McMaster University

Mathematics and Statistics

$$\int_M d\omega = \int_{\partial M} \omega$$

Mathematics 3A03
Real Analysis I

Instructor: David Earn

Lecture 14
Topology of $\mathbb{R}$ I
Monday 10 February 2025

Announcements

- Solutions to Assignment 2 have been reposted after correcting some errors (thanks to Kieran for spotting these).
  - There were typos in Q2(b) and Q4.
  - Q3 was incomplete because I assumed $f(x)$ was positive.
- Assignment 3 is posted on the course web site. Participation deadline is Monday 24 Feb 2025 @ 11:25 am.
- I reposted the slides for Lecture 13. Slide 79 now contains a sequence of hints for proving π is irrational.
- The midterm TEST is on Thursday 27 February 2025 @ 7:00pm in Hamilton Hall 302.
- The room is booked for 7:00–10:00 pm, but the intention is that a reasonable amount of time for the test is one hour. You will be given double time.

Topology of $\mathbb{R}$

Intervals

Open interval:

$$(a, b) = \{x : a < x < b\}$$

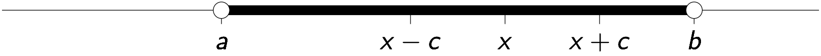
Closed interval:

$$[c, d] = \{x : c \le x \le d\}$$

Half-open interval:

$$(e, f] = \{x : e < x \le f\}$$

Interior point



Definition (Interior point)

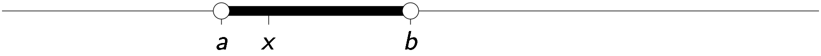
If $E \subseteq \mathbb{R}$ then x is an ***interior point*** of E if x lies in an open interval that is contained in E, *i.e.,*

$$\exists c > 0 \quad \ni \quad (x - c, x + c) \subset E.$$

Interior point examples

Set E	Interior points?
$(-1,1)$	
$[0,1]$	
$\mathbb{N}$	
$\mathbb{R}$	
$\mathbb{Q}$	
$(-1,1) \cup [0,1]$	
$(-1,1) \setminus \{\frac{1}{2}\}$	

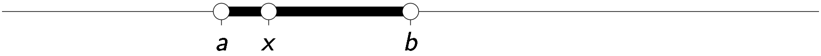
Neighbourhood



Definition (Neighbourhood)

A ***neighbourhood*** of a point $x \in \mathbb{R}$ is an open interval containing x.

Deleted neighbourhood

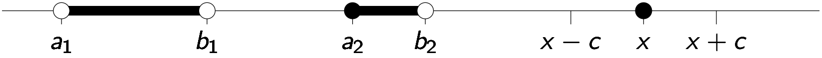


$$(a, b) \setminus \{x\}$$

Definition (Deleted neighbourhood)

A ***deleted neighbourhood*** of a point $x \in \mathbb{R}$ is a set formed by removing x from a neighbourhood of x.

Isolated point



$$E = (a_1, b_1) \cup [a_2, b_2) \cup \{x\}$$

Definition (Isolated point)

If $x \in E \subseteq \mathbb{R}$ then x is an ***isolated point*** of E if there is a neighbourhood of x for which the only point in E is x itself, *i.e.*,

$$\exists c > 0 \quad \ni \quad (x - c, x + c) \cap E \; = \; \{x\} \, .$$

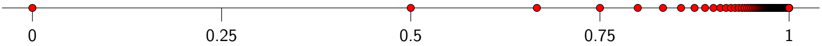
Poll

- Go to
  https://www.childsmath.ca/childsa/forms/main_login.php
- Click on Math 3A03
- Click on Take Class Poll
- Fill in poll **Topology: Isolated points**
- `Submit`.

