McMaster University

Mathematics and Statistics

$$\int_M d\omega = \int_{\partial M} \omega$$

Mathematics 3A03
Real Analysis I

Instructor: David Earn

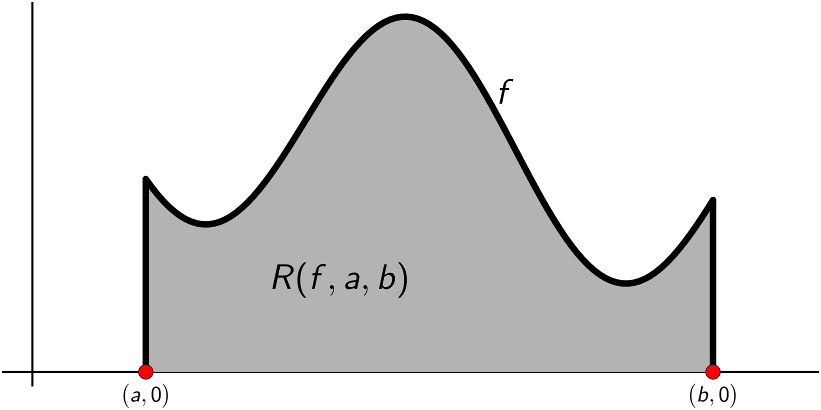
Lecture 7
Integration
Wednesday 22 January 2025

Announcements

- Solutions to Assignment 1 were posted last night.
- Kieran will have office hours tomorrow (Thursday) for two hours, 12:30–2:30 pm. (He will not have a Friday office hour this week.)

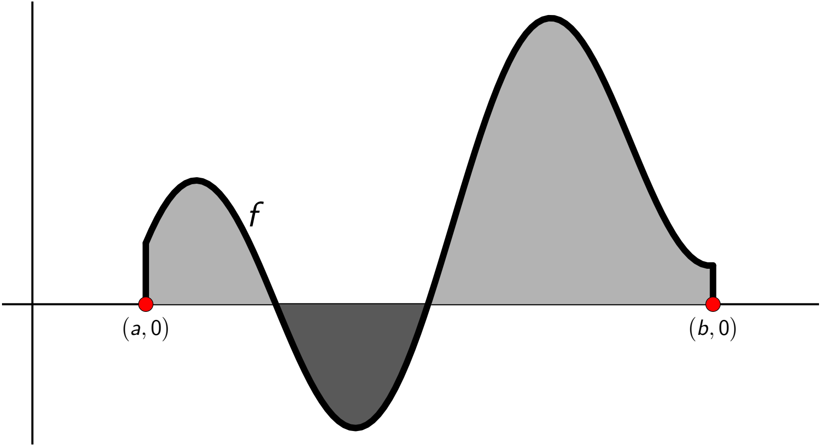
Integration

Integration



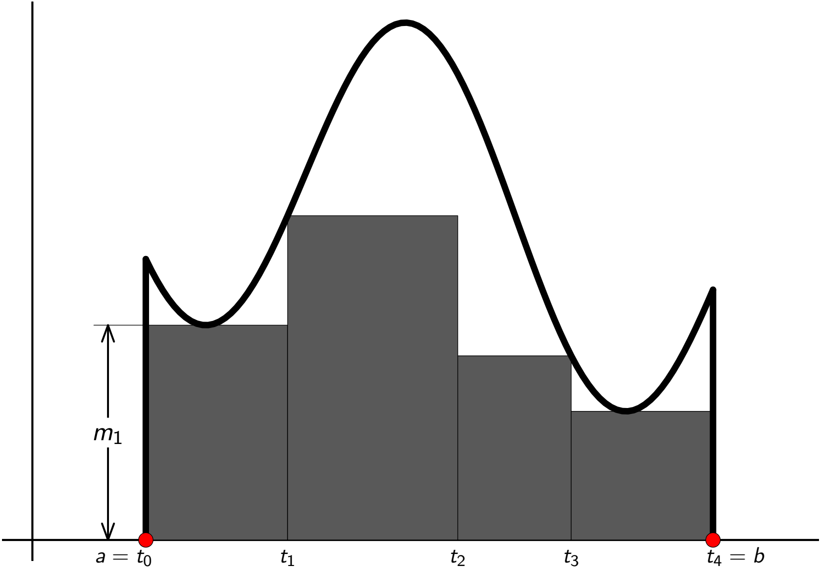
- "Area of region $R(f, a, b)$" is actually a very subtle concept.
- We will only scratch the surface of it (greater depth in Math 4A).
- Our treatment is similar to that in Michael Spivak's "Calculus" (2008); BS refer to this approach as the Darboux integral (BS §7.4, p. 225).
- The Darboux and Riemann approaches to the integral are equivalent.

