\alpha = \inf\{a : (a, x] \subset G\}, \qquad \beta = \sup\{b : [x, b) \subset G\}$$
  - Every component I_x contains a rational number, *i.e.*, $\exists r \in I_x \cap \mathbb{Q}$
  - But for any $y \in I_x$, we have $I_y = I_x$
  - $\therefore$ If $r_1, r_2 \in \mathbb{Q}$ and $r_1, r_2 \in I_x$ then $I_{r_1} = I_{r_2} = I_x$.
  - Components with different endpoints cannot overlap (they would contradict the inf and sup) so distinct components are disjoint
- $\therefore$ We can index (label) each component with a (unique) <u>rational</u> number
- $\therefore$ There are at most countably many intervals that make up G (*i.e.*, G is the union of a <u>sequence</u> of disjoint intervals)
- See textbook for details (TBB Theorem 4.15, p. 231).

Open vs. Closed Sets

Definition (Complement of a set of real numbers)

If $E \subseteq \mathbb{R}$ then the ***complement*** of E is the set

$$E^{\mathsf{c}} = \{x \in \mathbb{R} : x \notin E\} .$$

Theorem (Open vs. Closed)

If $E \subseteq \mathbb{R}$ then E is open iff E^{c} is closed.

(TBB Theorem 4.16)

Open vs. Closed Sets

Theorem (Properties of open sets of real numbers)

1. *The sets $\mathbb{R}$ and $\varnothing$ are open.*
2. *Any intersection of a finite number of open sets is open.*
3. *Any union of an arbitrary collection of open sets is open.*
4. *The complement of an open set is closed.*

(TBB Theorem 4.17)

Theorem (Properties of closed sets of real numbers)

1. *The sets $\mathbb{R}$ and $\varnothing$ are closed.*
2. *Any union of a finite number of closed sets is closed.*
3. *Any intersection of an arbitrary collection of closed sets is closed.*
4. *The complement of a closed set is open.*

(TBB Theorem 4.18)

Local vs. Global properties

Definition (Bounded function)

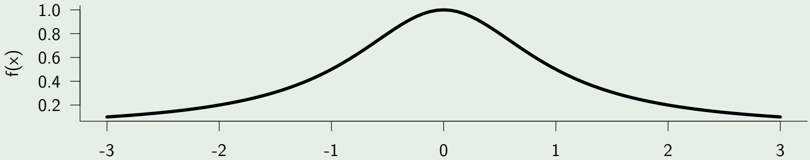
A real-valued function f is ***bounded*** on the set E if there exists $M > 0$ such that $|f(x)| \leq M$ for all $x \in E$.

(*i.e.*, the function f is bounded on E iff $\{f(x) : x \in E\}$ is a bounded set.)

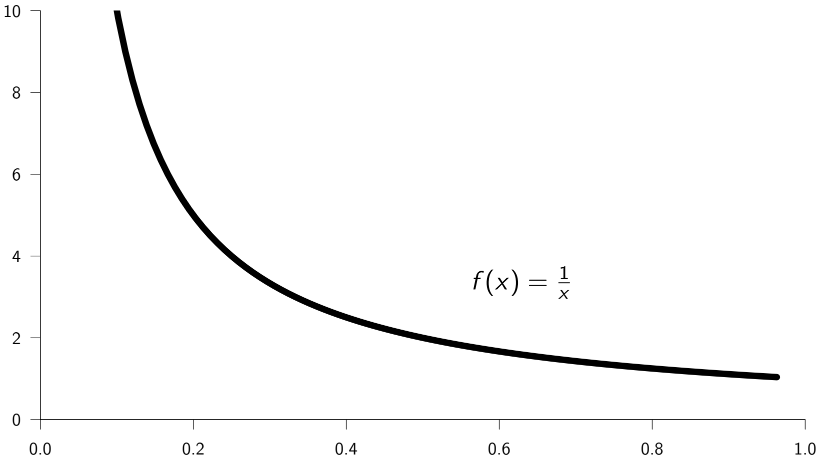
Note: This is a *global* property because there is a single bound M associated with the entire set E.

