Mathematics
and Statistics

$$\int_M d\omega = \int_{\partial M} \omega$$

Mathematics 3A03
Real Analysis I

Instructor: David Earn

Lecture 13
Topology of $\mathbb{R}$ I
Tuesday 1 October 2019

THINKING ABOUT GRADUATE SCHOOL?

JOIN US TO FIND OUT MORE AT THE GRAD INFO SESSION!

WHEN: THURSDAY OCTOBER 3, 2019
TIME: 5:30PM – 7:00PM
WHERE: HH/305 AND THE MATH CAFÉ

Matheus Grasselli will give general advice on applying to grad school.

Shui Feng will talk about graduate programs particular to statistics.

Tom Hurd will talk about graduate opportunities in financial math including PhiMac.

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Announcements

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Topology of $\mathbb{R}$

Intervals

Open interval:

$$(a, b) = \{x : a < x < b\}$$

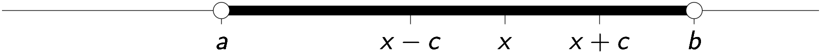
Closed interval:

$$[c, d] = \{x : c \leq x \leq d\}$$

Half-open interval:

$$(e, f] = \{x : e < x \leq f\}$$

Interior point



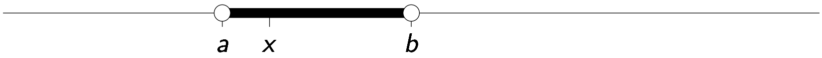
Definition (Interior point)

If $E \subseteq \mathbb{R}$ then x is an ***interior point*** of E if x lies in an open interval that is contained in E, *i.e.*, $\exists c > 0$ such that $(x - c, x + c) \subset E$.

Interior point examples

Set E	**Interior points?**
$(-1, 1)$	Every point
$[0, 1]$	Every point *except the endpoints*
$\mathbb{N}$	$\nexists$
$\mathbb{R}$	Every point
$\mathbb{Q}$	$\nexists$
$(-1, 1) \cup [0, 1]$	Every point *except 1*
$(-1, 1) \setminus \{\frac{1}{2}\}$	Every point

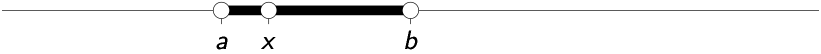
Neighbourhood



Definition (Neighbourhood)

A ***neighbourhood*** of a point $x \in \mathbb{R}$ is an open interval containing x.

Deleted neighbourhood

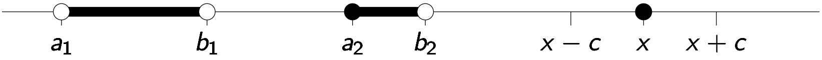


Definition (Deleted neighbourhood)

A ***deleted neighbourhood*** of a point $x \in \mathbb{R}$ is a set formed by removing x from a neighbourhood of x.

$$(a, b) \setminus \{x\}$$

Isolated point



$$E = (a_1, b_1) \cup [a_2, b_2) \cup \{x\}$$

Definition (Isolated point)

If $x \in E \subseteq \mathbb{R}$ then x is an ***isolated point*** of E if there is a neighbourhood of x for which the only point in E is x itself, *i.e.*, $\exists c > 0$ such that $(x - c, x + c) \cap E = \{x\}$.

Poll

- Go to https://www.childsmath.ca/childsa/forms/main_login.php
- Click on Math 3A03
- Click on Take Class Poll
- Fill in poll **Lecture 13: Isolated points**
- Submit.

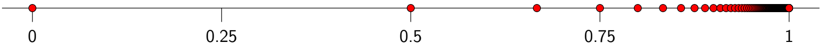
Isolated point examples

Set E	Isolated points?
$(-1,1)$	
$[0,1]$	
$\mathbb{N}$	
$\mathbb{R}$	
$\mathbb{Q}$	
$(-1,1) \cup [0,1]$	
$(-1,1) \setminus \{\frac{1}{2}\}$	

Isolated point examples

Set E	**Isolated points?**
$(-1,1)$	$\nexists$
$[0,1]$	$\nexists$
$\mathbb{N}$	Every point
$\mathbb{R}$	$\nexists$
$\mathbb{Q}$	$\nexists$
$(-1,1)\cup[0,1]$	$\nexists$
$(-1,1)\setminus\{\frac{1}{2}\}$	$\nexists$

Accumulation point



$$E = \left\{ 1 - \frac{1}{n} : n \in \mathbb{N} \right\}$$

Definition (Accumulation Point or Limit Point)

If $E \subseteq \mathbb{R}$ then x is an ***accumulation point*** or ***limit point*** of E if every neighbourhood of x contains infinitely many points of E,

$$\textit{i.e.,} \qquad \forall c > 0 \qquad (x - c, x + c) \cap (E \setminus \{x\}) \neq \varnothing \,.$$

Notes:

- It is possible but not necessary that $x \in E$.
- The shorthand condition is equivalent to saying that every deleted neighbourhood of x contains at least one point of E.

Poll

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- Submit.

