

13 Topology of \mathbb{R}

14 Topology of \mathbb{R} II

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16 Topology of \mathbb{R} IV



Mathematics
and Statistics

$$\int_M d\omega = \int_{\partial M} \omega$$

Mathematics 3A03 Real Analysis I

Instructor: David Earn

Lecture 13
Topology of \mathbb{R}
Thursday 5 February 2026

Announcements

- Solutions to [Assignment 2](#) have been posted.
- [Assignment 3](#) is posted on the course web site. Participation deadline is Tuesday 24 Feb 2025 @ 2:25pm.
- The midterm TEST is on Thursday 26 February 2026 @ 7:00pm in T13 123.
- The room is booked for 7:00–10:00 pm, but the intention is that a reasonable amount of time for the test is 90 minutes. You will be given double time.

Topology of \mathbb{R}

Intervals



Open interval:

$$(a, b) = \{x : a < x < b\}$$

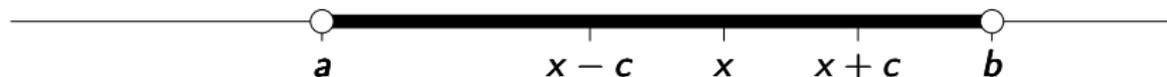
Closed interval:

$$[c, d] = \{x : c \leq x \leq d\}$$

Half-open interval:

$$(e, f] = \{x : e < x \leq f\}$$

Interior point



Definition (Interior point)

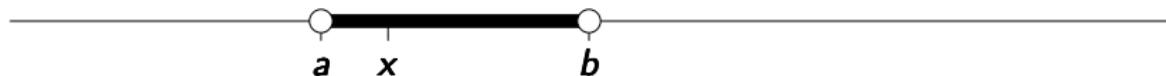
If $E \subseteq \mathbb{R}$ then x is an *interior point* of E if x lies in an open interval that is contained in E , i.e.,

$$\exists c > 0 \quad \vdash \quad (x - c, x + c) \subset E.$$

Interior point examples

Set E	Interior points?
$(-1, 1)$	
$[0, 1]$	
\mathbb{N}	
\mathbb{R}	
\mathbb{Q}	
$(-1, 1) \cup [0, 1]$	
$(-1, 1) \setminus \{\frac{1}{2}\}$	

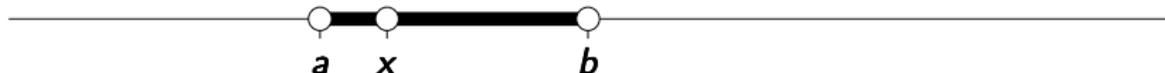
Neighbourhood



Definition (Neighbourhood)

A **neighbourhood** of a point $x \in \mathbb{R}$ is an open interval containing x .

Deleted neighbourhood



$$(a, b) \setminus \{x\}$$

Definition (Deleted neighbourhood)

A *deleted neighbourhood* of a point $x \in \mathbb{R}$ is a set formed by removing x from a neighbourhood of x .

Isolated point



$$E = (a_1, b_1) \cup [a_2, b_2) \cup \{x\}$$

Definition (Isolated point)

If $x \in E \subseteq \mathbb{R}$ then x is an **isolated point** of E if there is a neighbourhood of x for which the only point in E is x itself, *i.e.*,

$$\exists c > 0 \quad \text{) } (x - c, x + c) \cap E = \{x\}.$$

Poll

- Go to
https://www.childsmath.ca/childsa/forms/main_login.php
- Click on **Math 3A03**
- Click on **Take Class Poll**
- Fill in poll **Topology: Isolated points**
- .