Example

The function $f(x) = 1/(1 + x^2)$ is bounded on $\mathbb{R}$. *e.g.*, $M = 1$.

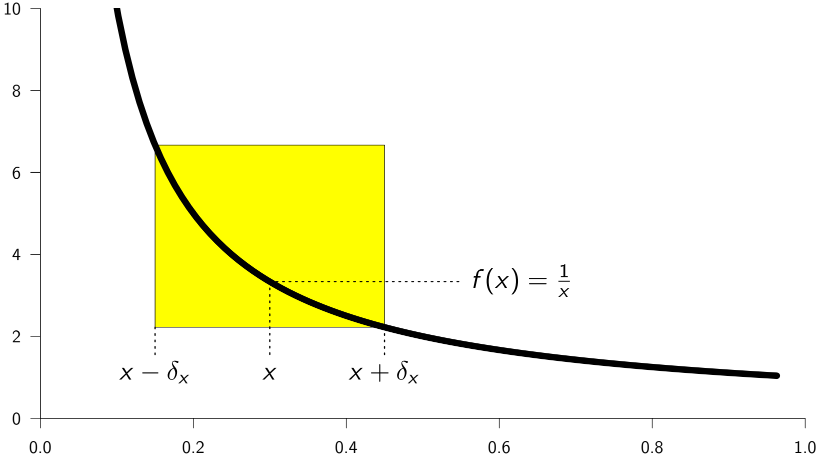


Local vs. Global properties



$f(x) = 1/x$ is not bounded on the interval $E = (0, 1)$.

Local vs. Global properties



$f(x) = 1/x$ is ***locally bounded*** on the interval $E = (0,1)$,
i.e., $\forall x \in E$, $\exists \delta_x, M_x > 0 \ni |f(t)| \le M_x \ \forall t \in (x - \delta_x, x + \delta_x)$.

Local vs. Global properties

Definition (Locally bounded at a point)

A real-valued function f is ***locally bounded*** at the point x if there is a neighbourhood of x in which f is bounded, *i.e.*, there exists $\delta_x > 0$ and $M_x > 0$ such that $|f(t)| \leq M_x$ for all $t \in (x - \delta_x, x + \delta_x)$.

Definition (Locally bounded on a set)

A real-valued function f is ***locally bounded*** on the set E if f is locally bounded at each point $x \in E$.

Note: The size of the neighbourhood (δ_x) and the local bound (M_x) depend on the point x.

Local vs. Global properties

Example (Function that is not even locally bounded)

Give an example of a function that is defined on the interval $(0,1)$ but is not locally bounded on $(0,1)$.

Let's construct a function $f(x)$ that is defined on $(0,1)$ but is not locally bounded at one point, say $x = \frac{1}{2}$.

$f(x)$ must blow up $x = \frac{1}{2}$. Let's make f look like $1/x$, but shifted so the blowup is at $x = \frac{1}{2}$.

$$f(x) = \begin{cases} \dfrac{1}{x - \frac{1}{2}} & x \neq \frac{1}{2}, \\ 0 & x = \frac{1}{2}. \end{cases}$$

Local vs. Global properties

Example (Function that is a mess near 0)

Give an example of a function $f(x)$ that is defined everywhere, yet in any neighbourhood of the origin there are infinitely many points at which f is not locally bounded.

Please do poll: Topology: Local boundedness

Consider $S(x) = \begin{cases} \sin\frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$

$S(x)$ is bounded on $\mathbb{R}$, hence locally bounded at every point.

Consider $T(x) = \begin{cases} \tan\frac{1}{x} & x \neq 0 \text{ and } \cos\frac{1}{x} \neq 0 \\ 0 & x = 0 \text{ or } \cos\frac{1}{x} = 0 \end{cases}$

$T(x)$ is not locally bounded at points where $\cos\frac{1}{x} = 0$, *i.e.*, for $\frac{1}{x} = \frac{\pi}{2} + n\pi$, $n \in \mathbb{Z}$. There are infinitely many such points in any neighbourhood of $x = 0$.