Accumulation point examples

Set E	**Accumulation points?**
$(-1,1)$	
$[0,1]$	
$\mathbb{N}$	
$\mathbb{R}$	
$\mathbb{Q}$	
$(-1,1)\cup[0,1]$	
$(-1,1)\setminus\{\frac{1}{2}\}$	
$\left\{1-\frac{1}{n} : n\in\mathbb{N}\right\}$	

Accumulation point examples

Set E	**Accumulation points?**
$(-1,1)$	$[-1,1]$
$[0,1]$	$[0,1]$
$\mathbb{N}$	$\nexists$
$\mathbb{R}$	$\mathbb{R}$
$\mathbb{Q}$	$\mathbb{R}$
$(-1,1) \cup [0,1]$	$[-1,1]$
$(-1,1) \setminus \{\frac{1}{2}\}$	$[-1,1]$
$\left\{1 - \frac{1}{n} : n \in \mathbb{N}\right\}$	$\{1\}$

McMaster University

Mathematics and Statistics

$$\int_M d\omega = \int_{\partial M} \omega$$

Mathematics 3A03
Real Analysis I

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Lecture 14

Topology of $\mathbb{R}$ II

Thursday 3 October 2019

Poll

- Go to https://www.childsmath.ca/childsa/forms/main_login.php
- Click on Math 3A03
- Click on Take Class Poll
- Fill in poll **Lecture 14: poll on polls**
- Submit.

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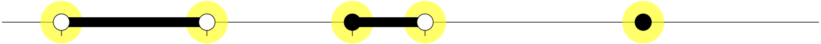
Topological concepts covered so far

- Interval
- Neighbourhood
- Deleted neighbourhood
- Interior point
- Isolated point
- Accumulation point

Boundary point



Definition (Boundary Point)

If $E \subseteq \mathbb{R}$ then x is a ***boundary point*** of E if every neighbourhood of x contains at least one point of E and at least one point not in E, *i.e.*,

$$\forall c > 0 \qquad (x-c, x+c) \cap E \neq \varnothing$$
$$\wedge \quad (x-c, x+c) \cap (\mathbb{R} \setminus E) \neq \varnothing \, .$$

Note: It is possible but not necessary that $x \in E$.

Definition (Boundary)

If $E \subseteq \mathbb{R}$ then the ***boundary*** of E, denoted ∂E, is the set of all boundary points of E.

Boundary point examples

Set E	**Boundary points?**
$(-1,1)$	$\{-1,1\}$
$[0,1]$	$\{0,1\}$
$\mathbb{N}$	$\mathbb{N}$
$\mathbb{R}$	$\nexists$
$\mathbb{Q}$	$\mathbb{R}$
$(-1,1)\cup[0,1]$	$\{-1,1\}$
$(-1,1)\setminus\{\frac{1}{2}\}$	$\{-1,\frac{1}{2},1\}$
$\left\{1-\frac{1}{n} : n\in\mathbb{N}\right\}$	$\left\{1-\frac{1}{n} : n\in\mathbb{N}\right\}\cup\{1\}$

Closed set

Definition (Closed set)

A set $E \subseteq \mathbb{R}$ is ***closed*** if it contains all of its accumulation points.

Definition (Closure of a set)

If $E \subseteq \mathbb{R}$ and E' is the set of accumulation points of E then $\overline{E} = E \cup E'$ is the ***closure*** of E.

Note: If the set E has no accumulation points, then E is closed because there are no accumulation points to check.

Open set

Definition (Open set)

A set $E \subseteq \mathbb{R}$ is ***open*** if every point of E is an interior point.

Definition (Interior of a set)

If $E \subseteq \mathbb{R}$ then the ***interior*** of E, denoted $\text{int}(E)$ or E°, is the set of all interior points of E.

Poll

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- Fill in poll **Lecture 14: Open or Closed**
- Submit.

Examples

Set E	Closed?	Open?	$\overline{E}$	E°	∂E
$(-1,1)$					
$[0,1]$					
$\mathbb{N}$					
$\mathbb{R}$					
$\varnothing$					
$\mathbb{Q}$					
$(-1,1)\cup[0,1]$					
$(-1,1)\setminus\{\frac{1}{2}\}$					
$\left\{1-\frac{1}{n} : n\in\mathbb{N}\right\}$					

Examples

Set E	Closed?	Open?	$\overline{E}$	E°	∂E
$(-1,1)$	NO	YES	$[-1,1]$	E	$\{-1,1\}$
$[0,1]$	YES	NO	E	$(0,1)$	$\{0,1\}$
$\mathbb{N}$	YES	NO	$\mathbb{N}$	$\varnothing$	$\mathbb{N}$
$\mathbb{R}$	YES	YES	$\mathbb{R}$	$\mathbb{R}$	$\varnothing$
$\varnothing$	YES	YES	$\varnothing$	$\varnothing$	$\varnothing$
$\mathbb{Q}$	NO	NO	$\mathbb{R}$	$\varnothing$	$\mathbb{R}$
$(-1,1)\cup[0,1]$	NO	NO	$[-1,1]$	$(-1,1)$	$\{-1,1\}$
$(-1,1)\setminus\{\frac{1}{2}\}$	NO	YES	$[-1,1]$	E	$\{-1,\frac{1}{2},1\}$
$\left\{1-\frac{1}{n} : n\in\mathbb{N}\right\}$	NO	NO	$E\cup\{1\}$	$\varnothing$	$E\cup\{1\}$

Component intervals of open sets

What does the most general open set look like?