Isolated point examples

Set E	Isolated points?
$(-1, 1)$	
$[0, 1]$	
\mathbb{N}	
\mathbb{R}	
\mathbb{Q}	
$(-1, 1) \cup [0, 1]$	
$(-1, 1) \setminus \{\frac{1}{2}\}$	

Accumulation point



$$E = \left\{ 1 - \frac{1}{n} : n \in \mathbb{N} \right\}$$

Definition (Accumulation Point or Limit Point or Cluster Point)

If $E \subseteq \mathbb{R}$ then x is an **accumulation point** of E if every neighbourhood of x contains infinitely many points of E ,

$$\text{i.e., } \forall c > 0 \quad (x - c, x + c) \cap (E \setminus \{x\}) \neq \emptyset.$$

Note:

- It is possible but not necessary that $x \in E$.
- The shorthand condition is equivalent to saying that every deleted neighbourhood of x contains at least one point of E .

Poll

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- Fill in poll **Topology: Accumulation points**
- .

Accumulation point examples

Set E	Accumulation points?
$(-1, 1)$	
$[0, 1]$	
\mathbb{N}	
\mathbb{R}	
\mathbb{Q}	
$(-1, 1) \cup [0, 1]$	
$(-1, 1) \setminus \{\frac{1}{2}\}$	
$\{1 - \frac{1}{n} : n \in \mathbb{N}\}$	

Boundary point



Definition (Boundary Point)

If $E \subseteq \mathbb{R}$ then x is a **boundary point** of E if every neighbourhood of x contains at least one point of E and at least one point not in E , i.e.,

$$\forall c > 0 \quad \begin{aligned} (x - c, x + c) \cap E &\neq \emptyset \\ \wedge \quad (x - c, x + c) \cap (\mathbb{R} \setminus E) &\neq \emptyset. \end{aligned}$$

Note: It is possible but not necessary that $x \in E$.

Definition (Boundary)

If $E \subseteq \mathbb{R}$ then the **boundary** of E , denoted ∂E , is the set of all boundary points of E .

Boundary point examples

Set E	Boundary points?
$(-1, 1)$	
$[0, 1]$	
\mathbb{N}	
\mathbb{R}	
\mathbb{Q}	
$(-1, 1) \cup [0, 1]$	
$(-1, 1) \setminus \{\frac{1}{2}\}$	
$\{1 - \frac{1}{n} : n \in \mathbb{N}\}$	

Closed set



Definition (Closed set)

A set $E \subseteq \mathbb{R}$ is **closed** if it contains all of its accumulation points.

Definition (Closure of a set)

If $E \subseteq \mathbb{R}$ and E' is the set of accumulation points of E then the **closure** of E is

$$\bar{E} = E \cup E'.$$

Note: If the set E has no accumulation points, then E is closed because there are no accumulation points to check.

Poll

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- Fill in poll **Topology: Open or Closed**
- .

Examples

Set E	Closed?	Open?	\bar{E}	E°	∂E
$(-1, 1)$					
$[0, 1]$					
\mathbb{N}					
\mathbb{R}					
\emptyset					
\mathbb{Q}					
$(-1, 1) \cup [0, 1]$					
$(-1, 1) \setminus \{\frac{1}{2}\}$					
$\{1 - \frac{1}{n} : n \in \mathbb{N}\}$					



Mathematics
and Statistics

$$\int_M d\omega = \int_{\partial M} \omega$$

Mathematics 3A03 Real Analysis I

Instructor: David Earn

Lecture 14
Topology of \mathbb{R}^n
Friday 6 February 2026

Announcements

- Today's Mathematics Colloquium (immediately after this class in HH-305) should be accessible to 3A students. Title "[Is mathematics obsolete?](#)"
- Solutions to [Assignment 2](#) have been posted.
- [Assignment 3](#) is posted on the course web site. Participation deadline is Tuesday 24 Feb 2025 @ 2:25pm.
- The midterm TEST is on Thursday 26 February 2026 @ 7:00pm in T13 123.
- The room is booked for 7:00–10:00 pm, but the intention is that a reasonable amount of time for the test is 90 minutes. You will be given double time.
- Last year's (winter 2025) midterm test and solutions are posted on the [course web site](#).

