McMaster University

Mathematics and Statistics

$$\int_M d\omega = \int_{\partial M} \omega$$

Mathematics 3A03
Real Analysis I

Instructor: David Earn

Lecture 13
Topology of $\mathbb{R}$ I
Tuesday 1 October 2019

THINKING ABOUT GRADUATE SCHOOL?

JOIN US TO FIND OUT MORE AT THE GRAD INFO SESSION!

WHEN: THURSDAY OCTOBER 3, 2019
TIME: 5:30PM – 7:00PM
WHERE: HH/305 AND THE MATH CAFÉ

Matheus Grasselli will give general advice on applying to grad school.

Shui Feng will talk about graduate programs particular to statistics.

Tom Hurd will talk about graduate opportunities in financial math including PhiMac.

Miroslav Lovric will give tips about applying to teachers' college.

PIZZA will be served! See you there!

Announcements

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Topology of $\mathbb{R}$

Intervals

Open interval:

$$(a, b) = \{x : a < x < b\}$$

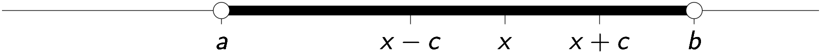
Closed interval:

$$[c, d] = \{x : c \leq x \leq d\}$$

Half-open interval:

$$(e, f] = \{x : e < x \leq f\}$$

Interior point



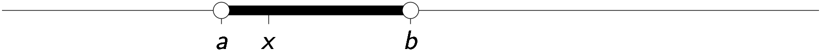
Definition (Interior point)

If $E \subseteq \mathbb{R}$ then x is an ***interior point*** of E if x lies in an open interval that is contained in E, *i.e.*, $\exists c > 0$ such that $(x - c, x + c) \subset E$.

Interior point examples

Set E	Interior points?
$(-1,1)$	Every point
$[0,1]$	Every point *except the endpoints*
$\mathbb{N}$	$\nexists$
$\mathbb{R}$	Every point
$\mathbb{Q}$	$\nexists$
$(-1,1)\cup[0,1]$	Every point *except 1*
$(-1,1)\setminus\{\frac{1}{2}\}$	Every point

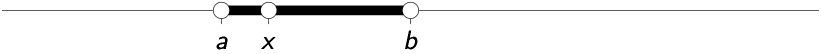
Neighbourhood



Definition (Neighbourhood)

A ***neighbourhood*** of a point $x \in \mathbb{R}$ is an open interval containing x.

Deleted neighbourhood

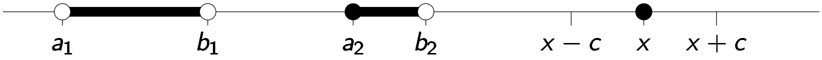


Definition (Deleted neighbourhood)

A ***deleted neighbourhood*** of a point $x \in \mathbb{R}$ is a set formed by removing x from a neighbourhood of x.

$$(a, b) \setminus \{x\}$$

Isolated point



$$E = (a_1, b_1) \cup [a_2, b_2) \cup \{x\}$$

Definition (Isolated point)

If $x \in E \subseteq \mathbb{R}$ then x is an ***isolated point*** of E if there is a neighbourhood of x for which the only point in E is x itself, *i.e.*, $\exists c > 0$ such that $(x - c, x + c) \cap E = \{x\}$.

Poll

- Go to https://www.childsmath.ca/childsa/forms/main_login.php
- Click on Math 3A03
- Click on Take Class Poll
- Fill in poll **Lecture 13: Isolated points**
- $\boxed{\texttt{Submit}}$.

