14 Topology of $\mathbb{R}$ I

15 Topology of $\mathbb{R}$ II

Mathematics
and Statistics

$$\int_M d\omega = \int_{\partial M} \omega$$

Mathematics 3A03
Real Analysis I

Instructor: David Earn

Lecture 14
Topology of $\mathbb{R}$ I
Monday 10 February 2025

Announcements

- Solutions to Assignment 2 have been reposted after correcting some errors (thanks to Kieran for spotting these).
    - There were typos in Q2(b) and Q4.
    - Q3 was incomplete because I assumed $f(x)$ was positive.
- Assignment 3 is posted on the course web site. Participation deadline is Monday 24 Feb 2025 @ 11:25 am.
- I reposted the slides for Lecture 13. Slide 79 now contains a sequence of hints for proving π is irrational.
- The midterm TEST is on Thursday 27 February 2025 @ 7:00pm in Hamilton Hall 302.
- The room is booked for 7:00–10:00 pm, but the intention is that a reasonable amount of time for the test is one hour. You will be given double time.

Topology of $\mathbb{R}$

Intervals

Open interval:

$$(a, b) = \{x : a < x < b\}$$

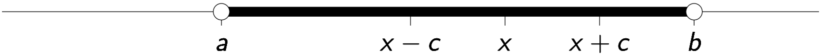
Closed interval:

$$[c, d] = \{x : c \leq x \leq d\}$$

Half-open interval:

$$(e, f] = \{x : e < x \leq f\}$$

Interior point



Definition (Interior point)

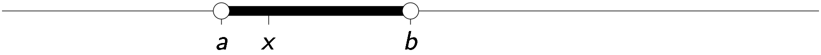
If $E \subseteq \mathbb{R}$ then x is an ***interior point*** of E if x lies in an open interval that is contained in E, *i.e.,*

$$\exists c > 0 \quad \ni \quad (x-c, x+c) \subset E.$$

Interior point examples

Set E	Interior points?
$(-1,1)$	
$[0,1]$	
$\mathbb{N}$	
$\mathbb{R}$	
$\mathbb{Q}$	
$(-1,1)\cup[0,1]$	
$(-1,1)\setminus\{\frac{1}{2}\}$	

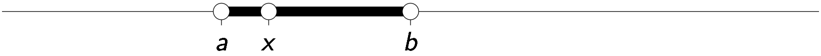
Neighbourhood



Definition (Neighbourhood)

A ***neighbourhood*** of a point $x \in \mathbb{R}$ is an open interval containing x.

Deleted neighbourhood

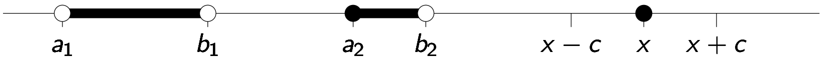


$$(a, b) \setminus \{x\}$$

Definition (Deleted neighbourhood)

A ***deleted neighbourhood*** of a point $x \in \mathbb{R}$ is a set formed by removing x from a neighbourhood of x.

Isolated point



$$E = (a_1, b_1) \cup [a_2, b_2) \cup \{x\}$$

Definition (Isolated point)

If $x \in E \subseteq \mathbb{R}$ then x is an ***isolated point*** of E if there is a neighbourhood of x for which the only point in E is x itself, *i.e.*,

$$\exists c > 0 \quad \ni \quad (x-c, x+c) \cap E \;=\; \{x\}\,.$$

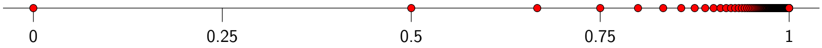
Poll

- Go to
  https://www.childsmath.ca/childsa/forms/main_login.php
- Click on Math 3A03
- Click on Take Class Poll
- Fill in poll **Topology: Isolated points**
- $\boxed{\texttt{Submit}}$.