Topological concepts covered so far

- Interval
- Neighbourhood
- Deleted neighbourhood
- Interior point
- Isolated point
- Accumulation point
- Boundary point
- Boundary
- Closed set
- Closure
- Open set
- Interior

Equivalent definitions

Example (Closure)

For $E \subseteq \mathbb{R}$, prove $E \cup E' = E \cup \partial E$, so \bar{E} can be defined either way.

Proof: We must show $E \cup E' \subseteq E \cup \partial E$ and $E \cup E' \supseteq E \cup \partial E$.

- \subseteq Suppose $x \in E \cup E'$. If $x \in E$ then $x \in E \cup A$ for any set A . In particular, $x \in E \cup \partial E$. Alternatively, suppose $x \notin E$, i.e., $x \in E^c$. Then, since $x \in E \cup E'$, it must be that $x \in E'$, which means that any neighbourhood of x contains a point of E . But $x \in E^c$, so any such neighbourhood also contains a point of E^c (namely x). Therefore, $x \in \partial E \subseteq E \cup \partial E$.
- \supseteq Suppose $x \in E \cup \partial E$. If $x \in E$ then $x \in E \cup A$ for any set A . In particular, $x \in E \cup E'$. Alternatively, suppose $x \notin E$, i.e., $x \in E^c$. Then, since $x \in E \cup \partial E$, it must be that $x \in \partial E$, which means that any neighbourhood of x contains a point of E . But that point is not x , since $x \notin E$. Thus, any *deleted* neighbourhood of x contains a point of E , i.e., $x \in E' \subseteq E \cup E'$. □

Question: In the proof above, did we use any properties of \mathbb{R} ?

Poll

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- Click on **Math 3A03**
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- Fill in poll **Topology: The most general type of open set**
- .

Component intervals of open sets

What does the most general **open set** look like?

Theorem (Component intervals)

If G is an open subset of \mathbb{R} and $G \neq \emptyset$ then there is a unique (possibly finite) sequence of disjoint open intervals $\{(a_n, b_n)\}$ such that

$$G = (a_1, b_1) \cup (a_2, b_2) \cup \cdots \cup (a_n, b_n) \cup \cdots ,$$

$$\text{i.e., } G = \bigcup_{n=1}^{\infty} (a_n, b_n) .$$

The open intervals (a_n, b_n) are said to be the **component intervals** of G .

(TBB [Theorem 4.15](#), p. 231)

Component intervals of open sets

Main ideas of proof of [component intervals theorem](#):

- $x \in G \implies x \in G^\circ \implies x \in (a, b) \subset G$ for some $a, b \in \mathbb{R}$
- \exists a largest neighbourhood of x in G : this **component of G** is $I_x = (\alpha, \beta)$, where $\alpha = \inf\{a : (a, x] \subset G\}$ and $\beta = \sup\{b : [x, b) \subset G\}$
- Components with different endpoints cannot overlap (they would contradict the inf and sup) so *distinct components are disjoint*
- Every component I_x *contains a rational number*, i.e., $\exists r \in I_x \cap \mathbb{Q}$
- But \mathbb{Q} is *countable*, and we can write an *explicit bijection* $q : \mathbb{N} \rightarrow \mathbb{Q}$
- \therefore We can *label* each component interval I of G with a *unique* $n \in \mathbb{N}$. The label of component I is $n_I = \min\{n \in \mathbb{N} : q(n) \in I\}$ (well-ordering)
- \therefore We can list the components of G in a *sequence* $\{I_1, I_2, I_3, \dots\}$
- \therefore There are at most countably many intervals that make up G i.e., *G is the union of a sequence of disjoint intervals*
- See [textbook](#) for a standard proof (TBB [Theorem 4.15](#), p. 231).

Note: (For those studying logic and set theory.) This argument relies ultimately on the [well-ordering principle](#) for \mathbb{N} (or, equivalently, the principle of mathematical induction for \mathbb{N}) but it does not require the [Axiom of Choice](#).