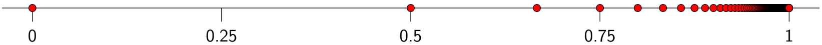
Isolated point examples

Set E	**Isolated points?**
$(-1,1)$	
$[0,1]$	
$\mathbb{N}$	
$\mathbb{R}$	
$\mathbb{Q}$	
$(-1,1)\cup[0,1]$	
$(-1,1)\setminus\{\frac{1}{2}\}$	

Isolated point examples

Set E	**Isolated points?**
$(-1,1)$	$\nexists$
$[0,1]$	$\nexists$
$\mathbb{N}$	Every point
$\mathbb{R}$	$\nexists$
$\mathbb{Q}$	$\nexists$
$(-1,1)\cup[0,1]$	$\nexists$
$(-1,1)\setminus\{\frac{1}{2}\}$	$\nexists$

Accumulation point



$$E = \left\{ 1 - \frac{1}{n} : n \in \mathbb{N} \right\}$$

Definition (Accumulation Point or Limit Point)

If $E \subseteq \mathbb{R}$ then x is an ***accumulation point*** or ***limit point*** of E if every neighbourhood of x contains infinitely many points of E,

$$\textit{i.e.,} \qquad \forall c > 0 \qquad (x - c, x + c) \cap (E \setminus \{x\}) \neq \varnothing\,.$$

Notes:

- It is possible but not necessary that $x \in E$.
- The shorthand condition is equivalent to saying that every deleted neighbourhood of x contains at least one point of E.

Poll

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- Fill in poll **Lecture 13: Accumulation points**
- $\boxed{\texttt{Submit}}$.

Accumulation point examples

Set E	**Accumulation points?**
$(-1,1)$	
$[0,1]$	
$\mathbb{N}$	
$\mathbb{R}$	
$\mathbb{Q}$	
$(-1,1) \cup [0,1]$	
$(-1,1) \setminus \{\frac{1}{2}\}$	
$\left\{1 - \frac{1}{n} : n \in \mathbb{N}\right\}$	

Accumulation point examples

Set E	Accumulation points?
$(-1,1)$	$[-1,1]$
$[0,1]$	$[0,1]$
$\mathbb{N}$	$\nexists$
$\mathbb{R}$	$\mathbb{R}$
$\mathbb{Q}$	$\mathbb{R}$
$(-1,1)\cup[0,1]$	$[-1,1]$
$(-1,1)\setminus\{\frac{1}{2}\}$	$[-1,1]$
$\left\{1-\frac{1}{n} : n\in\mathbb{N}\right\}$	$\{1\}$

McMaster University

Mathematics and Statistics

$$\int_M d\omega = \int_{\partial M} \omega$$

Mathematics 3A03
Real Analysis I

Instructor: David Earn

Lecture 14
Topology of $\mathbb{R}$ II
Thursday 3 October 2019

Poll

- Go to https://www.childsmath.ca/childsa/forms/main_login.php
- Click on Math 3A03
- Click on Take Class Poll
- Fill in poll **Lecture 14: poll on polls**
- `Submit`.

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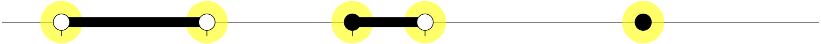
Topological concepts covered so far

- Interval
- Neighbourhood
- Deleted neighbourhood
- Interior point
- Isolated point
- Accumulation point

Boundary point



Definition (Boundary Point)

If $E \subseteq \mathbb{R}$ then x is a ***boundary point*** of E if every neighbourhood of x contains at least one point of E and at least one point not in E, *i.e.*,

$$\forall c > 0 \qquad (x-c, x+c) \cap E \neq \varnothing$$
$$\wedge \quad (x-c, x+c) \cap (\mathbb{R} \setminus E) \neq \varnothing\,.$$

Note: It is possible but not necessary that $x \in E$.

Definition (Boundary)

If $E \subseteq \mathbb{R}$ then the ***boundary*** of E, denoted ∂E, is the set of all boundary points of E.

Boundary point examples

Set E	**Boundary points?**
$(-1,1)$	$\{-1,1\}$
$[0,1]$	$\{0,1\}$
$\mathbb{N}$	$\mathbb{N}$
$\mathbb{R}$	$\nexists$
$\mathbb{Q}$	$\mathbb{R}$
$(-1,1)\cup[0,1]$	$\{-1,1\}$
$(-1,1)\setminus\{\frac{1}{2}\}$	$\{-1,\frac{1}{2},1\}$
$\left\{1-\frac{1}{n} : n\in\mathbb{N}\right\}$	$\left\{1-\frac{1}{n} : n\in\mathbb{N}\right\}\cup\{1\}$

Closed set

Definition (Closed set)

A set $E \subseteq \mathbb{R}$ is ***closed*** if it contains all of its accumulation points.