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Mathematics and Statistics

$$\int_M d\omega = \int_{\partial M} \omega$$

Mathematics 3A03
Real Analysis I

Instructor: David Earn

Lecture 16

Topology of $\mathbb{R}$ III

Friday 14 February 2025

Poll

- Go to https://www.childsmath.ca/childsa/forms/main_login.php
- Click on Math 3A03
- Click on Take Class Poll
- Fill in poll **Poll on polls**
- `Submit`.

Announcements

- Poll for Assignment 3 will be live after class today. Participation deadline is Monday 24 Feb 2025 @ 11:25 am.
- I improved the sketch of the component intervals theorem proof on slide 28.
- The midterm TEST is on Thursday 27 February 2025 @ 7:00pm in Hamilton Hall 302.
- The room is booked for 7:00–10:00 pm, but the intention is that a reasonable amount of time for the test is one hour. You will be given double time.

Poll

- Go to
  https://www.childsmath.ca/childsa/forms/main_login.php
- Click on Math 3A03
- Click on Take Class Poll
- Fill in poll **Topology: Locally bounded nowhere?**
- Submit.

Local vs. Global properties

Extra Challenge Problem:

Is there a function $f : \mathbb{R} \to \mathbb{R}$ that is not locally bounded anywhere?

Local vs. Global properties

- What condition(s) rule out such pathological behaviour?
- When does a property holding locally (near any given point in a set) imply that it holds globally (for the set as a whole)?
- For example: What condition(s) must a set $E \subseteq \mathbb{R}$ satisfy in order that a function f that is locally bounded on E is necessarily bounded on E?
- We will see that the condition we are seeking is that the set E must be "***compact***" ...

Compactness

Recall the Bolzano-Weierstrass theorem, for sequences of real numbers:

Theorem (Bolzano-Weierstrass theorem for sequences)

Every bounded sequence in $\mathbb{R}$ contains a convergent subsequence.

For *any set of real numbers*, we define:

Definition (Bolzano-Weierstrass property)

A set $E \subseteq \mathbb{R}$ is said to have the ***Bolzano-Weierstrass property*** iff any sequence of points chosen from E has a subsequence that converges to a point in E.

Compactness

Theorem (Bolzano-Weierstrass theorem for sets)

A set $E \subseteq \mathbb{R}$ has the Bolzano-Weierstrass property iff E is closed and bounded.

(TBB Theorem 4.21, p. 241)

Proof of $\Longleftarrow$.

Suppose E is closed and bounded. Let $\{x_n\}$ be a sequence in E. Since E is bounded, the usual Bolzano-Weierstrass theorem implies that there is a subsequence $\{x_{n_k}\}$ that converges. If $\{x_{n_k}\}$ is eventually constant, then its limit is a point in E, so we're done. Otherwise, $\{x_{n_k}\}$ must converge to an accumulation point of E. But E is closed, so it contains all its accumulation points, including the limit of $\{x_{n_k}\}$. Thus, again, we have a subsequence of $\{x_n\}$ that converges to a point in E. $\square$

Compactness

Theorem (Bolzano-Weierstrass theorem for sets)

A set $E \subseteq \mathbb{R}$ has the Bolzano-Weierstrass property iff E is closed and bounded.