Open vs. Closed Sets

Definition (Complement of a set of real numbers)

If $E \subseteq \mathbb{R}$ then the **complement** of E is the set

$$E^c = \{x \in \mathbb{R} : x \notin E\}.$$

Theorem (Open vs. Closed)

If $E \subseteq \mathbb{R}$ then E is open iff E^c is closed.

(TBB [Theorem 4.16](#))

Open vs. Closed Sets

Theorem (Properties of open sets of real numbers)

- 1 The sets \mathbb{R} and \emptyset are open.
- 2 Any *intersection* of a *finite* number of open sets is open.
- 3 Any *union* of an *arbitrary* collection of open sets is open.
- 4 The complement of an open set is closed.

(TBB Theorem 4.17)

Theorem (Properties of closed sets of real numbers)

- 1 The sets \mathbb{R} and \emptyset are closed.
- 2 Any *union* of a *finite* number of closed sets is closed.
- 3 Any *intersection* of an *arbitrary* collection of closed sets is closed.
- 4 The complement of a closed set is open.

(TBB Theorem 4.18)



Mathematics
and Statistics

$$\int_M d\omega = \int_{\partial M} \omega$$

Mathematics 3A03 Real Analysis I

Instructor: David Earn

Lecture 15
Topology of \mathbb{R}^n
Tuesday 10 February 2026

Announcements

- Solutions to [Assignment 2](#) have been re-posted (some typos have been fixed).
- [Assignment 3](#) is posted on the course web site. Participation deadline is Tuesday 24 Feb 2025 @ 2:25pm.
- The midterm TEST is on Thursday 26 February 2026 @ 7:00pm in T13 123.
- The room is booked for 7:00–10:00 pm, but the intention is that a reasonable amount of time for the test is 90 minutes. You will be given double time.
- Last year's (winter 2025) midterm test and solutions are posted on the [course web site](#).
- I have revised my [sketch](#) of the proof of the [component intervals theorem](#).
- For those who are interested in foundations of mathematics: the simplest result in real analysis that depends on the Axiom of Choice is the existence of a non-measurable set (take Math 4A...)

Local vs. Global properties

Definition (Bounded function)

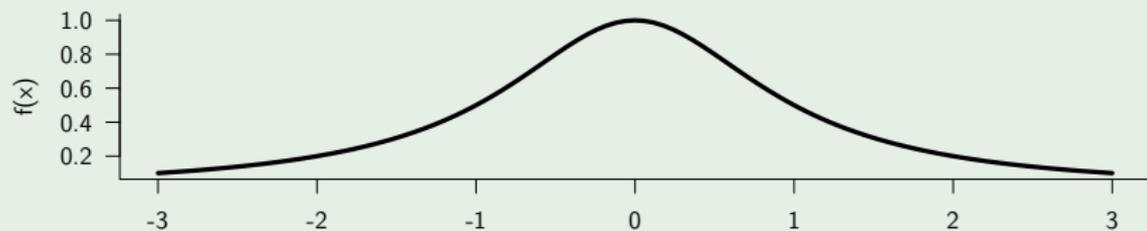
A real-valued function f is **bounded** on the set E if there exists $M > 0$ such that $|f(x)| \leq M$ for all $x \in E$.

(i.e., the function f is bounded on E iff $\{f(x) : x \in E\}$ is a bounded set.)