Isolated point examples

Set E	**Isolated points?**
$(-1,1)$	
$[0,1]$	
$\mathbb{N}$	
$\mathbb{R}$	
$\mathbb{Q}$	
$(-1,1)\cup[0,1]$	
$(-1,1)\setminus\{\frac{1}{2}\}$	

Accumulation point



$$E = \left\{1 - \frac{1}{n} : n \in \mathbb{N}\right\}$$

Definition (Accumulation Point or Limit Point or Cluster Point)

If $E \subseteq \mathbb{R}$ then x is an ***accumulation point*** of E if every neighbourhood of x contains infinitely many points of E,

$$\textit{i.e.,} \qquad \forall c > 0 \qquad (x-c, x+c) \cap (E \setminus \{x\}) \neq \varnothing\,.$$

Note:

- It is possible but not necessary that $x \in E$.
- The shorthand condition is equivalent to saying that every deleted neighbourhood of x contains at least one point of E.

Poll

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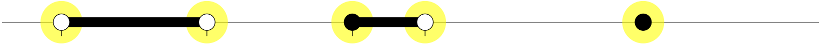
Accumulation point examples

Set E	**Accumulation points?**
$(-1,1)$	
$[0,1]$	
$\mathbb{N}$	
$\mathbb{R}$	
$\mathbb{Q}$	
$(-1,1)\cup[0,1]$	
$(-1,1)\setminus\{\frac{1}{2}\}$	
$\left\{1-\frac{1}{n} : n\in\mathbb{N}\right\}$	

Boundary point



Definition (Boundary Point)

If $E \subseteq \mathbb{R}$ then x is a ***boundary point*** of E if every neighbourhood of x contains at least one point of E and at least one point not in E, *i.e.*,

$$\forall c > 0 \qquad (x-c, x+c) \cap E \neq \varnothing \\ \wedge \quad (x-c, x+c) \cap (\mathbb{R} \setminus E) \neq \varnothing\,.$$

Note: It is possible but not necessary that $x \in E$.

Definition (Boundary)

If $E \subseteq \mathbb{R}$ then the ***boundary*** of E, denoted ∂E, is the set of all boundary points of E.

Boundary point examples

Set E	**Boundary points?**
$(-1,1)$	
$[0,1]$	
$\mathbb{N}$	
$\mathbb{R}$	
$\mathbb{Q}$	
$(-1,1)\cup[0,1]$	
$(-1,1)\setminus\{\frac{1}{2}\}$	
$\left\{1-\frac{1}{n} : n\in\mathbb{N}\right\}$	

Closed set

Definition (Closed set)

A set $E \subseteq \mathbb{R}$ is ***closed*** if it contains all of its accumulation points.

Definition (Closure of a set)

If $E \subseteq \mathbb{R}$ and E' is the set of accumulation points of E then the ***closure*** of E is

$$\overline{E} = E \cup E'.$$

Note: If the set E has no accumulation points, then E is closed because there are no accumulation points to check.

Open set

Definition (Open set)

A set $E \subseteq \mathbb{R}$ is ***open*** if every point of E is an interior point.

Definition (Interior of a set)

If $E \subseteq \mathbb{R}$ then the ***interior*** of E, denoted $\text{int}(E)$ or E°, is the set of all interior points of E.

Poll

- Go to https://www.childsmath.ca/childsa/forms/main_login.php
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- Fill in poll **Topology: Open or Closed**
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Examples

Set E	Closed?	Open?	$\overline{E}$	E°	∂E
$(-1,1)$					
$[0,1]$					
$\mathbb{N}$					
$\mathbb{R}$					
$\varnothing$					
$\mathbb{Q}$					
$(-1,1)\cup[0,1]$					
$(-1,1)\setminus\{\frac{1}{2}\}$					
$\left\{1-\frac{1}{n} : n\in\mathbb{N}\right\}$					