Integration

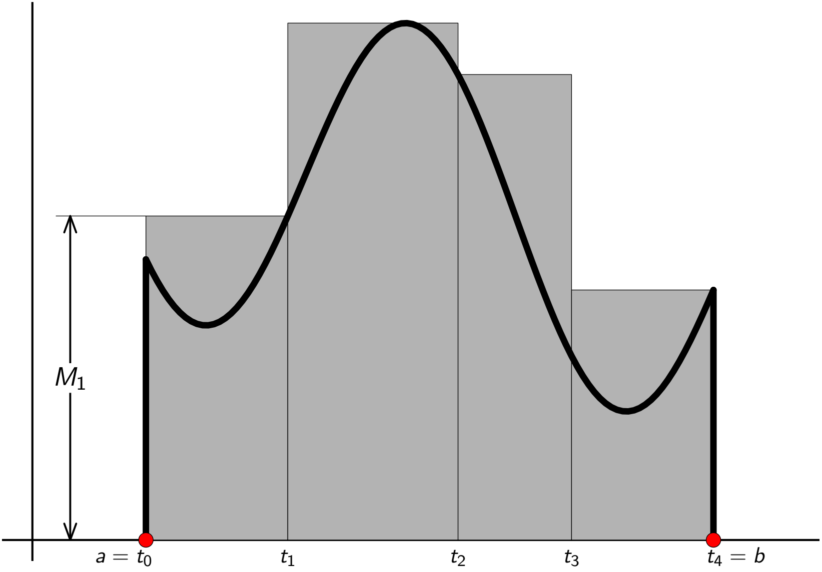


- Contribution to "area of $R(f, a, b)$" is positive or negative depending on whether f is positive or negative.