Definition (Closure of a set)

If $E \subseteq \mathbb{R}$ and E' is the set of accumulation points of E then $\overline{E} = E \cup E'$ is the ***closure*** of E.

Note: If the set E has no accumulation points, then E is closed because there are no accumulation points to check.

Open set

Definition (Open set)

A set $E \subseteq \mathbb{R}$ is ***open*** if every point of E is an interior point.

Definition (Interior of a set)

If $E \subseteq \mathbb{R}$ then the ***interior*** of E, denoted $\text{int}(E)$ or E°, is the set of all interior points of E.

Poll

- Go to https://www.childsmath.ca/childsa/forms/main_login.php
- Click on Math 3A03
- Click on Take Class Poll
- Fill in poll **Lecture 14: Open or Closed**
- $\boxed{\texttt{Submit}}$.

Examples

Set E	**Closed?**	**Open?**	$\overline{E}$	E°	∂E
$(-1,1)$					
$[0,1]$					
$\mathbb{N}$					
$\mathbb{R}$					
$\varnothing$					
$\mathbb{Q}$					
$(-1,1)\cup[0,1]$					
$(-1,1)\setminus\{\frac{1}{2}\}$					
$\left\{1-\frac{1}{n} : n\in\mathbb{N}\right\}$					

Examples

Set E	Closed?	Open?	$\overline{E}$	E°	∂E
$(-1,1)$	NO	YES	$[-1,1]$	E	$\{-1,1\}$
$[0,1]$	YES	NO	E	$(0,1)$	$\{0,1\}$
$\mathbb{N}$	YES	NO	$\mathbb{N}$	$\varnothing$	$\mathbb{N}$
$\mathbb{R}$	YES	YES	$\mathbb{R}$	$\mathbb{R}$	$\varnothing$
$\varnothing$	YES	YES	$\varnothing$	$\varnothing$	$\varnothing$
$\mathbb{Q}$	NO	NO	$\mathbb{R}$	$\varnothing$	$\mathbb{R}$
$(-1,1)\cup[0,1]$	NO	NO	$[-1,1]$	$(-1,1)$	$\{-1,1\}$
$(-1,1)\setminus\{\frac{1}{2}\}$	NO	YES	$[-1,1]$	E	$\{-1,\frac{1}{2},1\}$
$\left\{1-\frac{1}{n} : n\in\mathbb{N}\right\}$	NO	NO	$E\cup\{1\}$	$\varnothing$	$E\cup\{1\}$

McMaster University

Mathematics and Statistics

$$\int_M d\omega = \int_{\partial M} \omega$$

Mathematics 3A03
Real Analysis I

Instructor: David Earn

Lecture 15
Topology of $\mathbb{R}$ III
Friday 4 October 2019

Announcements

- Assignment 3 is posted, but more problems will be added over the weekend. **Due Tuesday 22 October 2019 at 2:25pm via crowdmark.**
- **Math 3A03 Test #1**
  Tuesday 29 October 2019, 5:30–7:00pm, in JHE 264
  (room is booked for 90 minutes; you should not feel rushed)

Topological concepts covered so far

- Interval
- Neighbourhood
- Deleted neighbourhood
- Interior point
- Isolated point
- Accumulation point
- Boundary point
- Boundary
- Closed set
- Closure
- Open set
- Interior

Poll

- Go to https://www.childsmath.ca/childsa/forms/main_login.php
- Click on Math 3A03
- Click on Take Class Poll
- Fill in poll **Lecture 15: The most general type of open set**
- $\boxed{\texttt{Submit}}$.

Component intervals of open sets

What does the most general open set look like?

