McMaster
University

**Mathematics
and Statistics**

$$\int_M d\omega = \int_{\partial M} \omega$$

Mathematics 3A03
Real Analysis I

Instructor: David Earn

Lecture 13
Topology of $\mathbb{R}$
Thursday 5 February 2026

Announcements

- Solutions to Assignment 2 have been posted.
- Assignment 3 is posted on the course web site. Participation deadline is Tuesday 24 Feb 2026 @ 2:25pm.
- The midterm TEST is on Thursday 26 February 2026 @ 7:00pm in T13 123.
- The room is booked for 7:00–10:00 pm, but the intention is that a reasonable amount of time for the test is 90 minutes. You will be given double time.

Topology of $\mathbb{R}$

Intervals

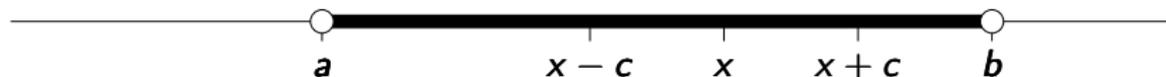*Open interval:*

$$(a, b) = \{x : a < x < b\}$$

Closed interval:

$$[c, d] = \{x : c \leq x \leq d\}$$

Half-open interval:

$$(e, f] = \{x : e < x \leq f\}$$
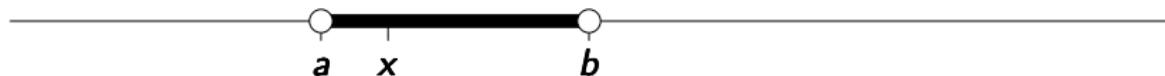
Interior point

Definition (Interior point)

If $E \subseteq \mathbb{R}$ then x is an *interior point* of E if x lies in an open interval that is contained in E, *i.e.*,

$$\exists c > 0 \quad \} \quad (x - c, x + c) \subset E.$$

Interior point examples

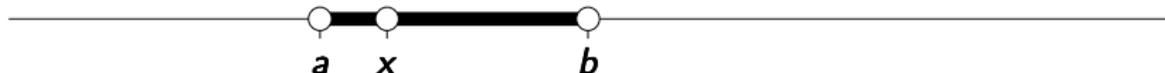| Set E	Interior points?
$(-1, 1)$	
$[0, 1]$	
$\mathbb{N}$	
$\mathbb{R}$	
$\mathbb{Q}$	
$(-1, 1) \cup [0, 1]$	
$(-1, 1) \setminus \{\frac{1}{2}\}$	

Neighbourhood

Definition (Neighbourhood)

A **neighbourhood** of a point $x \in \mathbb{R}$ is an open interval containing x.

Deleted neighbourhood

$$(a, b) \setminus \{x\}$$

Definition (Deleted neighbourhood)

A **deleted neighbourhood** of a point $x \in \mathbb{R}$ is a set formed by removing x from a neighbourhood of x.

Isolated point

$$E = (a_1, b_1) \cup [a_2, b_2] \cup \{x\}$$

Definition (Isolated point)

If $x \in E \subseteq \mathbb{R}$ then x is an **isolated point** of E if there is a
neighbourhood of x for which the only point in E is x itself, *i.e.*,

$$\exists c > 0 \quad \ni \quad (x - c, x + c) \cap E = \{x\}.$$

Poll

- Go to
  https://www.childsmath.ca/childsa/forms/main_login.php

- Click on Math 3A03

- Click on Take Class Poll

- Fill in poll **Topology: Isolated points**

- Submit .

Isolated point examples

Set E	Isolated points?
$(-1, 1)$	
$[0, 1]$	
$\mathbb{N}$	
$\mathbb{R}$	
$\mathbb{Q}$	
$(-1, 1) \cup [0, 1]$	
$(-1, 1) \setminus \{\frac{1}{2}\}$	

Accumulation point

$$E = \left\{ 1 - \frac{1}{n} : n \in \mathbb{N} \right\}$$

Definition (Accumulation Point or Limit Point or Cluster Point)

If $E \subseteq \mathbb{R}$ then x is an *accumulation point* of E if every neighbourhood of x contains infinitely many points of E,

$$\text{i.e.,} \qquad \forall c > 0 \qquad (x - c, x + c) \cap (E \setminus \{x\}) \neq \varnothing .$$

Note:

- It is possible but <u>not necessary</u> that $x \in E$.
- The shorthand condition is equivalent to saying that every <u>deleted neighbourhood</u> of x contains <u>at least one</u> point of E.

Poll

- Go to
  https://www.childsmath.ca/childsa/forms/main_login.php

- Click on Math 3A03

- Click on Take Class Poll

- Fill in poll **Topology: Accumulation points**

- Submit .