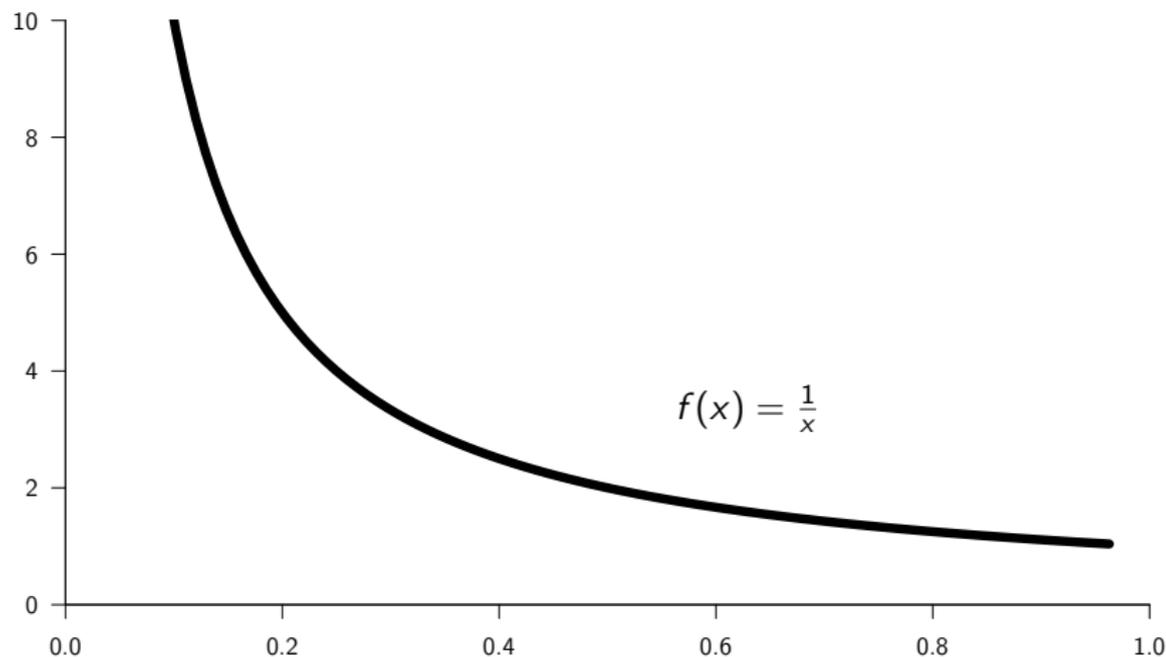
Note: This is a **global** property because there is a single bound M associated with the entire set E .

Example

The function $f(x) = 1/(1 + x^2)$ is bounded on \mathbb{R} . e.g., $M = 1$.