McMaster University

Mathematics and Statistics

$$\int_M d\omega = \int_{\partial M} \omega$$

Mathematics 3A03
Real Analysis I

Instructor: David Earn

Lecture 15
Topology of $\mathbb{R}$ II
Wednesday 12 February 2025

Announcements

- Solutions to Assignment 2 have been reposted after correcting some errors (thanks to Kieran for spotting these).
  - There were typos in Q2(b) and Q4.
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Topological concepts covered so far

- Interval
- Neighbourhood
- Deleted neighbourhood
- Interior point
- Isolated point
- Accumulation point
- Boundary point
- Boundary
- Closed set
- Closure
- Open set
- Interior

Equivalent definitions

Example (Closure)

For $E \subseteq \mathbb{R}$, prove $E \cup E' = E \cup \partial E$, so $\overline{E}$ can be defined either way.

Proof: We must show $E \cup E' \subseteq E \cup \partial E$ and $E \cup E' \supseteq E \cup \partial E$.

- $\subseteq$ Suppose $x \in E \cup E'$. If $x \in E$ then $x \in E \cup A$ for any set A. In particular, $x \in E \cup \partial E$. Alternatively, suppose $x \notin E$, *i.e.*, $x \in E^{c}$. Then, since $x \in E \cup E'$, it must be that $x \in E'$, which means that any neighbourhood of x contains a point of E. But $x \in E^{c}$, so any such neighbourhood also contains a point of E^{c} (namely x). Therefore, $x \in \partial E \subseteq E \cup \partial E$.
- $\supseteq$ Suppose $x \in E \cup \partial E$. If $x \in E$ then $x \in E \cup A$ for any set A. In particular, $x \in E \cup E'$. Alternatively, suppose $x \notin E$, *i.e.*, $x \in E^{c}$. Then, since $x \in E \cup \partial E$, it must be that $x \in \partial E$, which means that any neighbourhood of x contains a point of E. But that point is not x, since $x \notin E$. Thus, any *deleted* neighbourhood of x contains a point of E, *i.e.*, $x \in E' \subseteq E \cup E'$. $\square$

Question: In the proof above, did we use any properties of $\mathbb{R}$?

Poll

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- Fill in poll **Topology: The most general type of open set**
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Component intervals of open sets

What does the most general open set look like?

Theorem (Component intervals)

If G is an open subset of $\mathbb{R}$ and $G \neq \varnothing$ then there is a unique (possibly finite) sequence of disjoint open intervals $\{(a_n, b_n)\}$ such that

$$G = (a_1, b_1) \cup (a_2, b_2) \cup \cdots \cup (a_n, b_n) \cup \cdots ,$$

$$\textit{i.e.,} \qquad G = \bigcup_{n=1}^{\infty} (a_n, b_n) .$$

*The open intervals (a_n, b_n) are said to be the **component intervals** of G.*

(TBB Theorem 4.15, p. 231)

Component intervals of open sets

Main ideas of proof of component intervals theorem:

- $x \in G \implies x$ is an interior point of $G \implies$
  - some neighbourhood of x is contained in G, *i.e.*, $\exists c > 0$ such that $(x-c, x+c) \subseteq G$
  - $\exists$ a largest neighbourhood of x that is contained in G: this "***component of G***" is $I_x = (\alpha, \beta)$, where

$$\alpha = \inf\{a : (a, x] \subset G\}, \qquad \beta = \sup\{b : [x, b) \subset G\}$$

  - I_x contains a rational number, *i.e.*, $\exists r \in I_x \cap \mathbb{Q}$
- $\therefore$ We can index all the intervals I_x by rational numbers
- $\therefore$ There are are most countably many intervals that make up G (*i.e.*, G is the union of a sequence of intervals)
- We can choose a disjoint subsequence of these intervals whose union is all of G; see proof in TBB textbook for details (TBB Theorem 4.15, p. 231).