Accumulation point examples

Set E	Accumulation points?
$(-1, 1)$	
$[0, 1]$	
$\mathbb{N}$	
$\mathbb{R}$	
$\mathbb{Q}$	
$(-1, 1) \cup [0, 1]$	
$(-1, 1) \setminus \{\frac{1}{2}\}$	
$\left\{ 1 - \frac{1}{n} : n \in \mathbb{N} \right\}$	

Boundary point

Definition (Boundary Point)

If $E \subseteq \mathbb{R}$ then x is a **boundary point** of E if every neighbourhood of x contains at least one point of E and at least one point not in E, *i.e.*,

$$\forall c > 0 \qquad (x - c, x + c) \cap E \ \neq \ \varnothing$$
$$\wedge \quad (x - c, x + c) \cap (\mathbb{R} \setminus E) \ \neq \ \varnothing .$$

Note: It is possible but <u>not necessary</u> that $x \in E$.

Definition (Boundary)

If $E \subseteq \mathbb{R}$ then the **boundary** of E, denoted ∂E, is the set of all boundary points of E.

Boundary point examples

Set E	Boundary points?
$(-1, 1)$	
$[0, 1]$	
$\mathbb{N}$	
$\mathbb{R}$	
$\mathbb{Q}$	
$(-1, 1) \cup [0, 1]$	
$(-1, 1) \setminus \{\frac{1}{2}\}$	
$\left\{ 1 - \frac{1}{n} : n \in \mathbb{N} \right\}$	

Closed set

Definition (Closed set)

A set $E \subseteq \mathbb{R}$ is **closed** if it contains all of its accumulation points.

Definition (Closure of a set)

If $E \subseteq \mathbb{R}$ and E' is the set of accumulation points of E then the **closure** of E is

$$\overline{E} = E \cup E'.$$

Note: If the set E has no accumulation points, then E is closed because there are no accumulation points to check.

Open set

Definition (Open set)

A set $E \subseteq \mathbb{R}$ is **open** if every point of E is an interior point.

Definition (Interior of a set)

If $E \subseteq \mathbb{R}$ then the **interior** of E, denoted $\operatorname{int}(E)$ or E°, is the set of all interior points of E.

Poll

- Go to
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- Click on Math 3A03

- Click on Take Class Poll

- Fill in poll **Topology: Open or Closed**

- Submit .

Examples

Set E	Closed?	Open?	$\overline{E}$	E°	∂E
$(-1,1)$					
$[0,1]$					
$\mathbb{N}$					
$\mathbb{R}$					
$\varnothing$					
$\mathbb{Q}$					
$(-1,1) \cup [0,1]$					
$(-1,1) \setminus \{\frac{1}{2}\}$					
$\left\{ 1 - \frac{1}{n} : n \in \mathbb{N} \right\}$					

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Mathematics 3A03
Real Analysis I

Instructor: David Earn

Lecture 14
Topology of $\mathbb{R}$ II
Friday 6 February 2026

Announcements

- Today's Mathematics Colloquium (immediately after this class in HH-305) should be accessible to 3A students. Title "Is mathematics obselete?"

- Solutions to Assignment 2 have been posted.

- Assignment 3 is posted on the course web site. Participation deadline is Tuesday 24 Feb 2025 @ 2:25pm.

- The midterm TEST is on Thursday 26 February 2026 @ 7:00pm in T13 123.

- The room is booked for 7:00–10:00 pm, but the intention is that a reasonable amount of time for the test is 90 minutes. You will be given double time.

- Last year's (winter 2025) midterm test and solutions are posted on the course web site.

Topological concepts covered so far

- Interval

- Neighbourhood

- Deleted neighbourhood

- Interior point

- Isolated point

- Accumulation point

- Boundary point

- Boundary

- Closed set

- Closure

- Open set

- Interior

Equivalent definitions

Example (<u>Closure</u>)

For $E \subseteq \mathbb{R}$, prove $E \cup E' = E \cup \partial E$, so $\overline{E}$ can be defined either way.

Proof: We must show $E \cup E' \subseteq E \cup \partial E$ and $E \cup E' \supseteq E \cup \partial E$.

$\subseteq$ Suppose $x \in E \cup E'$. If $x \in E$ then $x \in E \cup A$ for any set A. In particular, $x \in E \cup \partial E$. Alternatively, suppose $x \notin E$, *i.e.,* $x \in E^c$. Then, since $x \in E \cup E'$, it must be that $x \in E'$, which means that any neighbourhood of x contains a point of E. But $x \in E^c$, so any such neighbourhood also contains a point of E^c (namely x). Therefore, $x \in \partial E \subseteq E \cup \partial E$.

$\supseteq$ Suppose $x \in E \cup \partial E$. If $x \in E$ then $x \in E \cup A$ for any set A. In particular, $x \in E \cup E'$. Alternatively, suppose $x \notin E$, *i.e.,* $x \in E^c$. Then, since $x \in E \cup \partial E$, it must be that $x \in \partial E$, which means that any neighbourhood of x contains a point of E. But that point is not x, since $x \notin E$. Thus, any *deleted* neighbourhood of x contains a point of E, *i.e.,* $x \in E' \subseteq E \cup E'$. $\square$

Question: In the proof above, did we use any properties of $\mathbb{R}$?