Proof of $\implies$ (Part 1)

Let's prove the contrapositive, *i.e.,* If E is either not bounded or not closed then E does not have the Bolzano-Weierstrass property. ***Suppose E is unbounded.*** In particular, suppose E is not bounded above (the argument is similar if E is not bounded below). We will construct a sequence in E that has no convergent subsequence. Pick a point $x_1 \in E$ such that $x_1 > 1$ (which is possible because E is not bounded above). Also, since E is not bounded above, we can find $x_2 \in E$ such that $x_2 > x_1 + 1 > 2$. More generally, given $x_k \in E$ we can find $x_{k+1} \in E$ such that $x_{k+1} > x_k + 1 > k + 1$. The sequence $\{x_n\}$ constructed in this way is increasing and diverges to ∞ (since $x_n > n$ for all $n \in \mathbb{N}$). Moreover, this is true of any subsequence of $\{x_n\}$. $\therefore$ ***E does not have the Bolzano-Weierstrass property.***

Compactness

Theorem (Bolzano-Weierstrass theorem for sets)

A set $E \subseteq \mathbb{R}$ has the Bolzano-Weierstrass property iff E is closed and bounded.

Proof of $\implies$ (Part 2)

Now ***suppose E is not closed.*** Then there must be a sequence $\{x_n\}$ in E such that $\{x_n\}$ converges to a point not in E. If E has the Bolzano-Weierstrass property, then $\{x_n\}$ has a subsequence that converges to a point in E. But every subsequence of a convergent sequence must converge to the same point as the full sequence, and the full sequence converges to a point not in E! Thus, if E is not closed ***then it does not have the Bolzano-Weierstrass property.*** □

Compactness

Theorem (Bolzano-Weierstrass theorem for sets)

A set $E \subseteq \mathbb{R}$ has the Bolzano-Weierstrass property iff E is closed and bounded.

Notes:

- Why do we need both *closed* and *bounded*? Why don't we need *closed* in the Bolzano-Weierstrass theorem for sequences?
  - Because the statement of the Bolzano-Weierstrass theorem for sequences doesn't require the limit of the convergent subsequence to be in the set!
- The Bolzano-Weierstrass theorem for sets implies that "If $E \subseteq \mathbb{R}$ is bounded then its closure $\overline{E}$ has the Bolzano-Weierstrass property".
  - The Bolzano-Weierstrass theorem for sequences is a special case of this statement because any convergent sequence together with its limit is a closed set.

Compactness

Definition (Open Cover)

Let $E \subseteq \mathbb{R}$ and let $\mathcal{U}$ be a family of open intervals. If for every $x \in E$ there exists at least one interval $U \in \mathcal{U}$ such that $x \in U$, *i.e.*,

$$E \subseteq \bigcup \{U : U \in \mathcal{U}\},$$

then $\mathcal{U}$ is called an ***open cover*** of E.

Example (Open covers of $\mathbb{N}$)

Give examples of open covers of $\mathbb{N}$.

- $\mathcal{U} = \left\{ \left(n - \frac{1}{2}, n + \frac{1}{2}\right) : n = 1, 2, \ldots \right\}$
- $\mathcal{U} = \{(0, \infty)\}$
- $\mathcal{U} = \{(0, \infty), \mathbb{R}, (\pi, 27)\}$

Compactness

Example (Open covers of $\{\frac{1}{n} : n \in \mathbb{N}\}$)

- $\mathcal{U} = \{(0,1),\ (0,2),\ \mathbb{R},\ (\pi, 27)\}$
- $\mathcal{U} = \{(0,2)\}$
- $\mathcal{U} = \left\{\left(\frac{1}{n},\ \frac{1}{n} + \frac{3}{4}\right) : n = 1, 2, \dots\right\}$

Example (Open covers of $[0,1]$)

- $\mathcal{U} = \{(-2,2)\}$
- $\mathcal{U} = \{(-\frac{1}{2}, \frac{1}{2}),\ (0,2)\}$
- $\mathcal{U} = \left\{\left(\frac{1}{n}, 2\right) : n = 1, 2, \dots\right\} \cup \left\{\left(-\frac{1}{2}, \frac{1}{2}\right)\right\}$

Poll

- Go to
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- Click on Math 3A03
- Click on Take Class Poll
- Fill in poll **Topology: Open covers of inverse squares**
- `Submit`.