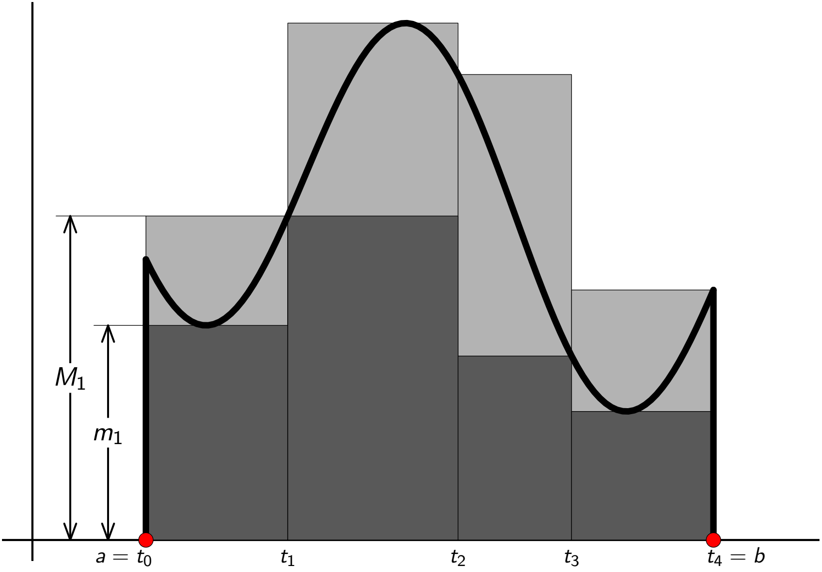
Lower sum



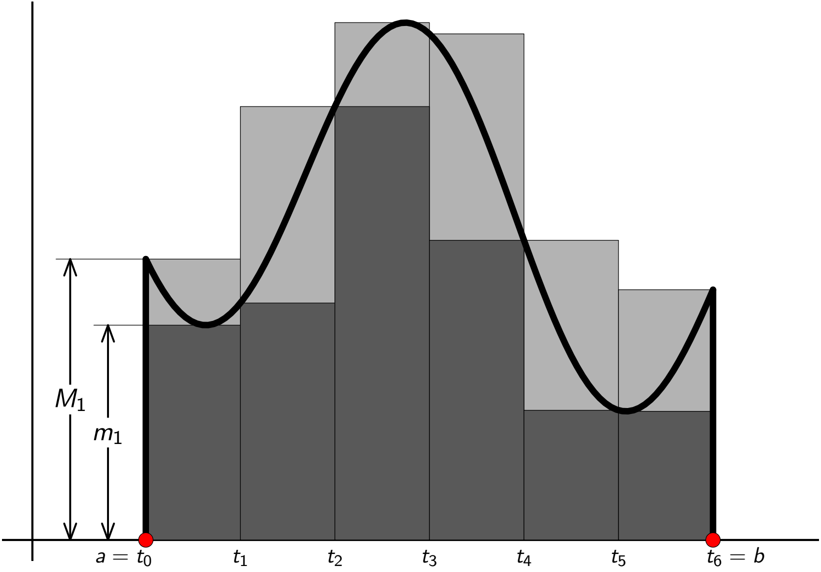
Upper sum



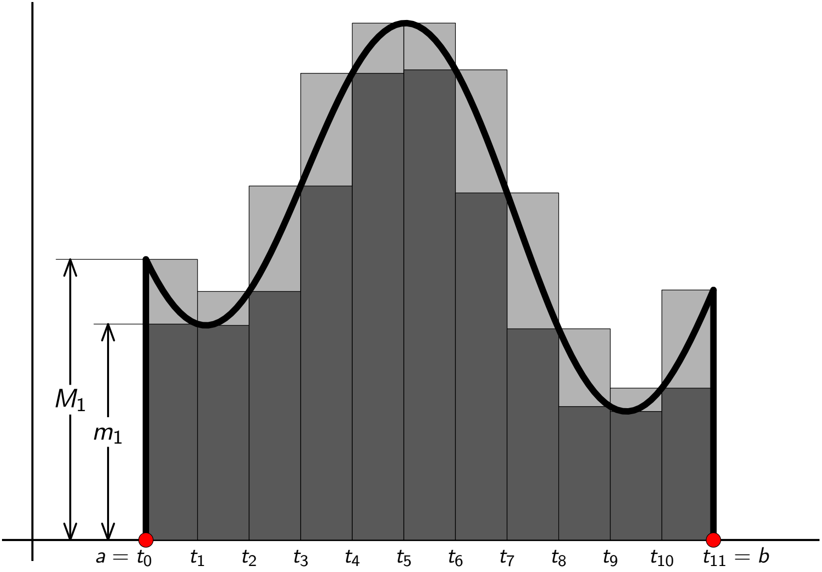
Lower and upper sums



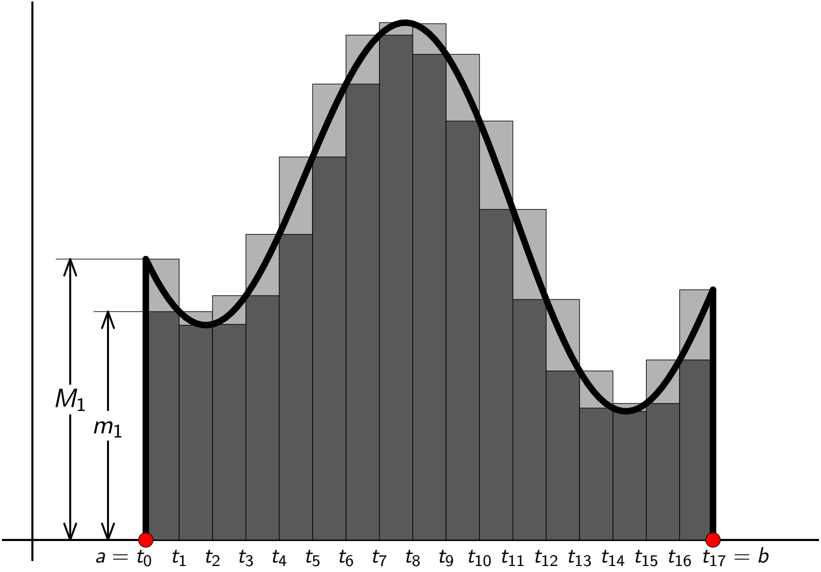
Lower and upper sums



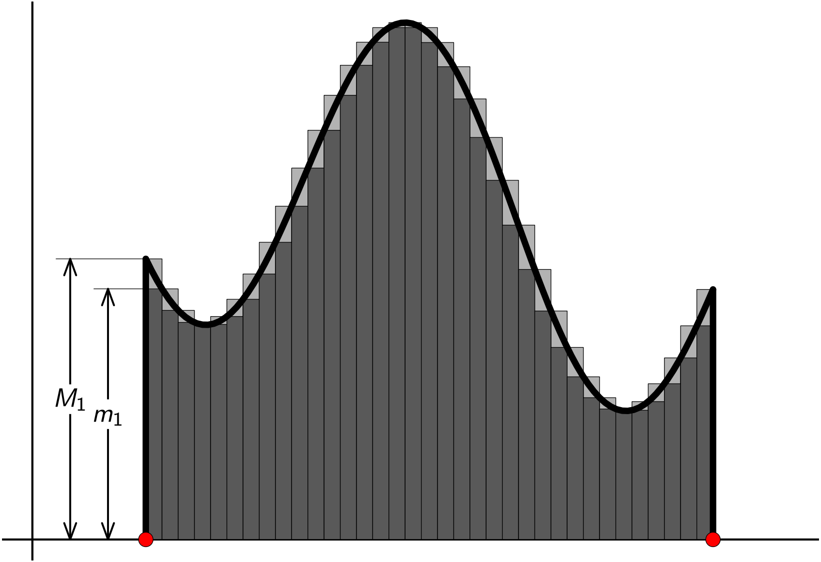
Lower and upper sums



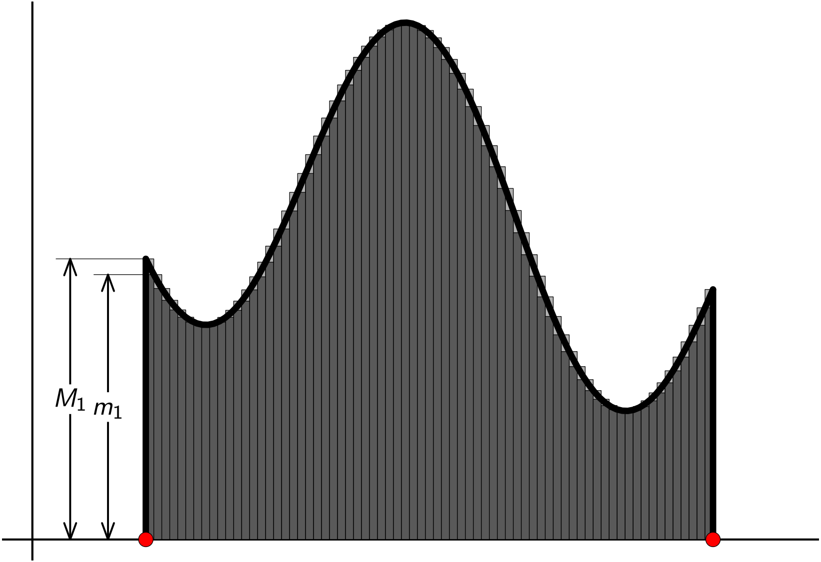
Lower and upper sums



Lower and upper sums



Lower and upper sums



Rigorous development of the integral

Definition (Partition)

Let $a < b$. A ***partition*** of the interval $[a, b]$ is a finite collection of points in $[a, b]$, one of which is a, and one of which is b.