Poll

- Go to
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- Click on Math 3A03

- Click on Take Class Poll

- Fill in poll **Topology: The most general type of open set**

- Submit .

Component intervals of open sets

What does the most general open set look like?

Theorem (Component intervals)

If G is an open subset of $\mathbb{R}$ and $G \neq \varnothing$ then there is a unique (possibly finite) sequence of disjoint open intervals $\{(a_n, b_n)\}$ such that

$$G = (a_1, b_1) \cup (a_2, b_2) \cup \cdots \cup (a_n, b_n) \cup \cdots,$$

$$i.e., \quad G = \bigcup_{n=1}^{\infty} (a_n, b_n).$$

*The open intervals (a_n, b_n) are said to be the **component intervals** of G.*

(TBB Theorem 4.15, p. 231)

Component intervals of open sets

Main ideas of proof of component intervals theorem:

- $x \in G \implies x \in G^\circ \implies x \in (a, b) \subset G$ for some $a, b \in \mathbb{R}$

- $\exists$ a <u>largest</u> neighbourhood of x in G: this **component of G** is $I_x = (\alpha, \beta)$, where $\alpha = \inf\{a : (a, x] \subset G\}$ and $\beta = \sup\{b : [x, b) \subset G\}$

- Components with different endpoints cannot overlap (they would contradict the inf and sup) so *distinct components are disjoint*

- Every component I_x *contains a rational number*, *i.e.*, $\exists r \in I_x \cap \mathbb{Q}$

- But $\mathbb{Q}$ is *countable*, and we can write an *explicit bijection* $q : \mathbb{N} \to \mathbb{Q}$

- $\therefore$ We can *label* each component interval I of G with a *unique* $n \in \mathbb{N}$. The label of component I is $n_I = \min\{n \in \mathbb{N} : q(n) \in I\}$ (well-ordering)

- $\therefore$ We can list the components of G in a *sequence* $\{I_1, I_2, I_3, \ldots\}$

- $\therefore$ There are at most countably many intervals that make up G *i.e.*, G is the union of a <u>sequence</u> of disjoint intervals

- See textbook for a standard proof (TBB Theorem 4.15, p. 231).

<u>*Note:*</u> (For those studying logic and set theory.) This argument relies ultimately on the well-ordering principle for ℕ (or, equivalently, the principle of mathematical induction for ℕ) but it does <u>not</u> require the Axiom of Choice.

Open vs. Closed Sets

Definition (Complement of a set of real numbers)

If $E \subseteq \mathbb{R}$ then the ***complement*** of E is the set

$$E^c = \{x \in \mathbb{R} : x \notin E\}.$$

Theorem (Open vs. Closed)

If $E \subseteq \mathbb{R}$ then E is open iff E^c is closed.

(TBB Theorem 4.16)

Open vs. Closed Sets

Theorem (Properties of open sets of real numbers)

1. *The sets $\mathbb{R}$ and $\varnothing$ are open.*
2. *Any intersection of a finite number of open sets is open.*
3. *Any union of an arbitrary collection of open sets is open.*
4. *The complement of an open set is closed.*

(TBB Theorem 4.17)

Theorem (Properties of closed sets of real numbers)

1. *The sets $\mathbb{R}$ and $\varnothing$ are closed.*
2. *Any union of a finite number of closed sets is closed.*
3. *Any intersection of an arbitrary collection of closed sets is closed.*
4. *The complement of a closed set is open.*

(TBB Theorem 4.18)

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Mathematics and Statistics

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Mathematics 3A03
Real Analysis I

Instructor: David Earn

Lecture 15
Topology of ℝ III
Tuesday 10 February 2026

Announcements

- Solutions to Assignment 2 have been re-posted (some typos have been fixed).

- Assignment 3 is posted on the course web site. Participation deadline is Tuesday 24 Feb 2025 @ 2:25pm.

- The midterm TEST is on Thursday 26 February 2026 @ 7:00pm in T13 123.

- The room is booked for 7:00–10:00 pm, but the intention is that a reasonable amount of time for the test is 90 minutes. You will be given double time.

- Last year's (winter 2025) midterm test and solutions are posted on the course web site.

- I have revised my sketch of the proof of the component intervals theorem.