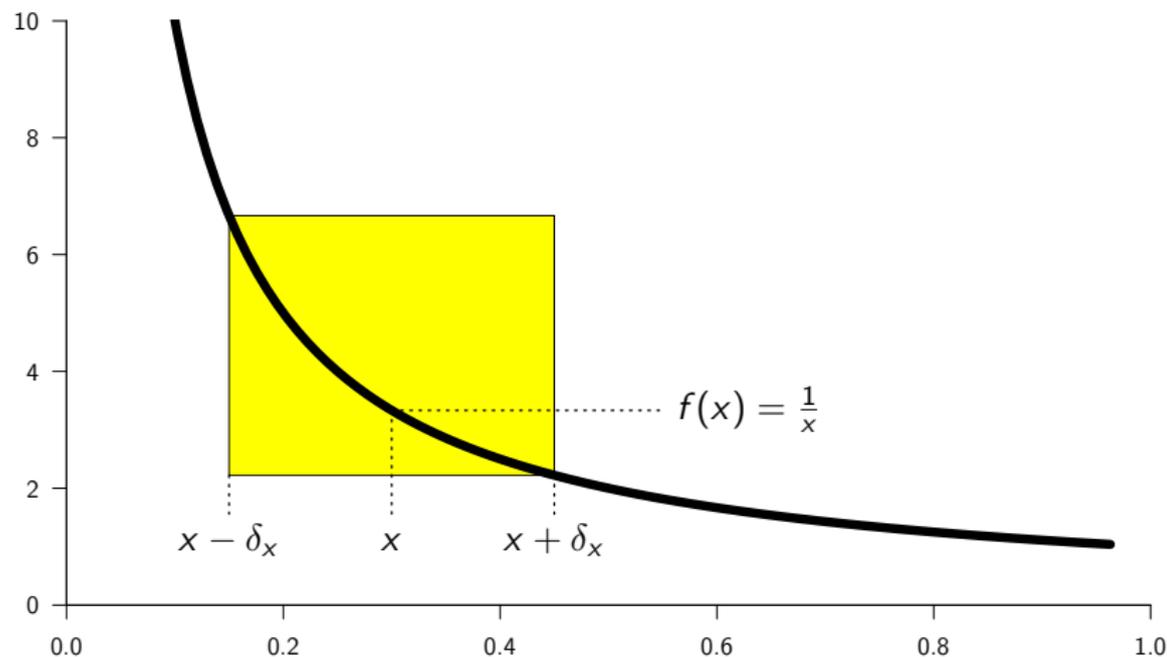


Local vs. Global properties



$f(x) = 1/x$ is not bounded on the interval $E = (0, 1)$.

Local vs. Global properties



$f(x) = 1/x$ is **locally bounded** on the interval $E = (0, 1)$,
i.e., $\forall x \in E, \exists \delta_x, M_x > 0 \mid |f(t)| \leq M_x \forall t \in (x - \delta_x, x + \delta_x)$.

Local vs. Global properties

Definition (Locally bounded at a point)

A real-valued function f is **locally bounded** at the point x if there is a neighbourhood of x in which f is bounded, *i.e.*, there exists $\delta_x > 0$ and $M_x > 0$ such that $|f(t)| \leq M_x$ for all $t \in (x - \delta_x, x + \delta_x)$.

Definition (Locally bounded on a set)

A real-valued function f is **locally bounded** on the set E if f is locally bounded at each point $x \in E$.

Note: The size of the neighbourhood (δ_x) and the local bound (M_x) depend on the point x .

Local vs. Global properties

Example (Function that is not even locally bounded)

Give an example of a function that is defined on the interval $(0, 1)$ but is not **locally bounded** on $(0, 1)$.

Let's construct a function $f(x)$ that is defined on $(0, 1)$ but is not locally bounded at one point, say $x = \frac{1}{2}$.

$f(x)$ must blow up $x = \frac{1}{2}$. Let's make f look like $1/x$, but shifted so the blowup is at $x = \frac{1}{2}$.

$$f(x) = \begin{cases} \frac{1}{x - \frac{1}{2}} & x \neq \frac{1}{2}, \\ 0 & x = \frac{1}{2}. \end{cases}$$

Local vs. Global properties

Example (Function that is a mess near 0)

Give an example of a function $f(x)$ that is defined everywhere, yet in any neighbourhood of the origin there are infinitely many points at which f is not locally bounded.

Please do poll: Topology: Local boundedness

$$\text{Consider } S(x) = \begin{cases} \sin \frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

$S(x)$ is bounded on \mathbb{R} , hence locally bounded at every point.

$$\text{Consider } T(x) = \begin{cases} \tan \frac{1}{x}, & x \neq 0 \text{ and } \cos \frac{1}{x} \neq 0 \\ 0, & x = 0 \text{ or } \cos \frac{1}{x} = 0 \end{cases}$$

$T(x)$ is not locally bounded at points where $\cos \frac{1}{x} = 0$, i.e., for $\frac{1}{x} = \frac{\pi}{2} + n\pi$, $n \in \mathbb{Z}$. There are infinitely many such points in any neighbourhood of $x = 0$.

Poll

- Go to
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- Click on **Take Class Poll**
- Fill in poll **Topology: Locally bounded nowhere?**
- .

Local vs. Global properties

Extra Challenge Problem:

Is there a function $f : \mathbb{R} \rightarrow \mathbb{R}$ that is not locally bounded anywhere?