Isolated point examples

Set E	**Isolated points?**
$(-1,1)$	
$[0,1]$	
$\mathbb{N}$	
$\mathbb{R}$	
$\mathbb{Q}$	
$(-1,1)\cup[0,1]$	
$(-1,1)\setminus\{\frac{1}{2}\}$	

Accumulation point



$$E = \left\{ 1 - \frac{1}{n} : n \in \mathbb{N} \right\}$$

Definition (Accumulation Point or Limit Point or Cluster Point)

If $E \subseteq \mathbb{R}$ then x is an ***accumulation point*** of E if every neighbourhood of x contains infinitely many points of E,

$$\textit{i.e.,} \qquad \forall c > 0 \qquad (x - c, x + c) \cap (E \setminus \{x\}) \neq \varnothing \, .$$

Note:

- It is possible but not necessary that $x \in E$.
- The shorthand condition is equivalent to saying that every deleted neighbourhood of x contains at least one point of E.

Poll

- Go to
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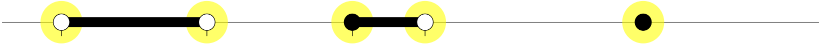
- Click on Math 3A03
- Click on Take Class Poll
- Fill in poll **Topology: Accumulation points**
- Submit.

Accumulation point examples

Set E	**Accumulation points?**
$(-1,1)$	
$[0,1]$	
$\mathbb{N}$	
$\mathbb{R}$	
$\mathbb{Q}$	
$(-1,1)\cup[0,1]$	
$(-1,1)\setminus\{\frac{1}{2}\}$	
$\left\{1-\frac{1}{n} : n\in\mathbb{N}\right\}$	

Boundary point



Definition (Boundary Point)

If $E \subseteq \mathbb{R}$ then x is a ***boundary point*** of E if every neighbourhood of x contains at least one point of E and at least one point not in E, *i.e.*,

$$\forall c > 0 \qquad (x-c, x+c) \cap E \neq \varnothing \quad \wedge \quad (x-c, x+c) \cap (\mathbb{R} \setminus E) \neq \varnothing\,.$$

Note: It is possible but not necessary that $x \in E$.

Definition (Boundary)

If $E \subseteq \mathbb{R}$ then the ***boundary*** of E, denoted ∂E, is the set of all boundary points of E.

Boundary point examples

Set E	Boundary points?
$(-1,1)$	
$[0,1]$	
$\mathbb{N}$	
$\mathbb{R}$	
$\mathbb{Q}$	
$(-1,1)\cup[0,1]$	
$(-1,1)\setminus\{\frac{1}{2}\}$	
$\left\{1-\frac{1}{n} : n\in\mathbb{N}\right\}$	

Closed set

Definition (Closed set)

A set $E \subseteq \mathbb{R}$ is ***closed*** if it contains all of its accumulation points.

Definition (Closure of a set)

If $E \subseteq \mathbb{R}$ and E' is the set of accumulation points of E then the ***closure*** of E is

$$\overline{E} = E \cup E'.$$

Note: If the set E has no accumulation points, then E is closed because there are no accumulation points to check.

Open set

Definition (Open set)

A set $E \subseteq \mathbb{R}$ is ***open*** if every point of E is an interior point.

Definition (Interior of a set)

If $E \subseteq \mathbb{R}$ then the ***interior*** of E, denoted $\text{int}(E)$ or E°, is the set of all interior points of E.

Poll

- Go to https://www.childsmath.ca/childsa/forms/main_login.php
- Click on Math 3A03
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- Fill in poll **Topology: Open or Closed**
- `Submit`.

Examples

Set E	Closed?	Open?	$\overline{E}$	E°	∂E
$(-1,1)$					
$[0,1]$					
$\mathbb{N}$					
$\mathbb{R}$					
$\varnothing$					
$\mathbb{Q}$					
$(-1,1)\cup[0,1]$					
$(-1,1)\setminus\{\frac{1}{2}\}$					
$\left\{1-\frac{1}{n} : n\in\mathbb{N}\right\}$					

McMaster University

Mathematics and Statistics

$$\int_M d\omega = \int_{\partial M} \omega$$

Mathematics 3A03
Real Analysis I

Instructor: David Earn

Lecture 15
Topology of $\mathbb{R}$ II
Wednesday 12 February 2025

Announcements (same as on Monday, 10 Feb 2025)