- For those who are interested in foundations of mathematics: the simplest result in real analysis that depends on the Axiom of Choice is the existence of a non-measurable set (take Math 4A…)

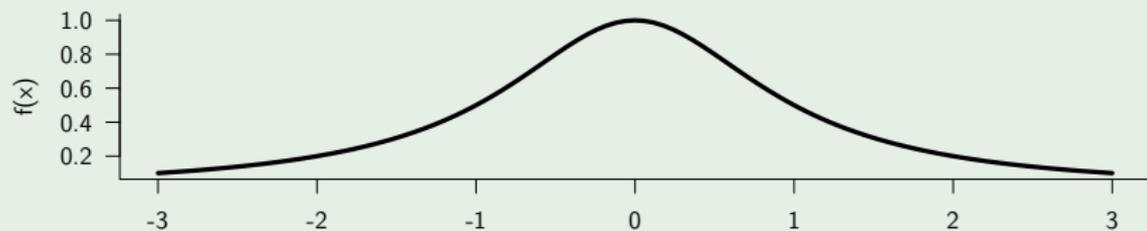# Local vs. Global properties

Definition (Bounded function)

A real-valued function f is **bounded** on the set E if there exists $M > 0$ such that $|f(x)| \leq M$ for all $x \in E$.

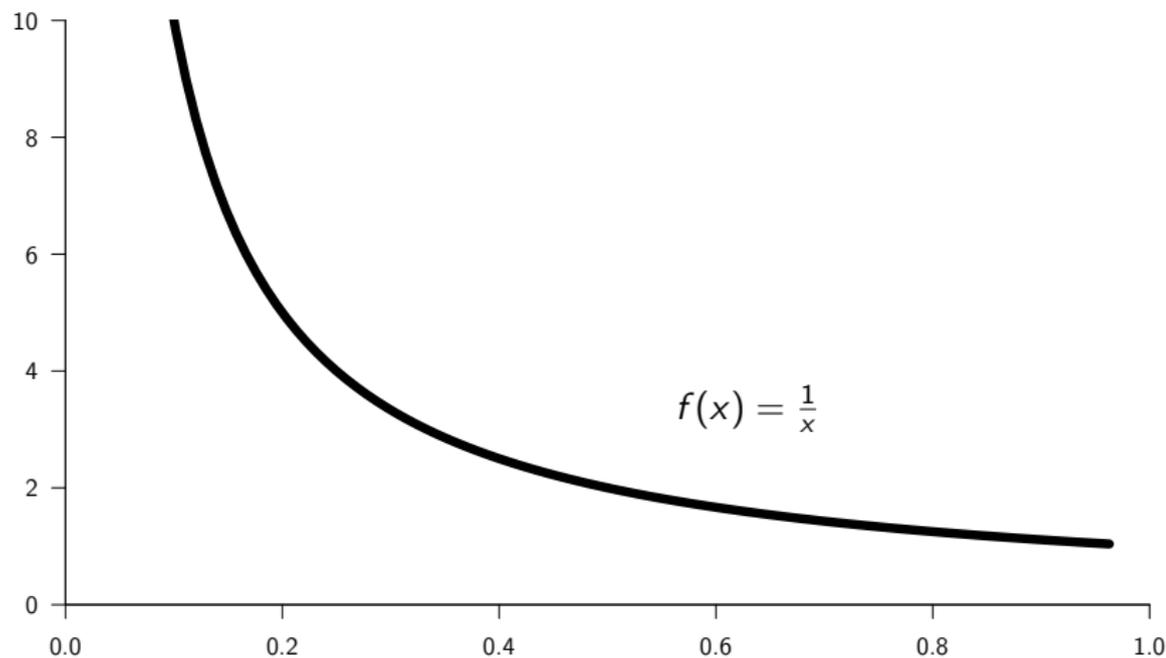(*i.e.*, the function f is bounded on E iff $\{f(x) : x \in E\}$ is a bounded set.)

Note: This is a *global* property because there is a single bound M associated with the entire set E.

Example

The function $f(x) = 1/(1 + x^2)$ is bounded on $\mathbb{R}$. *e.g.*, $M = 1$.