Compactness

Definition (Heine-Borel Property)

A set $E \subseteq \mathbb{R}$ is said to have the ***Heine-Borel property*** if every open cover of E can be reduced to a finite subcover. That is, if $\mathcal{U}$ is an open cover of E, then there exists a finite subfamily $\{U_1, U_2, \ldots, U_n\} \subseteq \mathcal{U}$, such that $E \subseteq U_1 \cup U_2 \cup \cdots \cup U_n$.

When does any open cover of a set E have a finite subcover?

Theorem (Heine-Borel Theorem)

A set $E \subseteq \mathbb{R}$ has the Heine-Borel property iff E is both closed and bounded.

(TBB pp. 249–250)

Compactness

Definition (Compact Set)

A set $E \subseteq \mathbb{R}$ is said to be ***compact*** if it has any of the following equivalent properties:

1. E is closed and bounded.
2. E has the Bolzano-Weierstrass property.
3. E has the Heine-Borel property.

Note: In spaces other than $\mathbb{R}$, these three properties are not necessarily equivalent. Usually the Heine-Borel property is taken as the definition of compactness.

Compactness

Example

Prove that the interval $(0, 1]$ is not compact by showing that it is not closed or not bounded.

Example

Prove that the interval $(0, 1]$ is not compact by showing that it does not have the Bolzano-Weierstrass property.

Example

Prove that the interval $(0, 1]$ is not compact by showing that it does not have the Heine-Borel property.

Poll

- Go to
  https://www.childsmath.ca/childsa/forms/main_login.php
- Click on Math 3A03
- Click on Take Class Poll
- Fill in poll **Topology: Compactness**
- `Submit`.

McMaster University

Mathematics and Statistics

$$\int_M d\omega = \int_{\partial M} \omega$$

Mathematics 3A03
Real Analysis I

Instructor: David Earn

Lecture 17
Topology of $\mathbb{R}$ IV
Monday 24 February 2025

Announcements

- The participation deadline for Assignment 3 was 11:25am today. Solutions were posted on Wednesday last week.
- Kieran's solutions to problems are now on the tutorials page of the course web site.
- All the polls are posted on the polls page of the course web site.
- The midterm TEST is on Thursday 27 February 2025 @ 7:00pm in Hamilton Hall 302.
- The room is booked for 7:00–10:00 pm, but the intention is that a reasonable amount of time for the test is one hour. The test is officially 90 minutes long, but you will have the full three hours if you want it.
- We will discuss the structure of the test at the end of today's class.

Topological concepts that we have covered

- Interval
- Neighbourhood
- Deleted neighbourhood
- Interior point
- Isolated point
- Accumulation point
- Boundary point
- Boundary
- Closed set
- Closure
- Open set
- Interior
- Complement
- Compact

Compactness

Classic non-trivial compactness argument:

Theorem (Compact $\implies$ bounded if locally bounded)

Let E be a compact subset of $\mathbb{R}$. If $f : E \to \mathbb{R}$ is locally bounded on E then f is bounded on E.

Proof via Bolzano-Weierstrass.

Suppose f is locally bounded on E, and that E satisfies the Bolzano-Weierstrass property. Suppose further, in order to derive a contradiction, that f is not bounded on E. Then there is some sequence $\{x_n\}$ in E such that $|f(x_n)| > n$ (otherwise $|f(x_n)| \le N$ for some $N \in \mathbb{N}$, so N would be a bound). By the Bolzano-Weierstrass property, $\{x_n\}$ has a subsequence $\{x_{n_k}\}$ such that $x_{n_k} \to L \in E$. Since f is locally bounded, there exist $\delta > 0$ and $M > 0$ such that $|f(x)| \le M$ for all $x \in (L - \delta, L + \delta)$. But $x_{n_k} \to L$, so for all sufficiently large k, $x_{n_k} \in (L - \delta, L + \delta)$. Yet for sufficiently large k, $|f(x_{n_k})| > n_k \ge k > M$. $\Rightarrow\Leftarrow$. Hence f must, in fact, be bounded on E. $\square$

Compactness

Theorem (Compact $\implies$ bounded if locally bounded)

Let E be a compact subset of $\mathbb{R}$. If $f : E \to \mathbb{R}$ is locally bounded on E then f is bounded on E.