- Solutions to Assignment 2 have been reposted after correcting some errors (thanks to Kieran for spotting these).
  - There were typos in Q2(b) and Q4.
  - Q3 was incomplete because I assumed $f(x)$ was positive.
- Assignment 3 is posted on the course web site. Participation deadline is Monday 24 Feb 2025 @ 11:25 am.
- I reposted the slides for Lecture 13. Slide 79 now contains a sequence of hints for proving π is irrational.
- The midterm TEST is on Thursday 27 February 2025 @ 7:00pm in Hamilton Hall 302.
- The room is booked for 7:00–10:00 pm, but the intention is that a reasonable amount of time for the test is one hour. You will be given double time.

Topological concepts covered so far

- Interval
- Neighbourhood
- Deleted neighbourhood
- Interior point
- Isolated point
- Accumulation point
- Boundary point
- Boundary
- Closed set
- Closure
- Open set
- Interior

Equivalent definitions

Example (Closure)

For $E \subseteq \mathbb{R}$, prove $E \cup E' = E \cup \partial E$, so $\overline{E}$ can be defined either way.

Proof: We must show $E \cup E' \subseteq E \cup \partial E$ and $E \cup E' \supseteq E \cup \partial E$.

- $\subseteq$ Suppose $x \in E \cup E'$. If $x \in E$ then $x \in E \cup A$ for any set A. In particular, $x \in E \cup \partial E$. Alternatively, suppose $x \notin E$, *i.e.*, $x \in E^{c}$. Then, since $x \in E \cup E'$, it must be that $x \in E'$, which means that any neighbourhood of x contains a point of E. But $x \in E^{c}$, so any such neighbourhood also contains a point of E^{c} (namely x). Therefore, $x \in \partial E \subseteq E \cup \partial E$.
- $\supseteq$ Suppose $x \in E \cup \partial E$. If $x \in E$ then $x \in E \cup A$ for any set A. In particular, $x \in E \cup E'$. Alternatively, suppose $x \notin E$, *i.e.*, $x \in E^{c}$. Then, since $x \in E \cup \partial E$, it must be that $x \in \partial E$, which means that any neighbourhood of x contains a point of E. But that point is not x, since $x \notin E$. Thus, any *deleted* neighbourhood of x contains a point of E, *i.e.*, $x \in E' \subseteq E \cup E'$. □

Question: In the proof above, did we use any properties of $\mathbb{R}$?

Poll

- Go to https://www.childsmath.ca/childsa/forms/main_login.php
- Click on Math 3A03
- Click on Take Class Poll
- Fill in poll **Topology: The most general type of open set**
- `Submit`.

Component intervals of open sets

What does the most general open set look like?

Theorem (Component intervals)

If G is an open subset of $\mathbb{R}$ and $G \neq \varnothing$ then there is a unique (possibly finite) sequence of disjoint open intervals $\{(a_n, b_n)\}$ such that

$$G = (a_1, b_1) \cup (a_2, b_2) \cup \cdots \cup (a_n, b_n) \cup \cdots ,$$

$$\textit{i.e.,} \qquad G = \bigcup_{n=1}^{\infty} (a_n, b_n) \, .$$

*The open intervals (a_n, b_n) are said to be the **component intervals** of G.*

(TBB Theorem 4.15, p. 231)

Component intervals of open sets

Main ideas of proof of component intervals theorem:

- $x \in G \implies x$ is an interior point of $G \implies$
  - some neighbourhood of x is contained in G, *i.e.*, $\exists c > 0$ such that $(x - c, x + c) \subseteq G$
  - $\exists$ a largest neighbourhood of x that is contained in G: this "***component of G***" is $I_x = (\alpha, \beta)$, where
    $$\alpha = \inf\{a : (a, x] \subset G\}, \qquad \beta = \sup\{b : [x, b) \subset G\}$$
  - Every component I_x contains a rational number, *i.e.*, $\exists r \in I_x \cap \mathbb{Q}$
  - But for any $y \in I_x$, we have $I_y = I_x$
  - $\therefore$ If $r_1, r_2 \in \mathbb{Q}$ and $r_1, r_2 \in I_x$ then $I_{r_1} = I_{r_2} = I_x$.
  - Components with different endpoints cannot overlap (they would contradict the inf and sup) so distinct components are disjoint
- $\therefore$ We can index (label) each component with a (unique) rational number
- $\therefore$ There are at most countably many intervals that make up G (*i.e.*, G is the union of a sequence of disjoint intervals)
- See textbook for details (TBB Theorem 4.15, p. 231).

Open vs. Closed Sets

Definition (Complement of a set of real numbers)

If $E \subseteq \mathbb{R}$ then the ***complement*** of E is the set

$$E^{\mathsf{c}} = \{x \in \mathbb{R} : x \notin E\}\,.$$

Theorem (Open vs. Closed)

If $E \subseteq \mathbb{R}$ then E is open iff E^{c} is closed.

(TBB Theorem 4.16)

Open vs. Closed Sets

Theorem (Properties of open sets of real numbers)

1. *The sets $\mathbb{R}$ and $\varnothing$ are open.*
2. *Any intersection of a finite number of open sets is open.*
3. *Any union of an arbitrary collection of open sets is open.*
4. *The complement of an open set is closed.*

(TBB Theorem 4.17)

Theorem (Properties of closed sets of real numbers)

1. *The sets $\mathbb{R}$ and $\varnothing$ are closed.*
2. *Any union of a finite number of closed sets is closed.*
3. *Any intersection of an arbitrary collection of closed sets is closed.*
4. *The complement of a closed set is open.*

(TBB Theorem 4.18)

Local vs. Global properties

Definition (Bounded function)

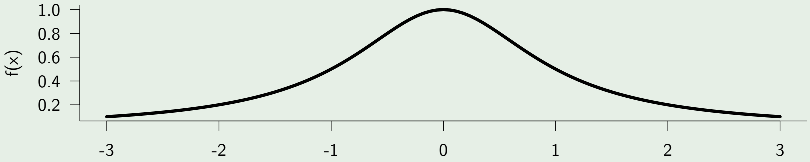
A real-valued function f is ***bounded*** on the set E if there exists $M > 0$ such that $|f(x)| \leq M$ for all $x \in E$.

(*i.e.*, the function f is bounded on E iff $\{f(x) : x \in E\}$ is a bounded set.)

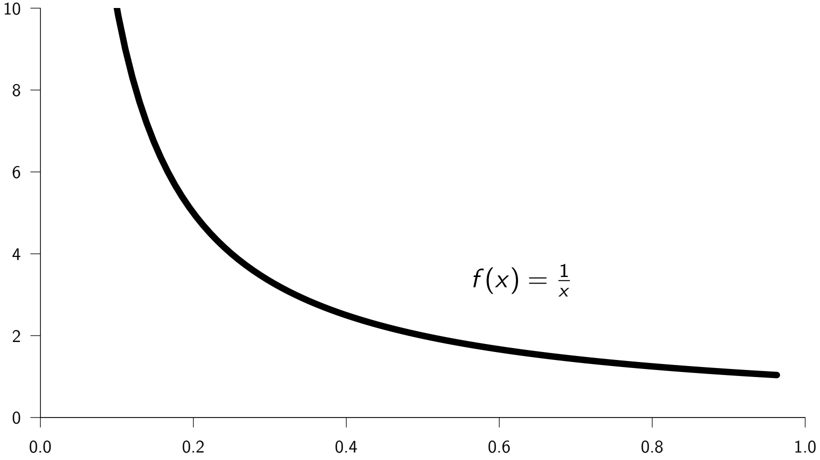
Note: This is a *global* property because there is a single bound M associated with the entire set E.