We normally label the points in a partition

$$a = t_0 < t_1 < \cdots < t_{n-1} < t_n = b\,,$$

so the i^{th} subinterval in the partition is

$$[t_{i-1}, t_i]\,.$$

Rigorous development of the integral

Definition (Lower and upper sums)

Suppose f is bounded on $[a, b]$ and $P = \{t_0, \ldots, t_n\}$ is a partition of $[a, b]$. Recalling the motivating sketch, let

$$m_i = \inf \{ f(x) : x \in [t_{i-1}, t_i] \},$$

$$M_i = \sup \{ f(x) : x \in [t_{i-1}, t_i] \}.$$

The lower sum of f for P, denoted by $L(f, P)$, is defined as

$$L(f, P) = \sum_{i=1}^{n} m_i(t_i - t_{i-1}).$$

The upper sum of f for P, denoted by $U(f, P)$, is defined as

$$U(f, P) = \sum_{i=1}^{n} M_i(t_i - t_{i-1}).$$

Rigorous development of the integral

Relationship between motivating sketch and rigorous definition of lower and upper sums:

- The lower and upper sums correspond to the total areas of rectangles lying below and above the graph of f in our motivating sketch.
- However, these sums have been defined precisely without any appeal to a concept of "area".
- The requirement that f be bounded on $[a, b]$ is essential in order to be sure that all the m_i and M_i are well-defined.
- It is also essential that the m_i and M_i be defined as inf's and sup's (rather than maxima and minima) because f was not assumed to be continuous.

Rigorous development of the integral

Relationship between motivating sketch and rigorous definition of lower and upper sums:

- Since $m_i \le M_i$ for each i, we have

$$m_i(t_i - t_{i-1}) \le M_i(t_i - t_{i-1}), \qquad i = 1, \ldots, n.$$

$\therefore$ For <u>any</u> partition P of $[a, b]$ we have

$$L(f, P) \le U(f, P),$$

because

$$L(f, P) = \sum_{i=1}^{n} m_i(t_i - t_{i-1}),$$
$$U(f, P) = \sum_{i=1}^{n} M_i(t_i - t_{i-1}).$$

Poll

- Go to
  https://www.childsmath.ca/childsa/forms/main_login.php
- Click on Math 3A03
- Click on Take Class Poll
- Fill in poll **Integrals: Lower and Upper Sums**
- Submit.

Rigorous development of the integral

Relationship between motivating sketch and rigorous definition of lower and upper sums:

- More generally, if P_1 and P_2 are any two partitions of $[a, b]$, it *ought* to be true that

$$L(f, P_1) \leq U(f, P_2),$$

  because $L(f, P_1)$ should be $\leq$ area of $R(f, a, b)$, and $U(f, P_2)$ should be $\geq$ area of $R(f, a, b)$.

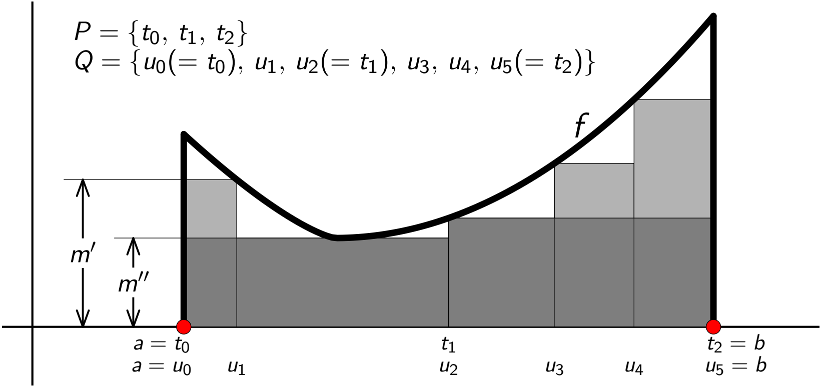
- But "ought to" and "should be" prove nothing, especially since we haven't yet even defined "area of $R(f, a, b)$".

- Before we can *define* "area of $R(f, a, b)$", we need to prove that $L(f, P_1) \leq U(f, P_2)$ for any partitions P_1, P_2 ...

Rigorous development of the integral

Lemma (Partition Lemma)

If partition $P \subseteq$ partition Q (i.e., if every point of P is also in Q), then $L(f,P) \le L(f,Q)$ *and* $U(f,P) \ge U(f,Q)$.



Rigorous development of the integral

Proof of Partition Lemma

As a first step, consider the special case in which the finer partition Q contains only one more point than P:

$$P = \{t_0, \ldots, t_n\},$$
$$Q = \{t_0, \ldots, t_{k-1}, u, t_k, \ldots, t_n\},$$

where

$$a = t_0 < t_1 < \cdots < t_{k-1} < u < t_k < \cdots < t_n = b\,.$$

Because $[t_{k-1}, t_k]$ is split by u, we have two lower bounds:

$$m' = \inf\{\, f(x) \,:\, x \in [t_{k-1}, u] \,\},$$
$$m'' = \inf\{\, f(x) \,:\, x \in [u, t_k] \,\}.$$

...continued...

Rigorous development of the integral

Proof of Partition Lemma (cont.)

Then $$L(f,P) = \sum_{i=1}^{n} m_i(t_i - t_{i-1}),$$

and $$L(f,Q) = \sum_{i=1}^{k-1} m_i(t_i - t_{i-1}) + m'(u - t_{k-1}) + m''(t_k - u) + \sum_{i=k+1}^{n} m_i(t_i - t_{i-1}).$$

$\therefore$ To prove $L(f,P) \le L(f,Q)$, it is enough to show

$$m_k(t_k - t_{k-1}) \le m'(u - t_{k-1}) + m''(t_k - u).$$

. . . continued. . .