Proof via Heine-Borel.

Since f is locally bounded, each $x \in E$ lies in some open interval U_x such that $|f(t)| \le M_x$ for all $t \in U_x$. The collection $\mathcal{U} = \{U_x : x \in E\}$ is an open cover of E. But E satisfies the Heine-Borel property, so $\mathcal{U}$ contains a finite subcover, say $\{U_{x_1}, \dots, U_{x_n}\}$. We can therefore find a bound for f on all of E. Let $M = \max\{M_{x_1}, \dots, M_{x_n}\}$. Then $|f(x)| \le M$ for all $x \in E$. □

Proof via Heine-Borel is much easier!

Compactness

Theorem (Compact $\implies$ bounded if locally bounded)

Let E be a compact subset of $\mathbb{R}$. If $f : E \to \mathbb{R}$ is locally bounded on E then f is bounded on E.

Example (Converse of above theorem)

Let $E \subseteq \mathbb{R}$. If every function $f : E \to \mathbb{R}$ that is locally bounded on E is bounded on E, then E is compact.

Note: The contrapositive of the converse is: If $E \subseteq \mathbb{R}$ is not compact then $\exists f : E \to \mathbb{R} \ni f$ is locally bounded on E but not bounded on E.

Compactness

Example ("bounded if locally bounded" $\Longrightarrow$ compact)

Consider the contrapositive: if $E \subseteq \mathbb{R}$ is not compact then there exists a function $f : E \to \mathbb{R}$ that is locally bounded but not bounded on E.

Suppose E is not compact. Then either E is not bounded or not closed.

Suppose first that E is not bounded, and let $f(x) = x$. Then f is locally bounded at any point $x \in E$, since f is bounded on any neighbourhood of x that has finite width; but $f(E) = E$ is an unbounded set, so f is an unbounded function on E.

Now suppose E is not closed. Then there is an accumulation point of E that is not in E, *i.e.*, there exists a sequence $\{x_n\} \subseteq E$ such that $x_n \to L \notin E$ as $n \to \infty$. Since $L \notin E$, for any $x \in E$ we have $x - L \neq 0$, so we can define $f(x) = \frac{1}{x-L}$ for all $x \in E$. But since L is a limit point of E, there are points in E that are arbitrarily close to L, hence points at which $|f(x)|$ is arbitrarily large. Thus, f is unbounded on E. □

Complements and Closures problem

Example

How many distinct sets can be obtained from $E = [0,1]$ by applying the complement and closure operations?

Consider this sequence of sets:

$$\begin{aligned} E_1 &= [0,1], \\ E_2 &= E_1^c = (-\infty,0) \cup (1,\infty), \\ E_3 &= \overline{E_2} = (-\infty,0] \cup [1,\infty), \\ E_4 &= E_3^c = (0,1), \\ E_5 &= \overline{E_4} = E_1. \end{aligned}$$

Is the answer 4 for any set $E \subseteq \mathbb{R}$?

Poll

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- Click on Take Class Poll
- Fill in poll **Topology: Complements and Closures**
- Submit.

Complements and Closures problem

Extra Challenge Problem

If $E \subseteq \mathbb{R}$, how many distinct sets can be obtained by taking complements or closures of E and its successors? Put another way, if $\{E_n\}$ is a sequence of sets produced by taking the complement or closure of the previous set, how many distinct sets can such a sequence contain? If the answer is finite, find a set E that generates the maximum number in this way.