Example

The function $f(x) = 1/(1 + x^2)$ is bounded on $\mathbb{R}$. *e.g.*, $M = 1$.

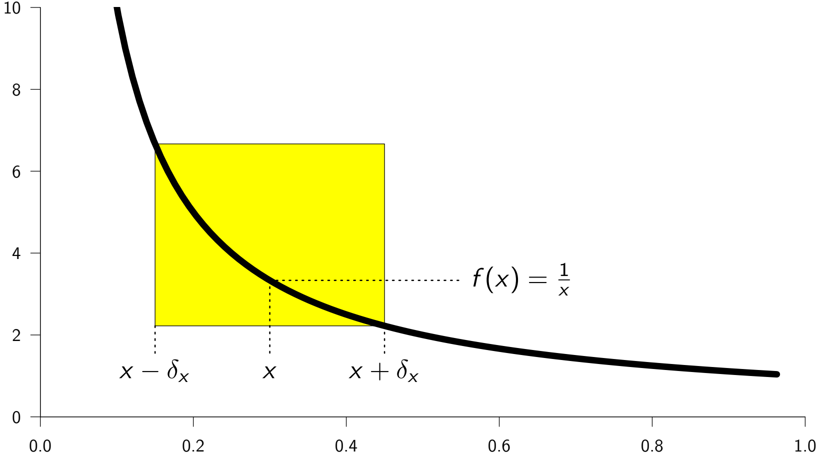


Local vs. Global properties



$f(x) = 1/x$ is not bounded on the interval $E = (0, 1)$.

Local vs. Global properties



$f(x) = 1/x$ is ***locally bounded*** on the interval $E = (0, 1)$,
i.e., $\forall x \in E$, $\exists \delta_x, M_x > 0 \ni |f(t)| \leq M_x \; \forall t \in (x - \delta_x, x + \delta_x)$.

Local vs. Global properties

Definition (Locally bounded at a point)

A real-valued function f is ***locally bounded*** at the point x if there is a neighbourhood of x in which f is bounded, *i.e.*, there exists $\delta_x > 0$ and $M_x > 0$ such that $|f(t)| \leq M_x$ for all $t \in (x - \delta_x, x + \delta_x)$.

Definition (Locally bounded on a set)

A real-valued function f is ***locally bounded*** on the set E if f is locally bounded at each point $x \in E$.

Note: The size of the neighbourhood (δ_x) and the local bound (M_x) depend on the point x.

Local vs. Global properties

Example (Function that is not even locally bounded)

Give an example of a function that is defined on the interval $(0, 1)$ but is not locally bounded on $(0, 1)$.

Let's construct a function $f(x)$ that is defined on $(0, 1)$ but is not locally bounded at one point, say $x = \frac{1}{2}$.

$f(x)$ must blow up $x = \frac{1}{2}$. Let's make f look like $1/x$, but shifted so the blowup is at $x = \frac{1}{2}$.

$$f(x) = \begin{cases} \dfrac{1}{x - \frac{1}{2}} & x \neq \frac{1}{2}, \\ 0 & x = \frac{1}{2}. \end{cases}$$

Local vs. Global properties

Example (Function that is a mess near 0)

Give an example of a function $f(x)$ that is defined everywhere, yet in any neighbourhood of the origin there are infinitely many points at which f is not locally bounded.

Please do poll: Topology: Local boundedness

Consider $S(x) = \begin{cases} \sin\frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$

$S(x)$ is bounded on $\mathbb{R}$, hence locally bounded at every point.

Consider $T(x) = \begin{cases} \tan\frac{1}{x} & x \neq 0 \text{ and } \cos\frac{1}{x} \neq 0 \\ 0 & x = 0 \text{ or } \cos\frac{1}{x} = 0 \end{cases}$

$T(x)$ is not locally bounded at points where $\cos\frac{1}{x} = 0$, *i.e.*, for $\frac{1}{x} = \frac{\pi}{2} + n\pi$, $n \in \mathbb{Z}$. There are infinitely many such points in any neighbourhood of $x = 0$.