Theorem (Component intervals)

If G is an open subset of $\mathbb{R}$ and $G \neq \varnothing$ then there is a unique (possibly finite) sequence of disjoint open intervals $\{(a_n, b_n)\}$ such that

$$G = (a_1, b_1) \cup (a_2, b_2) \cup \cdots \cup (a_n, b_n) \cup \cdots \,,$$

$$\textit{i.e.,} \qquad G = \bigcup_{n=1}^{\infty} (a_n, b_n)\,.$$

*The open intervals (a_n, b_n) are said to be the **component intervals** of G.*

(Textbook (TBB) Theorem 4.15, p. 231)

Component intervals of open sets

Main ideas of proof of component intervals theorem:

- $x \in G \implies x$ is an interior point of $G \implies$
  - some neighbourhood of x is contained in G, *i.e.*, $\exists c > 0$ such that $(x - c, x + c) \subseteq G$
  - $\exists$ a <u>largest</u> neighbourhood of x that is contained in G: this "***component of G***" is $I_x = (\alpha, \beta)$, where

$$\alpha = \inf\{a : (a, x] \subset G\}, \qquad \beta = \sup\{b : [x, b) \subset G\}$$

  - I_x contains a rational number, *i.e.*, $\exists r \in I_x \cap \mathbb{Q}$
- $\therefore$ We can index all the intervals I_x by <u>rational</u> numbers
- $\therefore$ There are are most countably many intervals that make up G (*i.e.*, G is the union of a sequence of intervals)
- We can choose a <u>disjoint</u> subsequence of these intervals whose union is all of G (see proof in textbook for details).

Open vs. Closed Sets

Definition (Complement of a set of real numbers)

If $E \subseteq \mathbb{R}$ then the ***complement*** of E is the set

$$E^{\mathsf{c}} = \{x \in \mathbb{R} : x \notin E\}\,.$$

Theorem (Open vs. Closed)

If $E \subseteq \mathbb{R}$ then E is open iff E^{c} is closed.

(Textbook (TBB) Theorem 4.16)

Open vs. Closed Sets

Theorem (Properties of open sets of real numbers)

1. *The sets $\mathbb{R}$ and $\varnothing$ are open.*
2. *Any intersection of a finite number of open sets is open.*
3. *Any union of an arbitrary collection of open sets is open.*
4. *The complement of an open set is closed.*

(Textbook (TBB) Theorem 4.17)

Theorem (Properties of closed sets of real numbers)

1. *The sets $\mathbb{R}$ and $\varnothing$ are closed.*
2. *Any union of a finite number of closed sets is closed.*
3. *Any intersection of an arbitrary collection of closed sets is closed.*
4. *The complement of a closed set is open.*

(Textbook (TBB) Theorem 4.18)

Local vs. Global properties

Definition (Bounded function)

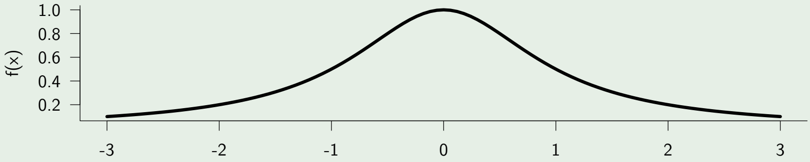
A real-valued function f is ***bounded*** on the set E if there exists $M > 0$ such that $|f(x)| \leq M$ for all $x \in E$.

(*i.e.*, the function f is bounded on E iff $\{f(x) : x \in E\}$ is a bounded set.)

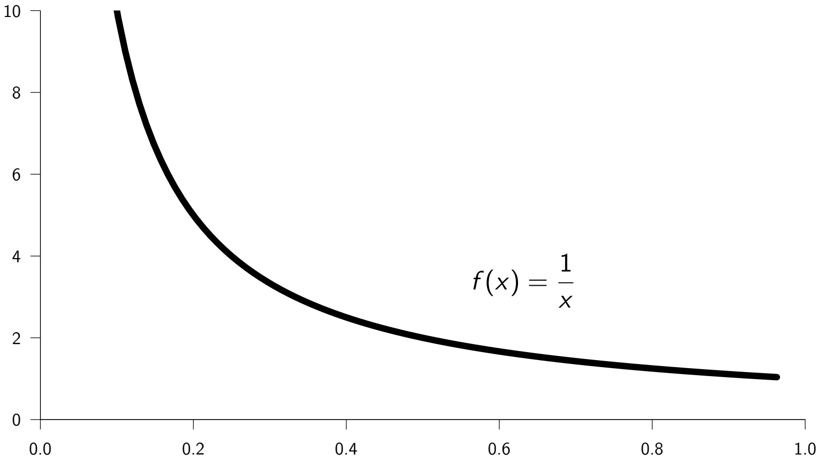
Note: This is a *global* property because there is a single bound M associated with the entire set E.