Midterm Test

What you need to know:

- Everything discussed in class, including all definitions/concepts and theorems/lemmas/corollaries.
- Everything in assignments and solutions to assignments. *Make sure you fully understand all the solutions to all the problems in all the assignments.*
- Most—but not all—of the material that you are responsible for is covered in the chapters of the textbooks indicated on the course web page. You are not responsible for material in the textbooks that was not mentioned in lectures, tutorials or assignments.
- It is essential that you understand how to use the definitions and theorems to construct proofs.

Midterm Test

Let's look at the test.

McMaster University

Mathematics and Statistics

$$\int_M d\omega = \int_{\partial M} \omega$$

Mathematics 3A03
Real Analysis I

Instructor: David Earn

Lecture 18
Examples; Q&A
Wednesday 26 February 2025

Announcements

- The participation deadline for Assignment 3 was 11:25am today. Solutions were posted on Wednesday last week.
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Poll

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- Fill in poll **Example: sup of sum**
- `Submit`.

Example (sup of sum)

Suppose $E \subseteq \mathbb{R}$, and f and g are defined and bounded on E. Prove that

$$\sup_{x\in E}\{f(x)+g(x)\} \le \sup_{x\in E}\{f(x)\} + \sup_{x\in E}\{g(x)\}.$$

Proof

$\sup_{x\in E}\{f(x)\}$ is an upper bound for $f(x)$ on E.

Consequently, $f(x) \le \sup_{x\in E}\{f(x)\}$ for all $x \in E$.

Similarly, $g(x) \le \sup_{x\in E}\{g(x)\}$ for all $x \in E$.

$$\therefore \quad f(x)+g(x) \le \sup_{x\in E}\{f(x)\} + \sup_{x\in E}\{g(x)\} \qquad \text{for all} \quad x \in E$$

$$\Longrightarrow \quad \sup_{x\in E}\left[f(x)+g(x)\right] \le \sup_{x\in E}\{f(x)\} + \sup_{x\in E}\{g(x)\} \qquad \square$$

Example (Integral of sum)

Prove that if f and g are integrable on $[a, b]$ then $f + g$ is integrable on $[a, b]$ and

$$\int_a^b (f + g) = \int_a^b f + \int_a^b g.$$

Proof

To establish that $f + g$ is integrable, we need to show that for any $\varepsilon > 0$ there is a partition P of $[a, b]$ such that

$$U(f + g, P) - L(f + g, P) < \varepsilon.$$

We will use the fact that f and g are both integrable.

. . . continued. . .

Proof of sum of integrals theorem (continued)

Since f is integrable, for any $\varepsilon > 0$ there is a partition of $[a, b]$, say P_f, such that

$$U(f, P_f) - L(f, P_f) < \frac{\varepsilon}{2}.$$

Similarly, since g is integrable, for any $\varepsilon > 0$ there is a partition of $[a, b]$, say P_g, such that

$$U(g, P_g) - L(g, P_g) < \frac{\varepsilon}{2}.$$

Create a finer partition $P = P_f \cup P_g$. Then

$$L(f, P_f) \le L(f, P) \le \int_a^b f \le U(f, P) \le U(f, P_f),$$

and similarly

$$L(g, P_g) \le L(g, P) \le \int_a^b g \le U(g, P) \le U(g, P_g).$$

...continued...

Proof of sum of integrals theorem (continued)

Consequently,

$$U(f,P) - L(f,P) \le U(f,P_f) - L(f,P_f) < \frac{\varepsilon}{2},$$
$$U(g,P) - L(g,P) \le U(f,P_g) - L(f,P_g) < \frac{\varepsilon}{2}.$$

Now recall the definition $U(f,P) = \sum_{i=1}^{n} M_i(t_i - t_{i-1})$, where $M_i = \sup\{(f(x) : t_{i-1} \le x \le t_i\}$, and recall the sup of sum example, which implies

$$U(f+g, P) \le U(f,P) + U(g,P). \qquad (*)$$

There is also a corresponding "inf of sum" result, which implies

$$L(f,P) + L(g,P) \le L(f+g, P) \qquad (**)$$

From the definition of lower and upper sums, we also know that

$$L(f+g, P) \le U(f+g, P). \qquad (***)$$

Putting together $(*)$ $(**)$, and $(***)$, we have

. . . continued. . .