Local vs. Global properties

- What condition(s) rule out such pathological behaviour?
- When does a property holding locally (near any given point in a set) imply that it holds globally (for the set as a whole)?
- For example: What condition(s) must a set $E \subseteq \mathbb{R}$ satisfy in order that a function f that is **locally bounded** on E is necessarily **bounded** on E ?
- We will see that the condition we are seeking is that the set E must be “**compact**” ...

Compactness

Recall the Bolzano-Weierstrass theorem, for sequences of real numbers:

Theorem (Bolzano-Weierstrass theorem for sequences)

Every bounded sequence in \mathbb{R} contains a convergent subsequence.

For *any set of real numbers*, we define:

Definition (Bolzano-Weierstrass property)

A set $E \subseteq \mathbb{R}$ is said to have the ***Bolzano-Weierstrass property*** iff any sequence of points chosen from E has a subsequence that converges to a point in E .

Compactness

Theorem (Bolzano-Weierstrass theorem for sets)

A set $E \subseteq \mathbb{R}$ has the *Bolzano-Weierstrass property* iff E is closed and bounded.

(TBB [Theorem 4.21](#), p. 241)

Proof of \Leftarrow .

Suppose E is closed and bounded. Let $\{x_n\}$ be a sequence in E . Since E is bounded, the usual Bolzano-Weierstrass theorem implies that there is a subsequence $\{x_{n_k}\}$ that converges. If $\{x_{n_k}\}$ is eventually constant, then its limit is a point in E , so we're done. Otherwise, $\{x_{n_k}\}$ must converge to an accumulation point of E . But E is closed, so it contains all its accumulation points, including the limit of $\{x_{n_k}\}$. Thus, again, we have a subsequence of $\{x_n\}$ that converges to a point in E . \square

Compactness

Theorem (Bolzano-Weierstrass theorem for sets)

A set $E \subseteq \mathbb{R}$ has the *Bolzano-Weierstrass property* iff E is closed and bounded.

Proof of \implies (Part 1)

Let's prove the contrapositive, *i.e.*, If E is either not bounded or not closed then E does not have the Bolzano-Weierstrass property. **Suppose E is unbounded.** In particular, suppose E is not bounded above (the argument is similar if E is not bounded below). We will construct a sequence in E that has no convergent subsequence. Pick a point $x_1 \in E$ such that $x_1 > 1$ (which is possible because E is not bounded above). Also, since E is not bounded above, we can find $x_2 \in E$ such that $x_2 > x_1 + 1 > 2$. More generally, given $x_k \in E$ we can find $x_{k+1} \in E$ such that $x_{k+1} > x_k + 1 > k + 1$. The sequence $\{x_n\}$ constructed in this way is increasing and diverges to ∞ (since $x_n > n$ for all $n \in \mathbb{N}$). Moreover, this is true of any subsequence of $\{x_n\}$. \therefore **E does not have the Bolzano-Weierstrass property.**

Compactness

Theorem (Bolzano-Weierstrass theorem for sets)

A set $E \subseteq \mathbb{R}$ has the *Bolzano-Weierstrass property* iff E is closed and bounded.

Proof of \implies (Part 2)

Now *suppose E is not closed*. Then there must be a sequence $\{x_n\}$ in E such that $\{x_n\}$ converges to a point not in E . If E has the Bolzano-Weierstrass property, then $\{x_n\}$ has a subsequence that converges to a point in E . But every subsequence of a convergent sequence must converge to the same point as the full sequence, and the full sequence converges to a point not in E ! Thus, if E is not closed *then it does not have the Bolzano-Weierstrass property*. □

Compactness

Theorem (Bolzano-Weierstrass theorem for sets)

A set $E \subseteq \mathbb{R}$ has the *Bolzano-Weierstrass property* iff E is closed and bounded.