Rigorous development of the integral

Proof of Partition Lemma (cont.)

Now note that since

$$\{\, f(x) \,:\, x \in [t_{k-1}, u] \,\} \quad \subseteq \quad \{\, f(x) \,:\, x \in [t_{k-1}, t_k] \,\},$$

the RHS might contain some additional *smaller* numbers, so we must have

$$\begin{aligned} m_k \quad &= \quad \inf \{\, f(x) \,:\, x \in [t_{k-1}, t_k] \,\} \\ &\leq \quad \inf \{\, f(x) \,:\, x \in [t_{k-1}, u] \,\} \quad = \quad m' \,. \end{aligned}$$

Thus, $m_k \leq m'$, and, similarly, $m_k \leq m''$.

$$\begin{aligned} \therefore \quad m_k(t_k - t_{k-1}) \quad &= \quad m_k(t_k - u + u - t_{k-1}) \\ &= \quad m_k(u - t_{k-1}) + m_k(t_k - u) \\ &\leq \quad m'(u - t_{k-1}) + m''(t_k - u) \,, \end{aligned}$$

. . . continued. . .

Rigorous development of the integral

Proof of Partition Lemma (cont.)

which proves (in this special case where Q contains only one more point than P) that $L(f,P) \leq L(f,Q)$.

We can now prove the general case by adding one point at a time.

If Q contains ℓ more points than P, define a sequence of partitions

$$P = P_0 \subset P_1 \subset \cdots \subset P_\ell = Q$$

such that P_{j+1} contains exactly one more point than P_j. Then

$$L(f,P) = L(f,P_0) \leq L(f,P_1) \leq \cdots \leq L(f,P_\ell) = L(f,Q)\,,$$

so $L(f,P) \leq L(f,Q)$.

(Proving $U(f,P) \geq U(f,Q)$ is similar: check!) □

Rigorous development of the integral

Theorem (Partition Theorem)

Let P_1 and P_2 be any two partitions of $[a, b]$. If f is bounded on $[a, b]$ then

$$L(f, P_1) \leq U(f, P_2).$$

Proof.

This is a straightforward consequence of the partition lemma.

Let $P = P_1 \cup P_2$, *i.e.*, P is the partition obtained by combining all the points of P_1 and P_2.

Then

$$L(f, P_1) \leq L(f, P) \leq U(f, P) \leq U(f, P_2).$$

□

Rigorous development of the integral

Important inferences that follow from the partition theorem:

- For <u>any</u> partition P', the upper sum $U(f,P')$ is an upper bound for the set of <u>all</u> lower sums $L(f,P)$.

  $$\therefore \quad \sup\{L(f,P) : P \text{ a partition of } [a,b]\} \le U(f,P') \qquad \forall P'$$

  $$\therefore \quad \sup\{L(f,P)\} \le \inf\{U(f,P)\}$$

  $\therefore$ For <u>any</u> partition P',

  $$L(f,P') \le \sup\{L(f,P)\} \le \inf\{U(f,P)\} \le U(f,P')$$

- If $\sup\{L(f,P)\} = \inf\{U(f,P)\}$ then we can define "area of $R(f,a,b)$" to be this number.
  - Is it possible that $\sup\{L(f,P)\} < \inf\{U(f,P)\}$?

Poll

- Go to
  https://www.childsmath.ca/childsa/forms/main_login.php
- Click on Math 3A03
- Click on Take Class Poll
- Fill in poll **Integrals:** $\sup\{L(f,P)\} < \inf\{U(f,P)\}$ **?**
- Submit.

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Mathematics and Statistics

$$\int_M d\omega = \int_{\partial M} \omega$$

Mathematics 3A03
Real Analysis I

Instructor: David Earn

Lecture 8
Integration II
Friday 24 January 2025

Announcements

- Assignment 2 will be posted either during, or soon after, the weekend.
- Kieran's office hours going forward are as follows:
  - Thursday 12:30–1:30 (Math Café)
  - Friday 12:30–1:30 (HH 207)

Poll

- Go to
  https://www.childsmath.ca/childsa/forms/main_login.php
- Click on Math 3A03
- Click on Take Class Poll
- Fill in poll **Integrals:** $\sup\{L(f,P)\} < \inf\{U(f,P)\}$ **?** **(AGAIN!)**
- `Submit`.

Rigorous development of the integral

Example

$\exists ?\ f : [a,b] \to \mathbb{R}$ (bounded) $\ni$ $\sup\{L(f,P)\} < \inf\{U(f,P)\}$

$$\text{Let} \qquad f(x) = \begin{cases} 1 & x \in \mathbb{Q} \cap [a,b], \\ 0 & x \in \mathbb{Q}^c \cap [a,b]. \end{cases}$$

Consider any partition P of $[a,b]$.
If $P = \{t_0, \dots, t_n\}$ then $m_i = 0\ \forall i \qquad (\because \quad [t_{i-1}, t_i] \cap \mathbb{Q}^c \neq \varnothing)$,
and $M_i = 1\ \forall i \qquad (\because \quad [t_{i-1}, t_i] \cap \mathbb{Q} \neq \varnothing)$.