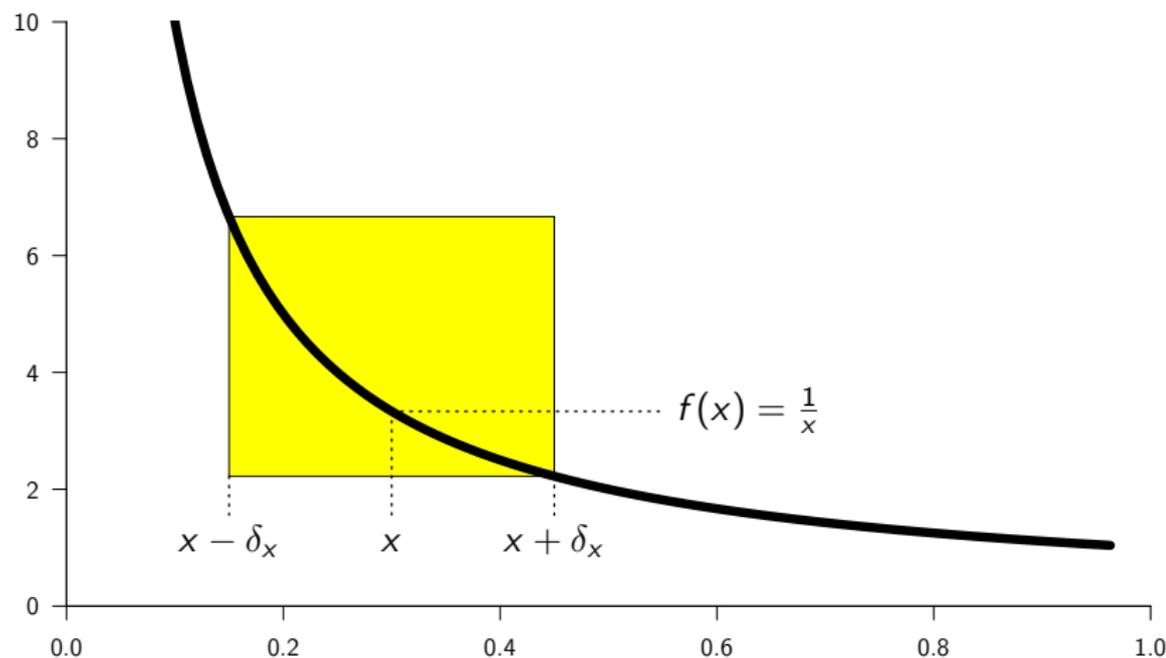# Local vs. Global properties

$f(x) = 1/x$ is <u>not</u> bounded on the interval $E = (0, 1)$.

Local vs. Global properties

$f(x) = 1/x$ is **locally bounded** on the interval $E = (0, 1)$,
i.e., $\forall x \in E$, $\exists \delta_x, M_x > 0 \ni |f(t)| \leq M_x \ \forall t \in (x - \delta_x, x + \delta_x)$.

Local vs. Global properties

Definition (Locally bounded at a point)

A real-valued function f is **locally bounded** at the point x if there is a neighbourhood of x in which f is bounded, *i.e.*, there exists $\delta_x > 0$ and $M_x > 0$ such that $|f(t)| \leq M_x$ for all $t \in (x - \delta_x, x + \delta_x)$.

Definition (Locally bounded on a set)

A real-valued function f is **locally bounded** on the set E if f is locally bounded at each point $x \in E$.

Note: The size of the neighbourhood (δ_x) and the local bound (M_x) depend on the point x.

Local vs. Global properties

Example (Function that is not even locally bounded)

Give an example of a function that is defined on the interval $(0, 1)$ but is <u>not</u> locally bounded on $(0, 1)$.

Let's construct a function $f(x)$ that is defined on $(0, 1)$ but is not locally bounded at one point, say $x = \frac{1}{2}$.

$f(x)$ must blow up $x = \frac{1}{2}$. Let's make f look like $1/x$, but shifted so the blowup is at $x = \frac{1}{2}$.

$$
f(x) = \begin{cases} \dfrac{1}{x - \frac{1}{2}} & x \neq \frac{1}{2}, \\[2mm] 0 & x = \frac{1}{2}. \end{cases}
$$

Local vs. Global properties

Example (Function that is a mess near 0)

Give an example of a function $f(x)$ that is defined everywhere, yet in <u>any</u> neighbourhood of the origin there are infinitely many points at which f is <u>not</u> locally bounded.

Please do poll: Topology: Local boundedness

Consider $S(x) = \begin{cases} \sin \frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

$S(x)$ is bounded on $\mathbb{R}$, hence locally bounded at every point.

Consider $T(x) = \begin{cases} \tan \frac{1}{x}, & x \neq 0 \text{ and } \cos \frac{1}{x} \neq 0 \\ 0, & x = 0 \text{ or } \cos \frac{1}{x} = 0 \end{cases}$

$T(x)$ is not locally bounded at points where $\cos \frac{1}{x} = 0$, *i.e.*, for $\frac{1}{x} = \frac{\pi}{2} + n\pi$, $n \in \mathbb{Z}$. There are infinitely many such points in any neighbourhood of $x = 0$.

Poll

- Go to
  https://www.childsmath.ca/childsa/forms/main_login.php

- Click on Math 3A03

- Click on Take Class Poll

- Fill in poll **Topology: Locally bounded nowhere?**

- ⟨Submit⟩.

Local vs. Global properties

Extra Challenge Problem:
Is there a function $f : \mathbb{R} \to \mathbb{R}$ that is <u>not</u>
locally bounded <u>anywhere</u>?

Local vs. Global properties

- What condition(s) rule out such pathological behaviour?

- When does a property holding locally (near any given point in a set) imply that it holds globally (for the set as a whole)?

- For example: What condition(s) must a set $E \subseteq \mathbb{R}$ satisfy in order that a function f that is locally bounded on E is necessarily bounded on E?

- We will see that the condition we are seeking is that the set E must be "***compact***" ...

Compactness

Recall the Bolzano-Weierstrass theorem, for sequences of real numbers:

Theorem (Bolzano-Weierstrass theorem for sequences)

Every bounded sequence in $\mathbb{R}$ contains a convergent subsequence.