Example

The function $f(x) = 1/(1 + x^2)$ is bounded on $\mathbb{R}$. *e.g.*, $M = 1$.

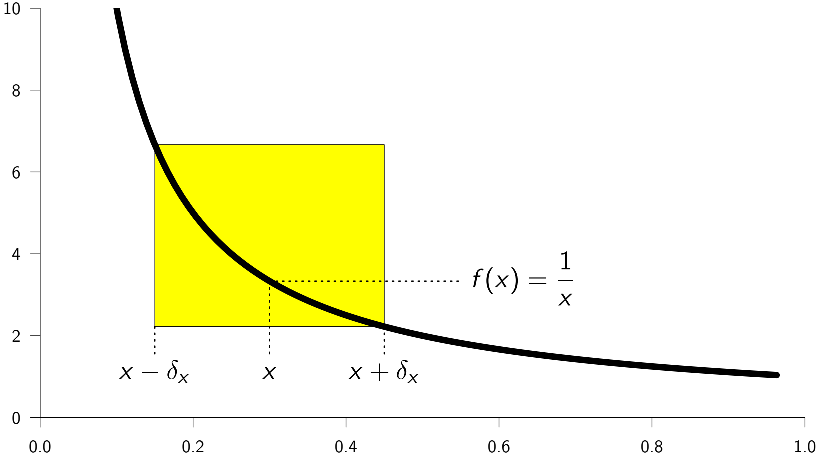


Local vs. Global properties



$f(x) = 1/x$ is <u>not</u> bounded on the interval $E = (0, 1)$.

Local vs. Global properties



$f(x) = 1/x$ is ***locally bounded*** on the interval $E = (0, 1)$,
i.e., $\forall x \in E$, $\exists \delta_x, M_x > 0 \ni |f(t)| \leq M_x \ \forall t \in (x - \delta_x, x + \delta_x)$.

Local vs. Global properties

Definition (Locally bounded at a point)

A real-valued function f is ***locally bounded*** at the point x if there is a neighbourhood of x in which f is bounded, *i.e.*, there exists $\delta_x > 0$ and $M_x > 0$ such that $|f(t)| \leq M_x$ for all $t \in (x - \delta_x, x + \delta_x)$.

Definition (Locally bounded on a set)

A real-valued function f is ***locally bounded*** on the set E if f is locally bounded at each point $x \in E$.

Note: The size of the neighbourhood (δ_x) and the local bound (M_x) depend on the point x.

Local vs. Global properties

Example (Function that is not even locally bounded)

Give an example of a function that is defined on the interval $(0, 1)$ but is not locally bounded on $(0, 1)$.

(solution on board)

Example (Function that is a mess near 0)

Give an example of a function $f(x)$ that is defined everywhere, yet in any neighbourhood of the origin there are infinitely many points at which f is not locally bounded.

(solution on board)

Extra Challenge Problem: Is there a function $f : \mathbb{R} \to \mathbb{R}$ that is not locally bounded anywhere?

McMaster University

Mathematics and Statistics

$$\int_M d\omega = \int_{\partial M} \omega$$

Mathematics 3A03
Real Analysis I

Instructor: David Earn

Lecture 16
Topology of $\mathbb{R}$ IV
Tuesday 8 October 2019

Announcements

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  Due Tuesday 22 October 2019 at 2:25pm via crowdmark.
- **Math 3A03 Test #1**
  Tuesday 29 October 2019, 5:30–7:00pm, in JHE 264
  (room is booked for 90 minutes; you should not feel rushed)

Local vs. Global properties

Example (Function that is not even locally bounded)

Give an example of a function that is defined on the interval $(0,1)$ but is not locally bounded on $(0,1)$.

(solution on board)

Example (Function that is a mess near 0)

Give an example of a function $f(x)$ that is defined everywhere, yet in any neighbourhood of the origin there are infinitely many points at which f is not locally bounded.