$\therefore \quad L(f,P) = 0 \quad$ and $\quad U(f,P) = b - a \quad$ for **any** partition P.

$\therefore \quad \sup\{L(f,P)\} = 0 < b - a = \inf\{U(f,P)\}$. □

Can we define "area of $R(f,a,b)$" for such a weird function?
Yes, but not in this course!

Rigorous development of the integral

Definition (Integrable)

A function $f : [a, b] \to \mathbb{R}$ is said to be ***integrable*** on $[a, b]$ if it is bounded on $[a, b]$ and

$$\sup\{L(f, P) : P \text{ a partition of } [a, b]\} = \inf\{U(f, P) : P \text{ a partition of } [a, b]\}\,.$$

In this case, this common number is called the ***integral*** of f on $[a, b]$ and is denoted

$$\int_a^b f$$

Note: If f is integrable then for any partition P we have

$$L(f, P) \le \int_a^b f \le U(f, P)\,,$$

and $\int_a^b f$ is the unique number with this property.

Rigorous development of the integral

- *Notation:*

$$\int_a^b f(x)\,dx \qquad \text{means precisely the same as} \qquad \int_a^b f\,.$$

  - The symbol "dx" has no meaning in isolation just as "$x \to$" has no meaning except in $\lim_{x\to a} f(x)$.

- It is not clear from the definition which functions are integrable.

- The definition of the integral does not itself indicate how to compute the integral of any given integrable function. So far, without a lot more effort, we can't say much more than these two things:
  1. If $f(x) \equiv c$ then f is integrable on $[a, b]$ and $\int_a^b f = c \cdot (b - a)$.
  2. The weird example function is <u>not</u> integrable.

Rigorous development of the integral

- Bartle and Sherbert refer to functions that are integrable according to our definition as ***Darboux integrable*** (BS §7.4, p. 225).
- BS develop the integral using one value of the function within each subinterval of a partition, rather than starting with upper and lower sums. They refer to functions that are integrable in this sense as ***Riemann integrable***.
- BS also prove (BS Theorem 7.4.11, p. 232) that a function is Riemann integrable if and only if it is Darboux integrable. So the two definitions are, in fact, equivalent.
- In Math 4A03 you will define ***Lebesgue integrable***, a more subtle concept that makes it possible to attach meaning to "area of $R(f, a, b)$" for the weird example function (among others), and to precisely characterize functions that are Riemann integrable.

Rigorous development of the integral

Theorem (Equivalent "ε-P" criterion for integrability)

A bounded function $f : [a, b] \to \mathbb{R}$ is integrable on $[a, b]$ iff for all $\varepsilon > 0$ there is a partition P of $[a, b]$ such that

$$U(f, P) - L(f, P) < \varepsilon\,.$$

(BS Theorem 7.4.8, p. 229)

Note: This theorem is just a restatement of the definition of integrability. It is often more convenient to work with $\varepsilon > 0$ than with sup's and inf's.

Rigorous development of the integral

Proof of equivalence of "sup = inf" and "ε-P" definitions of integrability.

($\Longrightarrow$) Suppose the bounded function f is integrable, *i.e.*,

$$\sup\{L(f,P) : P \text{ a partition of } [a,b]\}$$
$$= \inf\{U(f,P) : P \text{ a partition of } [a,b]\} = \int_a^b f$$

Given $\varepsilon > 0$, since $\displaystyle\int_a^b f$ is the least upper bound of the lower sums, there is a partition P_1 such that

$$\int_a^b f \;=\; \sup_{P'}\{L(f,P')\} \;<\; L(f,P_1) + \frac{\varepsilon}{2}\,,$$

i.e., such that
$$-L(f,P_1) \;<\; -\int_a^b f + \frac{\varepsilon}{2}\,. \qquad (\heartsuit)$$

. . . continued. . .

Rigorous development of the integral

Proof of equivalence of "sup = inf" and "ε-P" definitions of integrability.

Similarly, there is a partition P_2 such that

$$U(f,P_2) \; < \; \inf_{P'}\{U(f,P')\} + \frac{\varepsilon}{2} \; = \; \int_a^b f + \frac{\varepsilon}{2}\,. \qquad (\diamondsuit)$$

Therefore, putting together inequalities ($\diamondsuit$) and ($\heartsuit$), we have

$$U(f,P_2) - L(f,P_1) \; < \; \int_a^b f + \frac{\varepsilon}{2} - \int_a^b f + \frac{\varepsilon}{2} \; = \; \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \; = \; \varepsilon\,.$$

But that's not quite what we need. We need, for a *single* partition P,

$$U(f,P) - L(f,P) < \varepsilon.$$

How should we proceed? *Hint:* Recall the partition lemma ...

...continued...

Rigorous development of the integral

Proof of equivalence of "sup = inf" and "ε-P" definitions of integrability.

Let $P = P_1 \cup P_2$. Then the partition lemma implies that $L(f,P) \geq L(f,P_1)$, and $U(f,P) \leq U(f,P_2)$, so

$$\begin{aligned} U(f,P) - L(f,P) &\leq U(f,P_2) - L(f,P_1) \\ &< \int_a^b f + \frac{\varepsilon}{2} - \int_a^b f + \frac{\varepsilon}{2} = \varepsilon\,, \end{aligned}$$

which competes the proof that sup = inf $\implies$ ε-P.

($\Longleftarrow$) We now need to show that if a bounded function f satisfies the ε-P definition of integrability then it also satisfies the sup = inf definition of integrability.