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Mathematics and Statistics

$$\int_M d\omega = \int_{\partial M} \omega$$

Mathematics 3A03
Real Analysis I

Instructor: David Earn

Lecture 16
Topology of $\mathbb{R}$ III
Friday 14 February 2025

Poll

- Go to
  https://www.childsmath.ca/childsa/forms/main_login.php
- Click on Math 3A03
- Click on Take Class Poll
- Fill in poll **Poll on polls**
- `Submit`.

Announcements

- Poll for Assignment 3 will be live after class today. Participation deadline is Monday 24 Feb 2025 @ 11:25 am.
- I improved the sketch of the component intervals theorem proof on slide 28.
- The midterm TEST is on Thursday 27 February 2025 @ 7:00pm in Hamilton Hall 302.
- The room is booked for 7:00–10:00 pm, but the intention is that a reasonable amount of time for the test is one hour. You will be given double time.

Poll

- Go to
  https://www.childsmath.ca/childsa/forms/main_login.php
- Click on Math 3A03
- Click on Take Class Poll
- Fill in poll **Topology: Locally bounded nowhere?**
- Submit.

Local vs. Global properties

Extra Challenge Problem:

Is there a function $f : \mathbb{R} \to \mathbb{R}$ that is not locally bounded anywhere?

Local vs. Global properties

- What condition(s) rule out such pathological behaviour?
- When does a property holding locally (near any given point in a set) imply that it holds globally (for the set as a whole)?
- For example: What condition(s) must a set $E \subseteq \mathbb{R}$ satisfy in order that a function f that is locally bounded on E is necessarily bounded on E?
- We will see that the condition we are seeking is that the set E must be "***compact***" ...

Compactness

Recall the Bolzano-Weierstrass theorem, for sequences of real numbers:

Theorem (Bolzano-Weierstrass theorem for sequences)

Every bounded sequence in $\mathbb{R}$ contains a convergent subsequence.

For *any set of real numbers*, we define:

Definition (Bolzano-Weierstrass property)

A set $E \subseteq \mathbb{R}$ is said to have the ***Bolzano-Weierstrass property*** iff any sequence of points chosen from E has a subsequence that converges to a point in E.

Compactness

Theorem (Bolzano-Weierstrass theorem for sets)

A set $E \subseteq \mathbb{R}$ has the Bolzano-Weierstrass property iff E is closed and bounded.

(TBB Theorem 4.21, p. 241)

Proof of $\Longleftarrow$.

Suppose E is closed and bounded. Let $\{x_n\}$ be a sequence in E. Since E is bounded, the usual Bolzano-Weierstrass theorem implies that there is a subsequence $\{x_{n_k}\}$ that converges. If $\{x_{n_k}\}$ is eventually constant, then its limit is a point in E, so we're done. Otherwise, $\{x_{n_k}\}$ must converge to an accumulation point of E. But E is closed, so it contains all its accumulation points, including the limit of $\{x_{n_k}\}$. Thus, again, we have a subsequence of $\{x_n\}$ that converges to a point in E. $\square$

Compactness

Theorem (Bolzano-Weierstrass theorem for sets)

A set $E \subseteq \mathbb{R}$ has the Bolzano-Weierstrass property iff E is closed and bounded.

Proof of $\implies$ (Part 1)

Let's prove the contrapositive, *i.e.,* If E is either not bounded or not closed then E does not have the Bolzano-Weierstrass property. ***Suppose E is unbounded.*** In particular, suppose E is not bounded above (the argument is similar if E is not bounded below). We will construct a sequence in E that has no convergent subsequence. Pick a point $x_1 \in E$ such that $x_1 > 1$ (which is possible because E is not bounded above). Also, since E is not bounded above, we can find $x_2 \in E$ such that $x_2 > x_1 + 1 > 2$. More generally, given $x_k \in E$ we can find $x_{k+1} \in E$ such that $x_{k+1} > x_k + 1 > k + 1$. The sequence $\{x_n\}$ constructed in this way is increasing and diverges to ∞ (since $x_n > n$ for all $n \in \mathbb{N}$). Moreover, this is true of any subsequence of $\{x_n\}$. $\therefore$ ***E does not have the Bolzano-Weierstrass property.***

Compactness

Theorem (Bolzano-Weierstrass theorem for sets)

A set $E \subseteq \mathbb{R}$ has the Bolzano-Weierstrass property iff E is closed and bounded.