Notes:

- Why do we need both *closed* and *bounded*? Why don't we need *closed* in the *Bolzano-Weierstrass theorem* for sequences?
 - Because the statement of the *Bolzano-Weierstrass theorem* for sequences doesn't require the limit of the convergent subsequence to be in the set!
- The *Bolzano-Weierstrass theorem for sets* implies that "If $E \subseteq \mathbb{R}$ is bounded then its closure \overline{E} has the Bolzano-Weierstrass property".
 - The *Bolzano-Weierstrass theorem for sequences* is a special case of this statement because any convergent sequence together with its limit is a closed set.

Compactness

Definition (Open Cover)

Let $E \subseteq \mathbb{R}$ and let \mathcal{U} be a family of open intervals. If for every $x \in E$ there exists at least one interval $U \in \mathcal{U}$ such that $x \in U$, i.e.,

$$E \subseteq \bigcup \{U : U \in \mathcal{U}\},$$

then \mathcal{U} is called an *open cover* of E .

Example (Open covers of \mathbb{N})

Give examples of open covers of \mathbb{N} .

- $\mathcal{U} = \left\{ \left(n - \frac{1}{2}, n + \frac{1}{2} \right) : n = 1, 2, \dots \right\}$
- $\mathcal{U} = \{(0, \infty)\}$
- $\mathcal{U} = \{(0, \infty), \mathbb{R}, (\pi, 27)\}$

Compactness

Example (Open covers of $\{\frac{1}{n} : n \in \mathbb{N}\}$)

- $\mathcal{U} = \{(0, 1), (0, 2), \mathbb{R}, (\pi, 27)\}$
- $\mathcal{U} = \{(0, 2)\}$
- $\mathcal{U} = \left\{ \left(\frac{1}{n}, \frac{1}{n} + \frac{3}{4} \right) : n = 1, 2, \dots \right\}$

Example (Open covers of $[0, 1]$)

- $\mathcal{U} = \{(-2, 2)\}$
- $\mathcal{U} = \left\{ \left(-\frac{1}{2}, \frac{1}{2} \right), (0, 2) \right\}$
- $\mathcal{U} = \left\{ \left(\frac{1}{n}, 2 \right) : n = 1, 2, \dots \right\} \cup \left\{ \left(-\frac{1}{2}, \frac{1}{2} \right) \right\}$

Poll

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- Click on **Math 3A03**
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- Fill in poll **Topology: Open covers of inverse squares**
- .

Compactness

Definition (Heine-Borel Property)

A set $E \subseteq \mathbb{R}$ is said to have the *Heine-Borel property* if every open cover of E can be reduced to a finite subcover. That is, if \mathcal{U} is an open cover of E , then there exists a finite subfamily $\{U_1, U_2, \dots, U_n\} \subseteq \mathcal{U}$, such that $E \subseteq U_1 \cup U_2 \cup \dots \cup U_n$.

When does any open cover of a set E have a finite subcover?

Theorem (Heine-Borel Theorem)

A set $E \subseteq \mathbb{R}$ has the *Heine-Borel property* iff E is both *closed* and *bounded*.

(TBB pp. 249–250)

Compactness

Definition (Compact Set)

A set $E \subseteq \mathbb{R}$ is said to be **compact** if it has any of the following equivalent properties:

- 1 E is **closed** and bounded.
- 2 E has the **Bolzano-Weierstrass property**.
- 3 E has the **Heine-Borel property**.

Note: In spaces other than \mathbb{R} , these three properties are not necessarily equivalent. Usually the **Heine-Borel property** is taken as the definition of compactness.

Compactness

Example

Prove that the interval $(0, 1]$ is not compact by showing that it is not closed or not bounded.

Example

Prove that the interval $(0, 1]$ is not compact by showing that it does not have the Bolzano-Weierstrass property.

Example

Prove that the interval $(0, 1]$ is not compact by showing that it does not have the Heine-Borel property.

Poll

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- .



Mathematics
and Statistics

$$\int_M d\omega = \int_{\partial M} \omega$$

Mathematics 3A03 Real Analysis I

Instructor: David Earn

Lecture 16
Topology of \mathbb{R}^n IV
Thursday 12 February 2026