Open vs. Closed Sets

Definition (Complement of a set of real numbers)

If $E \subseteq \mathbb{R}$ then the ***complement*** of E is the set

$$E^{\mathrm{c}} = \{x \in \mathbb{R} : x \notin E\} \,.$$

Theorem (Open vs. Closed)

If $E \subseteq \mathbb{R}$ then E is open iff E^{c} is closed.

(TBB Theorem 4.16)

Open vs. Closed Sets

Theorem (Properties of open sets of real numbers)

1. *The sets $\mathbb{R}$ and $\varnothing$ are open.*
2. *Any intersection of a finite number of open sets is open.*
3. *Any union of an arbitrary collection of open sets is open.*
4. *The complement of an open set is closed.*

(TBB Theorem 4.17)

Theorem (Properties of closed sets of real numbers)

1. *The sets $\mathbb{R}$ and $\varnothing$ are closed.*
2. *Any union of a finite number of closed sets is closed.*
3. *Any intersection of an arbitrary collection of closed sets is closed.*
4. *The complement of a closed set is open.*

(TBB Theorem 4.18)

Local vs. Global properties

Definition (Bounded function)

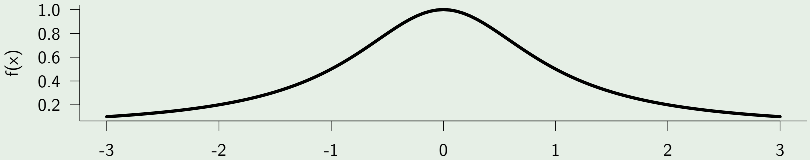
A real-valued function f is ***bounded*** on the set E if there exists $M > 0$ such that $|f(x)| \leq M$ for all $x \in E$.

(*i.e.*, the function f is bounded on E iff $\{f(x) : x \in E\}$ is a bounded set.)

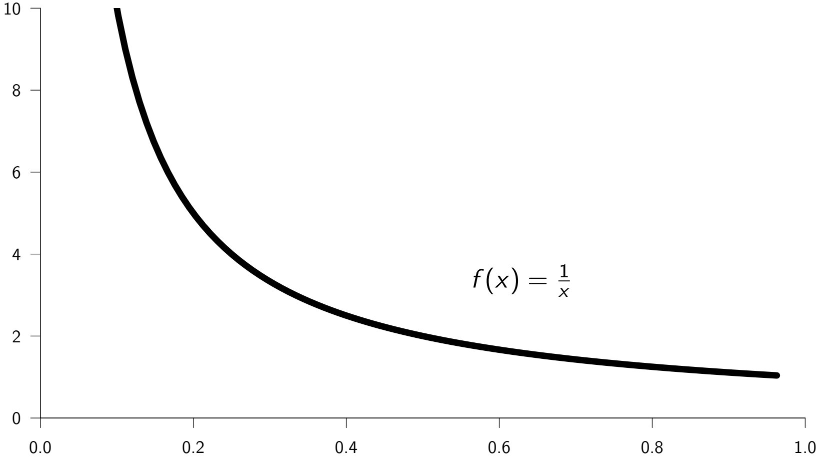
Note: This is a *global* property because there is a single bound M associated with the entire set E.

Example

The function $f(x) = 1/(1 + x^2)$ is bounded on $\mathbb{R}$. *e.g.*, $M = 1$.