Proof of $\implies$ (Part 2)

Now ***suppose E is not closed.*** Then there must be a sequence $\{x_n\}$ in E such that $\{x_n\}$ converges to a point not in E. If E has the Bolzano-Weierstrass property, then $\{x_n\}$ has a subsequence that converges to a point in E. But every subsequence of a convergent sequence must converge to the same point as the full sequence, and the full sequence converges to a point not in E! Thus, if E is not closed ***then it does not have the Bolzano-Weierstrass property.*** $\square$

Compactness

Theorem (Bolzano-Weierstrass theorem for sets)

A set $E \subseteq \mathbb{R}$ has the Bolzano-Weierstrass property iff E is closed and bounded.

Notes:

- Why do we need both *closed* and *bounded*? Why don't we need *closed* in the Bolzano-Weierstrass theorem for sequences?
  - Because the statement of the Bolzano-Weierstrass theorem for sequences doesn't require the limit of the convergent subsequence to be in the set!
- The Bolzano-Weierstrass theorem for sets implies that "If $E \subseteq \mathbb{R}$ is bounded then its closure $\overline{E}$ has the Bolzano-Weierstrass property".
  - The Bolzano-Weierstrass theorem for sequences is a special case of this statement because any convergent sequence together with its limit is a closed set.

Compactness

Definition (Open Cover)

Let $E \subseteq \mathbb{R}$ and let $\mathcal{U}$ be a family of open intervals. If for every $x \in E$ there exists at least one interval $U \in \mathcal{U}$ such that $x \in U$, *i.e.*,

$$E \subseteq \bigcup \{U : U \in \mathcal{U}\},$$

then $\mathcal{U}$ is called an ***open cover*** of E.

Example (Open covers of $\mathbb{N}$)

Give examples of open covers of $\mathbb{N}$.

- $\mathcal{U} = \left\{ \left(n - \frac{1}{2}, n + \frac{1}{2} \right) : n = 1, 2, \ldots \right\}$
- $\mathcal{U} = \{(0, \infty)\}$
- $\mathcal{U} = \{(0, \infty), \mathbb{R}, (\pi, 27)\}$

Compactness

Example (Open covers of $\{\frac{1}{n} : n \in \mathbb{N}\}$)

- $\mathcal{U} = \{(0,1),\ (0,2),\ \mathbb{R},\ (\pi,27)\}$
- $\mathcal{U} = \{(0,2)\}$
- $\mathcal{U} = \left\{\left(\frac{1}{n},\ \frac{1}{n}+\frac{3}{4}\right) : n = 1,2,\ldots\right\}$

Example (Open covers of $[0,1]$)

- $\mathcal{U} = \{(-2,2)\}$
- $\mathcal{U} = \{(-\frac{1}{2},\frac{1}{2}),\ (0,2)\}$
- $\mathcal{U} = \left\{\left(\frac{1}{n},2\right) : n = 1,2,\ldots\right\} \cup \left\{\left(-\frac{1}{2},\frac{1}{2}\right)\right\}$

Poll

- Go to
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- Click on Math 3A03
- Click on Take Class Poll
- Fill in poll **Topology: Open covers of inverse squares**
- Submit.

Compactness

Definition (Heine-Borel Property)

A set $E \subseteq \mathbb{R}$ is said to have the ***Heine-Borel property*** if every open cover of E can be reduced to a finite subcover. That is, if $\mathcal{U}$ is an open cover of E, then there exists a finite subfamily $\{U_1, U_2, \ldots, U_n\} \subseteq \mathcal{U}$, such that $E \subseteq U_1 \cup U_2 \cup \cdots \cup U_n$.

