McMaster University

Mathematics and Statistics

$$\int_M d\omega = \int_{\partial M} \omega$$

Mathematics 3A03
Real Analysis I

Instructor: David Earn

Lecture 13
Topology of $\mathbb{R}$
Thursday 5 February 2026

Announcements

- Solutions to Assignment 2 have been posted.
- Assignment 3 is posted on the course web site. Participation deadline is Tuesday 24 Feb 2025 @ 2:25pm.
- The midterm TEST is on Thursday 26 February 2026 @ 7:00pm in T13 123.
- The room is booked for 7:00–10:00 pm, but the intention is that a reasonable amount of time for the test is 90 minutes. You will be given double time.

Topology of $\mathbb{R}$

Intervals

Open interval:

$$(a, b) = \{x : a < x < b\}$$

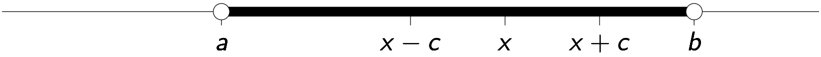
Closed interval:

$$[c, d] = \{x : c \leq x \leq d\}$$

Half-open interval:

$$(e, f] = \{x : e < x \leq f\}$$

Interior point



Definition (Interior point)

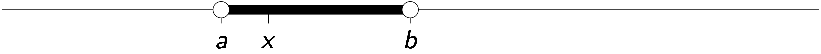
If $E \subseteq \mathbb{R}$ then x is an ***interior point*** of E if x lies in an open interval that is contained in E, *i.e.*,

$$\exists c > 0 \quad \ni \quad (x - c, x + c) \subset E.$$

Interior point examples

Set E	**Interior points?**
$(-1,1)$	
$[0,1]$	
$\mathbb{N}$	
$\mathbb{R}$	
$\mathbb{Q}$	
$(-1,1) \cup [0,1]$	
$(-1,1) \setminus \{\frac{1}{2}\}$	

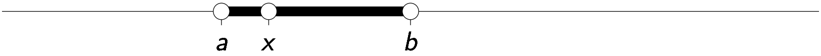
Neighbourhood



Definition (Neighbourhood)

A ***neighbourhood*** of a point $x \in \mathbb{R}$ is an open interval containing x.

Deleted neighbourhood

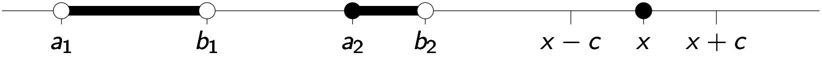


$$(a, b) \setminus \{x\}$$

Definition (Deleted neighbourhood)

A ***deleted neighbourhood*** of a point $x \in \mathbb{R}$ is a set formed by removing x from a neighbourhood of x.

Isolated point



$$E = (a_1, b_1) \cup [a_2, b_2) \cup \{x\}$$

Definition (Isolated point)

If $x \in E \subseteq \mathbb{R}$ then x is an ***isolated point*** of E if there is a neighbourhood of x for which the only point in E is x itself, *i.e.*,

$$\exists c > 0 \quad \ni \quad (x - c, x + c) \cap E \; = \; \{x\}\,.$$

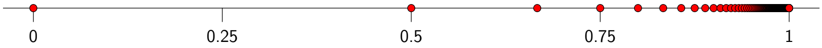
Poll

- Go to https://www.childsmath.ca/childsa/forms/main_login.php
- Click on Math 3A03
- Click on Take Class Poll
- Fill in poll **Topology: Isolated points**
- `Submit`.