For *any set of real numbers*, we define:

Definition (Bolzano-Weierstrass property)

A set $E \subseteq \mathbb{R}$ is said to have the ***Bolzano-Weierstrass property*** iff any sequence of points chosen from E has a subsequence that converges to a point in E.

Compactness

Theorem (Bolzano-Weierstrass theorem for sets)

A set $E \subseteq \mathbb{R}$ has the Bolzano-Weierstrass property iff E is closed and bounded.

(TBB Theorem 4.21, p. 241)

Proof of $\Longleftarrow$.

Suppose E is closed and bounded. Let $\{x_n\}$ be a sequence in E. Since E is bounded, the usual Bolzano-Weierstrass theorem implies that there is a subsequence $\{x_{n_k}\}$ that converges. If $\{x_{n_k}\}$ is eventually constant, then its limit is a point in E, so we're done. Otherwise, $\{x_{n_k}\}$ must converge to an accumulation point of E. But E is closed, so it contains all its accumulation points, including the limit of $\{x_{n_k}\}$. Thus, again, we have a subsequence of $\{x_n\}$ that converges to a point in E. $\qquad\qquad\square$

Compactness

Theorem (Bolzano-Weierstrass theorem for sets)

A set $E \subseteq \mathbb{R}$ has the Bolzano-Weierstrass property iff E is closed and bounded.

Proof of $\implies$ (Part 1)

Let's prove the contrapositive, *i.e.,* If E is either not bounded or not closed then E does not have the Bolzano-Weierstrass property. ***Suppose E is unbounded.*** In particular, suppose E is not bounded above (the argument is similar if E is not bounded below). We will construct a sequence in E that has no convergent subsequence. Pick a point $x_1 \in E$ such that $x_1 > 1$ (which is possible because E is not bounded above). Also, since E is not bounded above, we can find $x_2 \in E$ such that $x_2 > x_1 + 1 > 2$. More generally, given $x_k \in E$ we can find $x_{k+1} \in E$ such that $x_{k+1} > x_k + 1 > k + 1$. The sequence $\{x_n\}$ constructed in this way is increasing and diverges to ∞ (since $x_n > n$ for all $n \in \mathbb{N}$). Moreover, this is true of any subsequence of $\{x_n\}$. $\therefore$ E ***does not have the Bolzano-Weierstrass property.***

Compactness

Theorem (Bolzano-Weierstrass theorem for sets)

A set $E \subseteq \mathbb{R}$ has the Bolzano-Weierstrass property iff E is closed and bounded.

Proof of $\implies$ (Part 2)

Now **suppose E is not closed.** Then there must be a sequence $\{x_n\}$ in E such that $\{x_n\}$ converges to a point <u>not</u> in E. If E has the Bolzano-Weierstrass property, then $\{x_n\}$ has a subsequence that converges to a point in E. But every subsequence of a convergent sequence must converge to the same point as the full sequence, and the full sequence converges to a point <u>not</u> in E! Thus, if E is not closed **then it does not have the Bolzano-Weierstrass property.** $\qquad\qquad\square$

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Mathematics 3A03
Real Analysis I

Instructor: David Earn

Lecture 16
Topology of $\mathbb{R}$ IV
Thursday 12 February 2026

Announcements

- The participation deadline for Assignment 3 is Tuesday 24 Feb 2025 @ 2:25pm.
- The midterm TEST is on Thursday 26 February 2026 @ 7:00pm in T13 123.
- The room is booked for 7:00–10:00 pm, but the intention is that a reasonable amount of time for the test is 90 minutes. You will be given double time.
- Last year's (winter 2025) midterm test and solutions are posted on the course web site.

Compactness

Theorem (Bolzano-Weierstrass theorem for sets)

A set $E \subseteq \mathbb{R}$ has the Bolzano-Weierstrass property iff E is closed and bounded.

Notes:

- Why do we need both *closed* and *bounded*? Why don't we need *closed* in the Bolzano-Weierstrass theorem for sequences?
  - Because the statement of the Bolzano-Weierstrass theorem for sequences doesn't require the limit of the convergent subsequence to be in the set!
- The Bolzano-Weierstrass theorem for sets implies that "If $E \subseteq \mathbb{R}$ is bounded then its closure $\overline{E}$ has the Bolzano-Weierstrass property".
  - The Bolzano-Weierstrass theorem for sequences is a special case of this statement because any convergent sequence together with its limit is a closed set.

Compactness

Definition (Open Cover)

Let $E \subseteq \mathbb{R}$ and let $\mathcal{U}$ be a family of open intervals. If for every $x \in E$ there exists at least one interval $U \in \mathcal{U}$ such that $x \in U$, *i.e.,*

$$E \subseteq \bigcup \{U : U \in \mathcal{U}\},$$

then $\mathcal{U}$ is called an ***open cover*** of E.

Example (Open covers of $\mathbb{N}$)

Give examples of open covers of $\mathbb{N}$.