When does any open cover of a set E have a finite subcover?

Theorem (Heine-Borel Theorem)

A set $E \subseteq \mathbb{R}$ has the Heine-Borel property iff E is both closed and bounded.

(TBB pp. 249–250)

Compactness

Definition (Compact Set)

A set $E \subseteq \mathbb{R}$ is said to be ***compact*** if it has any of the following equivalent properties:

1. E is closed and bounded.
2. E has the Bolzano-Weierstrass property.
3. E has the Heine-Borel property.

Note: In spaces other than $\mathbb{R}$, these three properties are not necessarily equivalent. Usually the Heine-Borel property is taken as the definition of compactness.

Compactness

Example

Prove that the interval $(0,1]$ is not compact by showing that it is not closed or not bounded.

Example

Prove that the interval $(0,1]$ is not compact by showing that it does not have the Bolzano-Weierstrass property.

Example

Prove that the interval $(0,1]$ is not compact by showing that it does not have the Heine-Borel property.

Poll

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Compactness

Classic non-trivial compactness argument:

Theorem (Compact $\implies$ bounded if locally bounded)

Let E be a compact subset of $\mathbb{R}$. If $f : E \to \mathbb{R}$ is locally bounded on E then f is bounded on E.

Proof via Bolzano-Weierstrass.

E satisfies the Bolzano-Weierstrass property, *i.e.*, any sequence in E has a subsequence that converges to a point in E. Suppose, in order to derive a contradiction, that f is not bounded on E. Then there is some sequence $\{x_n\}$ in E such that $|f(x_n)| > n$ (otherwise $|f(x_n)| \le N$ for some $N \in \mathbb{N}$, so N would be a bound). By the Bolzano-Weierstrass property, $\{x_n\}$ has a subsequence $\{x_{n_k}\}$ such that $x_{n_k} \to L \in E$. Since f is locally bounded, there exist $\delta > 0$ and $M > 0$ and such that $|f(x)| \le M$ for all $x \in (L - \delta, L + \delta)$. But $x_{n_k} \to L$, so for all sufficiently large k, $x_{n_k} \in (L - \delta, L + \delta)$. Yet for sufficiently large k, $|f(x_{n_k})| > n_k \ge k > M$. $\Rightarrow\Leftarrow$. Hence f must, in fact, be bounded on E. □

Compactness

Theorem (Compact $\implies$ bounded if locally bounded)

Let E be a compact subset of $\mathbb{R}$. If $f : E \to \mathbb{R}$ is locally bounded on E then f is bounded on E.

Proof via Heine-Borel.

Since f is locally bounded, each $x \in E$ lies in some open interval U_x such that $|f(t)| \le M_x$ for all $t \in U_x$. The collection $\mathcal{U} = \{U_x : x \in E\}$ is an open cover of E. But E satisfies the Heine-Borel property, so $\mathcal{U}$ contains a finite subcover, say $\{U_{x_1}, \ldots, U_{x_n}\}$. We can therefore find a bound for f on all of E. Let $M = \max\{M_{x_1}, \ldots, M_{x_n}\}$. Then $|f(x)| \le M$ for all $x \in E$. □

Proof via Heine-Borel is much easier!

Compactness

Theorem (Compact $\implies$ bounded if locally bounded)

Let E be a compact subset of $\mathbb{R}$. If $f : E \to \mathbb{R}$ is locally bounded on E then f is bounded on E.

Example (Converse of above theorem)

Let $E \subseteq \mathbb{R}$. If every function $f : E \to \mathbb{R}$ that is locally bounded on E is bounded on E, then E is compact.

Note: Contrapositive of converse is: If $E \subseteq \mathbb{R}$ is not compact then $\exists f : E \to \mathbb{R} \ni f$ is locally bounded on E but not bounded on E.