(solution on board)

Extra Challenge Problem: Is there a function $f : \mathbb{R} \to \mathbb{R}$ that is not locally bounded anywhere?

Poll

- Go to https://www.childsmath.ca/childsa/forms/main_login.php
- Click on Math 3A03
- Click on Take Class Poll
- Fill in poll **Lecture 16: Local boundedness**
- $\boxed{\texttt{Submit}}$.

Local vs. Global properties

- What condition(s) rule out such pathological behaviour?
- When does a property holding locally (near any given point in a set) imply that it holds globally (for the set as a whole)?
- For example: What condition(s) must a set $E \subseteq \mathbb{R}$ satisfy in order that a function f that is locally bounded on E is necessarily bounded on E?
- We will see that the condition we are seeking is that the set E must be "***compact***" . . .

Compactness

Recall the Bolzano-Weierstrass theorem, which we proved when investigating sequences of real numbers:

Theorem (Bolzano-Weierstrass theorem for sequences)

Every bounded sequence in $\mathbb{R}$ contains a convergent subsequence.

For *any set of real numbers*, we define:

Definition (Bolzano-Weierstrass property)

A set $E \subseteq \mathbb{R}$ is said to have the ***Bolzano-Weierstrass property*** iff any sequence of points chosen from E has a subsequence that converges to a point in E.

Compactness

Theorem (Bolzano-Weierstrass theorem for sets)

A set $E \subseteq \mathbb{R}$ has the Bolzano-Weierstrass property iff E is closed and bounded.

(Textbook (TBB) Theorem 4.21, p. 241)

Proof of $\Longleftarrow$.

Suppose E is closed and bounded. Let $\{x_n\}$ be a sequence in E. Since E is bounded, the usual Bolzano-Weierstrass theorem implies that there is a subsequence $\{x_{n_k}\}$ that converges. If $\{x_{n_k}\}$ is eventually constant, then its limit is a point in E, so we're done. Otherwise, $\{x_{n_k}\}$ must converge to an accumulation point of E. But E is closed, so it contains all its accumulation points, including the limit of $\{x_{n_k}\}$. Thus, again, we have a subsequence of $\{x_n\}$ that converges to a point in E. $\square$

Compactness

Theorem (Bolzano-Weierstrass theorem for sets)

A set $E \subseteq \mathbb{R}$ has the Bolzano-Weierstrass property iff E is closed and bounded.

Proof of $\implies$ (Part 1)

Let's prove the contrapositive, *i.e.,* If E is either not bounded or not closed then E does not have the Bolzano-Weierstrass property. ***Suppose E is unbounded.*** In particular, suppose E is not bounded above (the argument is similar if E is not bounded below). We will construct a sequence in E that has no convergent subsequence. Pick a point $x_1 \in E$ such that $x_1 > 1$ (which is possible because E is not bounded above). Also, since E is not bounded above, we can find $x_2 \in E$ such that $x_2 > x_1 + 1 > 2$. More generally, given $x_k \in E$ we can find $x_{k+1} \in E$ such that $x_{k+1} > x_k + 1 > k + 1$. The sequence $\{x_n\}$ constructed in this way is increasing and diverges to ∞ (since $x_n > n$ for all $n \in \mathbb{N}$). Moreover, this is true of any subsequence of $\{x_n\}$. $\therefore$ ***E does not have the Bolzano-Weierstrass property.***

Compactness

Theorem (Bolzano-Weierstrass theorem for sets)

A set $E \subseteq \mathbb{R}$ has the Bolzano-Weierstrass property iff E is closed and bounded.

Proof of $\implies$ (Part 2)

Now ***suppose E is not closed.*** Then there must be a sequence $\{x_n\}$ in E such that $\{x_n\}$ converges to a point not in E. If E has the Bolzano-Weierstrass property, then $\{x_n\}$ has a subsequence that converges to a point in E. But every subsequence of a convergent sequence must converge to the same point as the full sequence, and the full sequence converges to a point not in E! Thus, if E is not closed ***then it does not have the Bolzano-Weierstrass property.*** $\square$

Compactness

Theorem (Bolzano-Weierstrass theorem for sets)

A set $E \subseteq \mathbb{R}$ has the Bolzano-Weierstrass property iff E is closed and bounded.