Proof of sum of integrals theorem (continued)

$$
\begin{aligned}
L(f,P)+L(g,P) &\le L(f+g,\,P) \\
&\le U(f+g,\,P) \le U(f,P)+U(g,P)
\end{aligned}
\qquad (\spadesuit)
$$

from which it follows that

$$
\begin{aligned}
U(f+g,\,P) - L(f+g,\,P) \\
&\le \Big(U(f,P)+U(g,P)\Big) - \Big(L(f,P)+L(g,P)\Big) \\
&= \Big(U(f,P)-L(f,P)\Big) + \Big(U(g,P)-L(g,P)\Big) \\
&< \frac{\varepsilon}{2}+\frac{\varepsilon}{2} = \varepsilon.
\end{aligned}
$$

$\therefore$ $f+g$ is integrable. Consequently, in ($\spadesuit$), $\int_a^b (f+g)$ is the unique number that lies between $L(f+g,P)$ and $U(f+g,P)$ for any partition P. Similarly, $\int_a^b f + \int_a^b g$ is the unique number that lies between the outermost quantities in ($\spadesuit$) for all P. Therefore, $\int_a^b (f+g) = \int_a^b f + \int_a^b g$, as required. □

Example (Characteristics of open sets)

Prove the fundamental properties of open sets.

Proof that $\mathbb{R}$ is open.

$$x \in \mathbb{R} \quad \Longrightarrow \quad x \in (x-1, x+1) \subset \mathbb{R} \quad \Longrightarrow \quad x \in \mathbb{R}^\circ.$$

$\therefore$ Every $x \in \mathbb{R}$ is an interior point of $\mathbb{R}$, *i.e.*, $\mathbb{R}$ is open. □

Proof that $\varnothing$ is open.

Since there are no points in $\varnothing$, every point in $\varnothing$ is an interior point of $\varnothing$, so $\varnothing$ is open. □

Proof that any union of open sets is open.

Suppose $\mathcal{U}$ is a collection of open sets, and $x \in \bigcup_{U \in \mathcal{U}} U$. Then $x \in U$ for some $U \in \mathcal{U}$,*i.e.*, $x \in U$ for some open set $U \subseteq \bigcup_{U \in \mathcal{U}} U$. So $\bigcup_{U \in \mathcal{U}} U$ is open. □

Proof that any finite intersection of open sets is open.

Suppose U_1 and U_2 are open. If U_1 and U_2 are disjoint, then their intersection is $\varnothing$, which is open. If $U_1 \cap U_2 \neq \varnothing$ then let $x \in U_1 \cap U_2$. Since $x \in U_1$, $\exists \delta_1 > 0 \ni (x - \delta_1, x + \delta_1) \subseteq U_1$. Similarly, since $x \in U_2$, $\exists \delta_2 > 0 \ni (x - \delta_2, x + \delta_2) \subseteq U_2$.

Let $\delta = \min\{\delta_1, \delta_2\}$. Then $(x - \delta, x + \delta) \subseteq U_1 \cap U_2$. So x is an interior point of $U_1 \cap U_2$. Hence $U_1 \cap U_2$ is open.

The result for <u>any</u> finite intersection follows by induction. Since

$$\bigcap_{i=1}^{n} U_i = \left(\bigcap_{i=1}^{n-1} U_i \right) \bigcap U_n \,,$$

we can apply the induction hypothesis together with the result for the intersection of two open sets to infer the result for n open sets. $\square$

DON'T FORGET THE TEST TOMORROW!

- The midterm TEST is on **TOMORROW:**
  Thursday 27 February 2025 @ 7:00pm in Hamilton Hall 302.

- I have an office hour this afternoon, 2:00-3:00pm.

GOOD LUCK!