- $\mathcal{U} = \left\{ \left(n - \frac{1}{2}, n + \frac{1}{2} \right) : n = 1, 2, \ldots \right\}$
- $\mathcal{U} = \{(0, \infty)\}$
- $\mathcal{U} = \{(0, \infty), \mathbb{R}, (\pi, 27)\}$

Compactness

Example (Open covers of $\{\frac{1}{n} : n \in \mathbb{N}\}$)

- $\mathcal{U} = \{(0,1),\, (0,2),\, \mathbb{R},\, (\pi, 27)\}$
- $\mathcal{U} = \{(0,2)\}$
- $\mathcal{U} = \left\{ \left(\frac{1}{n},\, \frac{1}{n} + \frac{3}{4} \right) : n = 1, 2, \ldots \right\}$

Example (Open covers of $[0,1]$)

- $\mathcal{U} = \{(-2,2)\}$
- $\mathcal{U} = \{(-\frac{1}{2}, \frac{1}{2}),\, (0,2)\}$
- $\mathcal{U} = \left\{ \left(\frac{1}{n}, 2 \right) : n = 1, 2, \ldots \right\} \cup \left\{ \left(-\frac{1}{2}, \frac{1}{2} \right) \right\}$

Poll

- Go to
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- Click on Math 3A03

- Click on Take Class Poll

- Fill in poll **Topology: Open covers of inverse squares**

- Submit .

Compactness

Definition (Heine-Borel Property)

A set $E \subseteq \mathbb{R}$ is said to have the ***Heine-Borel property*** if every open cover of E can be reduced to a <u>finite</u> subcover. That is, if $\mathcal{U}$ is an open cover of E, then there exists a <u>finite</u> subfamily $\{U_1, U_2, \ldots, U_n\} \subseteq \mathcal{U}$, such that $E \subseteq U_1 \cup U_2 \cup \cdots \cup U_n$.

When does <u>any</u> open cover of a set E have a <u>finite</u> subcover?

Theorem (Heine-Borel Theorem)

A set $E \subseteq \mathbb{R}$ has the Heine-Borel property iff E is both closed and bounded.

(TBB pp. 249–250)

Compactness

Definition (Compact Set)

A set $E \subseteq \mathbb{R}$ is said to be ***compact*** if it has any of the following equivalent properties:

1 E is closed and bounded.

2 E has the Bolzano-Weierstrass property.

3 E has the Heine-Borel property.

Note: In spaces other than $\mathbb{R}$, these three properties are <u>not</u> necessarily equivalent. Usually the Heine-Borel property is taken as the definition of compactness.

Compactness

Example

Prove that the interval $(0, 1]$ is <u>not</u> compact by showing that it is <u>not</u> closed or <u>not</u> bounded.

Example

Prove that the interval $(0, 1]$ is <u>not</u> compact by showing that it does <u>not</u> have the Bolzano-Weierstrass property.

Example

Prove that the interval $(0, 1]$ is <u>not</u> compact by showing that it does <u>not</u> have the Heine-Borel property.

Poll

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- Click on Math 3A03

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- Fill in poll **Topology: Compactness**

- Submit.

Topological concepts that we have covered

- Interval

- Neighbourhood

- Deleted neighbourhood

- Interior point

- Isolated point

- Accumulation point

- Boundary point

- Boundary

- Closed set

- Closure

- Open set

- Interior

- Complement

- Compact

Compactness

Classic <u>non-trivial</u> compactness argument:

Theorem (Compact $\implies$ bounded if locally bounded)

Let E be a compact subset of $\mathbb{R}$. If $f : E \to \mathbb{R}$ is locally bounded on E then f is bounded on E.

Proof via Bolzano-Weierstrass.

Suppose f is locally bounded on E, and that E satisfies the Bolzano-Weierstrass property. Suppose further, in order to derive a contradiction, that f is <u>not</u> bounded on E. Then there is some sequence $\{x_n\}$ in E such that $|f(x_n)| > n$ (otherwise $|f(x_n)| \leq N$ for some $N \in \mathbb{N}$, so N would be a bound). By the Bolzano-Weierstrass property, $\{x_n\}$ has a subsequence $\{x_{n_k}\}$ such that $x_{n_k} \to L \in E$. Since f is locally bounded, there exist $\delta > 0$ and $M > 0$ such that $|f(x)| \leq M$ for all $x \in (L - \delta, L + \delta)$. But $x_{n_k} \to L$, so for all sufficiently large k, $x_{n_k} \in (L - \delta, L + \delta)$. Yet for sufficiently large k, $|f(x_{n_k})| > n_k \geq k > M$. $\Rightarrow\Leftarrow$. Hence f must, in fact, be bounded on E. $\qquad\square$

Compactness

Theorem (Compact $\implies$ bounded if locally bounded)

Let E be a compact subset of $\mathbb{R}$. If $f : E \to \mathbb{R}$ is locally bounded on E then f is bounded on E.