Notes:

- Why do we need both *closed* and *bounded*? Why didn't we need *closed* in the original version of the Bolzano-Weierstrass theorem (for sequences)?
  - Because we didn't require the limit of the convergent subsequence to be in the set!
- The Bolzano-Weierstrass theorem for sets implies that "If $E \subseteq \mathbb{R}$ is bounded then its closure $\overline{E}$ has the Bolzano-Weierstrass property".
  - The original Bolzano-Weierstrass theorem for sequences is a special case of this statement because any convergent sequence together with its limit is a closed set.
- We assumed implicitly in the proof that $E \neq \varnothing$. Was that OK?

Compactness

Definition (Open Cover)

Let $E \subseteq \mathbb{R}$ and let $\mathcal{U}$ be a family of open intervals. If for every $x \in E$ there exists at least one interval $U \in \mathcal{U}$ such that $x \in U$, *i.e.*,

$$E \subseteq \bigcup \{U : U \in \mathcal{U}\} ,$$

then $\mathcal{U}$ is called an ***open cover*** of E.

Example (Open covers of $\mathbb{N}$)

Give examples of open covers of $\mathbb{N}$.

- $\mathcal{U} = \left\{ \left(n - \frac{1}{2}, n + \frac{1}{2} \right) : n = 1, 2, \ldots \right\}$
- $\mathcal{U} = \{(0, \infty)\}$
- $\mathcal{U} = \{(0, \infty), \mathbb{R}, (\pi, 27)\}$

Compactness

Example (Open covers of $\{\frac{1}{n} : n \in \mathbb{N}\}$)

- $\mathcal{U} = \{(0,1),\ (0,2),\ \mathbb{R},\ (\pi,27)\}$
- $\mathcal{U} = \{(0,2)\}$
- $\mathcal{U} = \left\{\left(\frac{1}{n},\ \frac{1}{n}+\frac{3}{4}\right) : n = 1,2,\ldots\right\}$

Example (Open covers of $[0,1]$)

- $\mathcal{U} = \{(-2,2)\}$
- $\mathcal{U} = \{(-\frac{1}{2},\frac{1}{2}),\ (0,2)\}$
- $\mathcal{U} = \left\{\left(\frac{1}{n},2\right) : n = 1,2,\ldots\right\} \cup \left\{\left(-\frac{1}{2},\frac{1}{2}\right)\right\}$

Poll

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- Fill in poll **Lecture 16: Open covers**
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Mathematics and Statistics

$$\int_M d\omega = \int_{\partial M} \omega$$

Mathematics 3A03
Real Analysis I

Instructor: David Earn

Lecture 17
Topology of $\mathbb{R}$ V
Thursday 10 October 2019

Announcements

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- **Math 3A03 Test #1**
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  (room is booked for 90 minutes; you should not feel rushed)
- Math 3A03 **Final Exam**: Fri 6 Dec 2019, 9:00am–11:30am
  Location: `MDCL 1105`

Compactness

Definition (Heine-Borel Property)

A set $E \subseteq \mathbb{R}$ is said to have the ***Heine-Borel property*** if every open cover of E can be reduced to a finite subcover. That is, if $\mathcal{U}$ is an open cover of E, then there exists a finite subfamily $\{U_1, U_2, \ldots, U_n\} \subseteq \mathcal{U}$, such that $E \subseteq U_1 \cup U_2 \cup \cdots \cup U_n$.

When does any open cover of a set E have a finite subcover?

Theorem (Heine-Borel Theorem)

A set $E \subseteq \mathbb{R}$ has the Heine-Borel property iff E is both closed and bounded.

(Textbook (TBB) pp. 249–250)

Compactness

Definition (Compact Set)

A set $E \subseteq \mathbb{R}$ is said to be ***compact*** if it has any of the following equivalent properties:

1. E is closed and bounded.
2. E has the Bolzano-Weierstrass property.
3. E has the Heine-Borel property.

Note: In spaces other than $\mathbb{R}$, these three properties are not necessarily equivalent. Usually the Heine-Borel property is taken as the definition of compactness.