Given $\varepsilon > 0$, we can choose a partition P (depending on ε) such that

$$U(f,P) - L(f,P) < \varepsilon\,.$$

. . . continued. . .

Rigorous development of the integral

Proof of equivalence of "$\sup = \inf$" and "ε-P" definitions of integrability.

Now, for any partition, and in particular for P, we have

$$L(f,P) \ \le \ \sup_{P'}\{L(f,P')\} \le \inf_{P'}\{U(f,P')\} \le U(f,P)\,,$$

We can temporarily write this more simply as

$$L \ \le \ S \ \le \ I \ \le \ U$$

Subtracting S from this chain of inequalities implies

$$L - S \ \le 0 \ \le \ I - S \ \le \ U - S$$

Now note that $L \le S$ implies $U - S \le U - L$, so we have

$$0 \ \le \ I - S \ \le \ U - L$$

i.e., $\quad 0 \ \le \ \inf_{P'}\{U(f,P')\} - \sup_{P'}\{L(f,P')\} \ \le \ U(f,P) - L(f,P) \ < \ \varepsilon\,.$

But by hypothesis, such a partition P can be found for any given $\varepsilon > 0$.

Therefore, $\inf_{P'}\{U(f,P')\} = \sup_{P'}\{L(f,P')\}$. □

Rigorous development of the integral

Example

Suppose $b > 0$ and $f(x) = x$ for all $x \in \mathbb{R}$. Prove, using only the definition of the integral via sup $=$ inf or ε-P, that

$$\int_0^b f = \frac{b^2}{2}\,.$$

(This exercise should help you appreciate the Fundamental Theorem of Calculus.)

Note: If working through the above example doesn't convince you of the power of the Fundamental Theorem of Calculus, try computing $\int_0^b x^2\,dx$ directly from the definition of the integral.

McMaster University

Mathematics and Statistics

$$\int_M d\omega = \int_{\partial M} \omega$$

Mathematics 3A03
Real Analysis I

Instructor: David Earn

Lecture 9
Integration III
Monday 27 January 2025

Announcements

- Assignment 2 has been posted on the course web site.
  - The participation deadline is Monday 3 Feb 2025 @ 11:25am.
- On Friday this week, the class will be a Q&A session with the TA. It's a great opportunity to ask questions about Assignment 2, or anything else.

Last time. . .

Rigorous development of the integral:

- Definition: integrable.
- Example: non-integrable function.
- Theorem: Equivalent "ε-P" definition of integrable.
- *Note:* The different equivalent definitions are most convenient in different contexts, *e.g.*,
  - Proving non-integrability of the weird example was easiest using the sup-inf definition.
  - Computing the value of $\int_0^b x\,\mathrm{d}x$ is easiest using the ε-P definition.

Poll

- Go to https://www.childsmath.ca/childsa/forms/main_login.php
- Click on Math 3A03
- Click on Take Class Poll
- Fill in poll **Integrals: Integrable vs Continuous vs Differentiable**
- `Submit`.

Integral theorems

Theorem (continuous $\Longrightarrow$ integrable)

If f is continuous on $[a, b]$ then f is integrable on $[a, b]$.

Rough work to prepare for proof:

$$U(f,P) - L(f,P) = \sum_{i=1}^{n} (M_i - m_i)(t_i - t_{i-1})$$

Given $\varepsilon > 0$, choose a partition P that is so fine that $M_i - m_i < \varepsilon$ for all i (possible because f is continuous and bounded). Then

$$U(f,P) - L(f,P) < \varepsilon \sum_{i=1}^{n} (t_i - t_{i-1}) = \varepsilon(b-a)\,.$$

Not quite what we want. So choose the partition P such that $M_i - m_i < \varepsilon/(b-a)$ for all i. To get that, choose P such that

$$|f(x) - f(y)| < \frac{\varepsilon}{2(b-a)} \qquad \text{if } |x-y| < \max_{1 \le i \le n} (t_i - t_{i-1}),$$

which we can do because f is uniformly continuous on $[a, b]$.

Integral theorems

Proof that continuous $\implies$ integrable (cont.)

Since f is continuous on the closed interval $[a, b]$, it is bounded on $[a, b]$ (which is the first requirement to be integrable on $[a, b]$).

Also, since f is continuous on $[a, b]$, it is uniformly continuous on $[a, b]$. $\therefore \forall \varepsilon > 0 \ \exists \delta > 0$ such that $\forall x, y \in [a, b]$,

$$|x - y| < \delta \quad \implies \quad |f(x) - f(y)| < \frac{\varepsilon}{2(b-a)} .$$

Now choose a partition of $[a, b]$ such that the length of each subinterval $[t_{i-1}, t_i]$ is less than δ, *i.e.*, $t_i - t_{i-1} < \delta$. Then, for any $x, y \in [t_{i-1}, t_i]$, we have $|x - y| < \delta$ and therefore

...continued...