Isolated point examples

Set E	**Isolated points?**
$(-1, 1)$	
$[0, 1]$	
$\mathbb{N}$	
$\mathbb{R}$	
$\mathbb{Q}$	
$(-1, 1) \cup [0, 1]$	
$(-1, 1) \setminus \{\frac{1}{2}\}$	

Accumulation point



$$E = \left\{1 - \frac{1}{n} : n \in \mathbb{N}\right\}$$

Definition (Accumulation Point or Limit Point or Cluster Point)

If $E \subseteq \mathbb{R}$ then x is an ***accumulation point*** of E if every neighbourhood of x contains infinitely many points of E,

$$i.e., \qquad \forall c > 0 \qquad (x-c, x+c) \cap (E \setminus \{x\}) \neq \varnothing\,.$$

Note:

- It is possible but not necessary that $x \in E$.
- The shorthand condition is equivalent to saying that every deleted neighbourhood of x contains at least one point of E.

Poll

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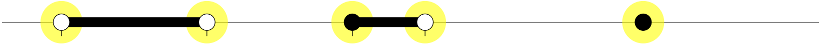
  https://www.childsmath.ca/childsa/forms/main_login.php
- Click on Math 3A03
- Click on Take Class Poll
- Fill in poll **Topology: Accumulation points**
- Submit.

Accumulation point examples

Set E	**Accumulation points?**
$(-1,1)$	
$[0,1]$	
$\mathbb{N}$	
$\mathbb{R}$	
$\mathbb{Q}$	
$(-1,1)\cup[0,1]$	
$(-1,1)\setminus\{\frac{1}{2}\}$	
$\left\{1-\frac{1}{n} : n\in\mathbb{N}\right\}$	

Boundary point



Definition (Boundary Point)

If $E \subseteq \mathbb{R}$ then x is a ***boundary point*** of E if every neighbourhood of x contains at least one point of E and at least one point not in E, *i.e.*,

$$\forall c > 0 \qquad (x-c, x+c) \cap E \neq \varnothing \\ \wedge \quad (x-c, x+c) \cap (\mathbb{R} \setminus E) \neq \varnothing\,.$$

Note: It is possible but not necessary that $x \in E$.

Definition (Boundary)

If $E \subseteq \mathbb{R}$ then the ***boundary*** of E, denoted ∂E, is the set of all boundary points of E.

Boundary point examples

Set E	Boundary points?
$(-1,1)$	
$[0,1]$	
$\mathbb{N}$	
$\mathbb{R}$	
$\mathbb{Q}$	
$(-1,1)\cup[0,1]$	
$(-1,1)\setminus\{\frac{1}{2}\}$	
$\left\{1-\frac{1}{n} : n\in\mathbb{N}\right\}$	

Closed set

Definition (Closed set)

A set $E \subseteq \mathbb{R}$ is ***closed*** if it contains all of its accumulation points.

Definition (Closure of a set)

If $E \subseteq \mathbb{R}$ and E' is the set of accumulation points of E then the ***closure*** of E is

$$\overline{E} = E \cup E'.$$

Note: If the set E has no accumulation points, then E is closed because there are no accumulation points to check.

Open set

Definition (Open set)

A set $E \subseteq \mathbb{R}$ is ***open*** if every point of E is an interior point.

Definition (Interior of a set)

If $E \subseteq \mathbb{R}$ then the ***interior*** of E, denoted $\text{int}(E)$ or E°, is the set of all interior points of E.

Poll

- Go to https://www.childsmath.ca/childsa/forms/main_login.php
- Click on Math 3A03
- Click on Take Class Poll
- Fill in poll **Topology: Open or Closed**
- `Submit`.

Examples

Set E	Closed?	Open?	$\overline{E}$	E°	∂E
$(-1,1)$					
$[0,1]$					
$\mathbb{N}$					
$\mathbb{R}$					
$\varnothing$					
$\mathbb{Q}$					
$(-1,1)\cup[0,1]$					
$(-1,1)\setminus\{\frac{1}{2}\}$					
$\left\{1-\frac{1}{n} : n\in\mathbb{N}\right\}$					

Mathematics and Statistics

$$\int_M d\omega = \int_{\partial M} \omega$$

Mathematics 3A03
Real Analysis I

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Lecture 14
Topology of $\mathbb{R}$ II
Friday 6 February 2026