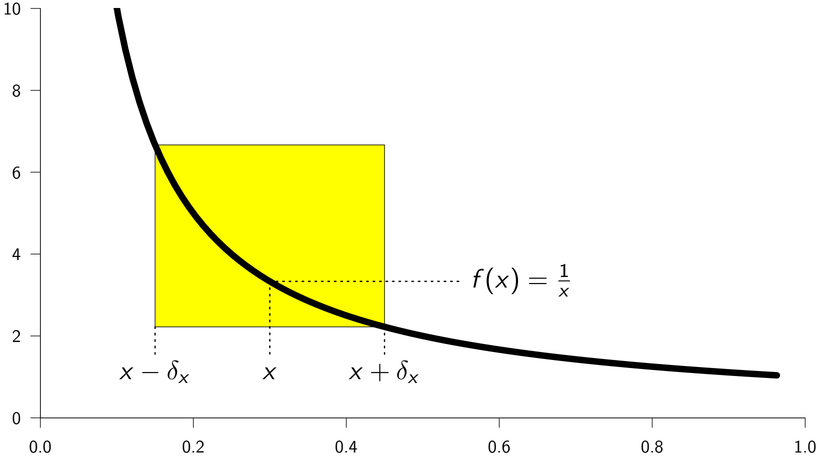


Local vs. Global properties



$f(x) = 1/x$ is not bounded on the interval $E = (0,1)$.

Local vs. Global properties



$f(x) = 1/x$ is ***locally bounded*** on the interval $E = (0,1)$,
i.e., $\forall x \in E$, $\exists \delta_x, M_x > 0 \ni |f(t)| \le M_x \; \forall t \in (x - \delta_x, x + \delta_x)$.

Local vs. Global properties

Definition (Locally bounded at a point)

A real-valued function f is ***locally bounded*** at the point x if there is a neighbourhood of x in which f is bounded, *i.e.*, there exists $\delta_x > 0$ and $M_x > 0$ such that $|f(t)| \leq M_x$ for all $t \in (x - \delta_x, x + \delta_x)$.

Definition (Locally bounded on a set)

A real-valued function f is ***locally bounded*** on the set E if f is locally bounded at each point $x \in E$.

Note: The size of the neighbourhood (δ_x) and the local bound (M_x) depend on the point x.

Local vs. Global properties

Example (Function that is not even locally bounded)

Give an example of a function that is defined on the interval $(0,1)$ but is not locally bounded on $(0,1)$.

Let's construct a function $f(x)$ that is defined on $(0,1)$ but is not locally bounded at one point, say $x = \frac{1}{2}$.

$f(x)$ must blow up $x = \frac{1}{2}$. Let's make f look like $1/x$, but shifted so the blowup is at $x = \frac{1}{2}$.

$$f(x) = \begin{cases} \dfrac{1}{x - \frac{1}{2}} & x \neq \frac{1}{2}, \\ 0 & x = \frac{1}{2}. \end{cases}$$

Local vs. Global properties

Example (Function that is a mess near 0)

Give an example of a function $f(x)$ that is defined everywhere, yet in <u>any</u> neighbourhood of the origin there are infinitely many points at which f is <u>not</u> locally bounded.

Please do poll: Topology: Local boundedness

Consider $S(x) = \begin{cases} \sin\frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$

$S(x)$ is bounded on $\mathbb{R}$, hence locally bounded at every point.

Consider $T(x) = \begin{cases} \tan\frac{1}{x} & \cos x \neq 0 \\ 0 & \cos x = 0 \end{cases}$

$T(x)$ is not locally bounded at points where $\cos x = 0$, *i.e.*, for $x = \frac{\pi}{2} + n\pi$, $n \in \mathbb{Z}$. There are infinitely many such points in any neighbourhood of $x = 0$.

Poll

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- Fill in poll **Topology: Locally bounded nowhere?**
- `Submit`.

Local vs. Global properties

Extra Challenge Problem:

Is there a function $f : \mathbb{R} \to \mathbb{R}$ that is <u>not</u> locally bounded <u>anywhere</u>?

Local vs. Global properties

- What condition(s) rule out such pathological behaviour?
- When does a property holding locally (near any given point in a set) imply that it holds globally (for the set as a whole)?
- For example: What condition(s) must a set $E \subseteq \mathbb{R}$ satisfy in order that a function f that is locally bounded on E is necessarily bounded on E?
- We will see that the condition we are seeking is that the set E must be "***compact***" . . .