Proof via Heine-Borel.

Since f is locally bounded, each $x \in E$ lies in some open interval U_x such that $|f(t)| \le M_x$ for all $t \in U_x$. The collection $\mathcal{U} = \{U_x : x \in E\}$ is an open cover of E. But E satisfies the Heine-Borel property, so $\mathcal{U}$ contains a <u>finite</u> subcover, say $\{U_{x_1}, \ldots, U_{x_n}\}$. We can therefore find a bound for f on <u>all</u> of E. Let $M = \max\{M_{x_1}, \ldots, M_{x_n}\}$. Then $|f(x)| \le M$ for all $x \in E$. $\square$

Proof via Heine-Borel is much easier!

McMaster
University

Mathematics
and Statistics

$$\int_M d\omega = \int_{\partial M} \omega$$

Mathematics 3A03
Real Analysis I

Instructor: David Earn

Lecture 17
Topology of $\mathbb{R}$ V
Friday 13 February 2026

Announcements

- The participation deadline for Assignment 3 is Tuesday 24 Feb 2025 @ 2:25pm.
- The midterm TEST is on Thursday 26 February 2026 @ 7:00pm in T13 123.
- The room is booked for 7:00–10:00 pm, but the intention is that a reasonable amount of time for the test is 90 minutes. You will be given double time.
- Last year's (winter 2025) midterm test and solutions are posted on the course web site.

Compactness

Theorem (Compact $\implies$ bounded if locally bounded)

Let E be a compact subset of $\mathbb{R}$. If $f : E \to \mathbb{R}$ is locally bounded on E then f is bounded on E.

Example (Converse of above theorem)

Let $E \subseteq \mathbb{R}$. If every function $f : E \to \mathbb{R}$ that is locally bounded on E is bounded on E, then E is compact.

<u>Note</u>: The contrapositive of the converse is: If $E \subseteq \mathbb{R}$ is <u>not</u> compact then $\exists f : E \to \mathbb{R} \,\not\!\ni\, f$ is locally bounded on E but <u>not</u> bounded on E.

Compactness

Example ("bounded if locally bounded" $\implies$ compact)

Consider the contrapositive: if $E \subseteq \mathbb{R}$ is not compact then there exists a function $f : E \to \mathbb{R}$ that is locally bounded but not bounded on E.

Suppose E is not compact. Then either E is not bounded or not closed.

Suppose first that E is not bounded, and let $f(x) = x$. Then f is locally bounded at any point $x \in E$, since f is bounded on any neighbourhood of x that has finite width; but $f(E) = E$ is an unbounded set, so f is an unbounded function on E.

Now suppose E is not closed. Then there is an accumulation point of E that not in E, *i.e.*, there exists a sequence $\{x_n\} \subseteq E$ such that $x_n \to L \notin E$ as $n \to \infty$. Since $L \notin E$, for any $x \in E$ we have $x - L \neq 0$, so we can define $f(x) = \frac{1}{x-L}$ for all $x \in E$. But since L is a limit point of E, there are points in E that are arbitrarily close to L, hence points at which $|f(x)|$ is arbitrarily large. Thus, f is unbounded on E. $\qquad\square$

Complements and Closures problem

Example

How many distinct sets can be obtained from $E = [0, 1]$ by applying the complement and closure operations?

Consider this sequence of sets:

$$
\begin{aligned}
E_1 &= [0, 1], \\
E_2 &= E_1^c = (-\infty, 0) \cup (1, \infty), \\
E_3 &= \overline{E_2} = (-\infty, 0] \cup [1, \infty), \\
E_4 &= E_3^c = (0, 1), \\
E_5 &= \overline{E_4} = E_1.
\end{aligned}
$$

Is the answer 4 for any set $E \subseteq \mathbb{R}$?

Poll

- Go to
  https://www.childsmath.ca/childsa/forms/main_login.php

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- Click on Take Class Poll

- Fill in poll **Topology: Complements and Closures**

- Submit .

Complements and Closures problem

Extra Challenge Problem

If $E \subseteq \mathbb{R}$, how many distinct sets can be obtained by taking
complements or closures of E and its successors? Put another
way, if $\{E_n\}$ is a sequence of sets produced by taking the
complement or closure of the previous set, how many distinct
sets can such a sequence contain? If the answer is finite, find
a set E that generates the maximum number in this way.

Midterm Test

What you need to know:

- Everything discussed in class, including all definitions/concepts and theorems/lemmas/corollaries.

- Everything in assignments and solutions to assignments. *Make sure you fully understand all the solutions to all the problems in all the assignments.*

- Most—but <u>not all</u>—of the material that you are responsible for is covered in the chapters of the textbooks indicated on the course web page. You are <u>not</u> responsible for material in the textbooks that was <u>not</u> mentioned in lectures, tutorials or assignments.

- It is essential that you understand how to use the definitions and theorems to construct proofs.

Instructor: David Earn   Mathematics 3A03   Real Analysis I