Compactness

Example

Prove that the interval $(0,1]$ is not compact by showing that it is not closed or not bounded.

(solution on board)

Example

Prove that the interval $(0,1]$ is not compact by showing that it does not have the Bolzano-Weierstrass property.

(solution on board)

Example

Prove that the interval $(0,1]$ is not compact by showing that it does not have the Heine-Borel property.

(solution on board)

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Compactness

Classic non-trivial compactness argument:

Theorem (Compact $\implies$ bounded if locally bounded)

Let E be a compact subset of $\mathbb{R}$. If $f : E \to \mathbb{R}$ is locally bounded on E then f is bounded on E.

Proof via Bolzano-Weierstrass.

E satisfies the Bolzano-Weierstrass property, *i.e.*, any sequence in E has a subsequence that converges to a point in E. Suppose, in order to derive a contradiction, that f is not bounded on E. Then there is some sequence $\{x_n\}$ in E such that $|f(x_n)| > n$ (otherwise $|f(x_n)| \leq N$ for some $N \in \mathbb{N}$, so N would be a bound). By the Bolzano-Weierstrass property, $\{x_n\}$ has a subsequence $\{x_{n_k}\}$ such that $x_{n_k} \to L \in E$. Since f is locally bounded, there exist $\delta > 0$ and $M > 0$ and such that $|f(x)| \leq M$ for all $x \in (L-\delta, L+\delta)$. But $x_{n_k} \to L$, so for all sufficiently large k, $x_{n_k} \in (L-\delta, L+\delta)$. Yet for sufficiently large k, $|f(x_{n_k})| > n_k \geq k > M$. $\Rightarrow\Leftarrow$. Hence f must, in fact, be bounded on E. $\square$

Compactness

Theorem (Compact $\implies$ bounded if locally bounded)

Let E be a compact subset of $\mathbb{R}$. If $f : E \to \mathbb{R}$ is locally bounded on E then f is bounded on E.

Proof via Heine-Borel.

Since f is locally bounded, each $x \in E$ lies in some open interval U_x such that $|f(t)| \le M_x$ for all $t \in U_x$. The collection $\mathcal{U} = \{U_x : x \in E\}$ is an open cover of E. But E satisfies the Heine-Borel property, so $\mathcal{U}$ contains a finite subcover, say $\{U_{x_1}, \dots, U_{x_n}\}$. We can therefore find a bound for f on all of E. Let $M = \max\{M_{x_1}, \dots, M_{x_n}\}$. Then $|f(x)| \le M$ for all $x \in E$. $\square$

Proof via Heine-Borel is much easier!

Compactness

Theorem (Compact $\implies$ bounded if locally bounded)

Let E be a compact subset of $\mathbb{R}$. If $f : E \to \mathbb{R}$ is locally bounded on E then f is bounded on E.

Example (Converse of above theorem)

Let $E \subseteq \mathbb{R}$. If every function $f : E \to \mathbb{R}$ that is locally bounded on E is bounded on E, then E is compact.

Note: Contrapositive of converse is: If $E \subseteq \mathbb{R}$ is not compact then $\exists f : E \to \mathbb{R} \ni f$ is locally bounded on E but not bounded on E.

Complements and Closures problem

Example

How many distinct sets can be obtained from $E = [0,1]$ by applying the complement and closure operations?

Consider this sequence of sets: $E_1 = [0,1]$,
$E_2 = E_1^c = (-\infty,0) \cup (1,\infty)$, $E_3 = \overline{E_2} = (-\infty,0] \cup [1,\infty)$,
$E_4 = E_3^c = (0,1)$, $E_5 = \overline{E_4} = E_1$.

Is the answer 4 for any set $E \subseteq \mathbb{R}$?

Extra Challenge Problem

If $E \subseteq \mathbb{R}$, how many distinct sets can be obtained by taking complements or closures of E and its successors? Put another way, if $\{E_n\}$ is a sequence of sets produced by taking the complement or closure of the previous set, how many distinct sets can such a sequence contain? If the answer is finite, find a set E that generates the maximum number in this way.

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