Integral theorems

Proof that continuous $\Longrightarrow$ integrable (cont.)

$$|f(x) - f(y)| < \frac{\varepsilon}{2(b-a)} \qquad \forall x, y \in [t_{i-1}, t_i]\,.$$

$$\therefore \qquad M_i - m_i \;\le\; \frac{\varepsilon}{2(b-a)} < \frac{\varepsilon}{b-a} \qquad i = 1, \dots, n.$$

Since this is true for all i, it follows that

$$\begin{aligned} U(f,P) - L(f,P) &= \sum_{i=1}^{n} (M_i - m_i)(t_i - t_{i-1}) \\ &< \frac{\varepsilon}{b-a} \sum_{i=1}^{n} (t_i - t_{i-1}) = \frac{\varepsilon}{b-a}(b-a) = \varepsilon\,. \end{aligned}$$

$\square$

Properties of the integral

Theorem (Integral segmentation)

Let $a < c < b$. If f is integrable on $[a, b]$, then f is integrable on $[a, c]$ and on $[c, b]$. Conversely, if f is integrable on $[a, c]$ and $[c, b]$ then f is integrable on $[a, b]$. Finally, if f is integrable on $[a, b]$ then

$$\int_a^b f = \int_a^c f + \int_c^b f\,. \qquad (\heartsuit)$$

(a good exercise)

This theorem motivates these definitions:

$$\int_a^a f = 0 \qquad \text{and} \quad \text{if } a > b \quad \text{then} \qquad \int_a^b f = -\int_b^a f\,.$$

Then $(\heartsuit)$ holds for any $a, b, c \in \mathbb{R}$.

Properties of the integral

Theorem (Algebra of integrals – a.k.a. $\int_a^b$ is a linear operator)

If f and g are integrable on $[a, b]$ and $c \in \mathbb{R}$ then $f + g$ and cf are integrable on $[a, b]$ and

1. $\displaystyle\int_a^b (f+g) = \int_a^b f + \int_a^b g;$
2. $\displaystyle\int_a^b cf = c\int_a^b f.$

(proofs are relatively easy; good exercises) (BS Theorem 7.1.5, p. 204)

Properties of the integral

Theorem (Integral of a product)

If f and g are integrable on $[a, b]$ then fg is integrable on $[a, b]$.

(compared to integral of a sum, proof is much harder; tough exercise)

Note:

- There is no "product rule" for integrals. While f and g integrable does imply fg integrable, we cannot write the integral of the product fg in terms of the integrals of the factors f and g.
- The closest we can come to a product formula is integration by parts, which arises from the Fundamental Theorem of Calculus together with the product rule for *derivatives*.

Properties of the integral

Lemma (Integral bounds)

Suppose f is integrable on $[a,b]$. If $m \le f(x) \le M$ for all $x \in [a,b]$ then

$$m(b-a) \le \int_a^b f \le M(b-a)\,.$$

Proof.

For any partition P, we must have $m \le m_i\ \forall i$ and $M \ge M_i\ \forall i$.

$$\therefore \quad m(b-a) \le L(f,P) \le U(f,P) \le M(b-a) \qquad \forall P$$

$$\therefore \quad m(b-a) \le \sup\{L(f,P)\} = \int_a^b f = \inf\{U(f,P)\} \le M(b-a)\,.$$

□

Properties of the integral

Theorem (Integrals are continuous)

If f is integrable on $[a, b]$ and F is defined on $[a, b]$ by

$$F(x) = \int_a^x f\,,$$

then F is continuous on $[a, b]$.

Proof

Let's first consider $x_0 \in [a, b)$ and show F is continuous from above at x_0, *i.e.*, $\lim_{x \to x_0^+} F(x) = F(x_0)$. If $x \in (x_0, b]$ then

$$(\heartsuit) \qquad \Longrightarrow \qquad F(x) - F(x_0) \;=\; \int_a^x f - \int_a^{x_0} f \;=\; \int_{x_0}^x f\,. \qquad (*)$$

. . . continued. . .

Properties of the integral

Proof that integrals are continuous (cont.)

Since f is integrable on $[a, b]$, it is bounded on $[a, b]$, so $\exists M > 0$ such that

$$-M \le f(x) \le M \qquad \forall x \in [a, b],$$

from which the integral bounds lemma implies

$$-M(x - x_0) \; \le \; \int_{x_0}^{x} f \le M(x - x_0)\,,$$

$$\therefore \quad (*) \implies -M(x - x_0) \; \le \; F(x) - F(x_0) \; \le \; M(x - x_0)\,.$$

$\therefore$ For any $\varepsilon > 0$, we can ensure $|F(x) - F(x_0)| < \varepsilon$ by requiring $0 \le x - x_0 < \varepsilon/M$, which proves $\lim_{x\to x_0^+} F(x) = F(x_0)$.

A similar argument starting from $x_0 \in (a, b]$ and $x \in [a, x_0)$ yields $\lim_{x\to x_0^-} F(x) = F(x_0)$. Thus, "integrals are continuous". $\square$

Mathematics and Statistics

$$\int_M d\omega = \int_{\partial M} \omega$$

Mathematics 3A03
Real Analysis I

Instructor: David Earn

Lecture 10
Integration IV
Wednesday 29 January 2025

Announcements

- Assignment 2 has been posted on the course web site.
  - The participation deadline is Monday 3 Feb 2025 @ 11:25am.
- On Friday this week, the class will be a Q&A session with the TA. It's a great opportunity to ask questions about Assignment 2, or anything else.
- The poll for Assignment 2 participation will open after class today until 11:25am on Monday.
- I have an office hour today, 2:00-3:00 pm.

Last time. . .

Rigorous development of the integral:

- continuous $\implies$ integrable.
- Integral segmentation.
- Algebra of integrals.
- Integral bounds lemma.
- Integrals are continuous.

Fundamental Theorem of Calculus

Theorem (First Fundamental Theorem of Calculus – FFTC)

Let f be integrable on $[a, b]$, and define F on $[a, b]$ by

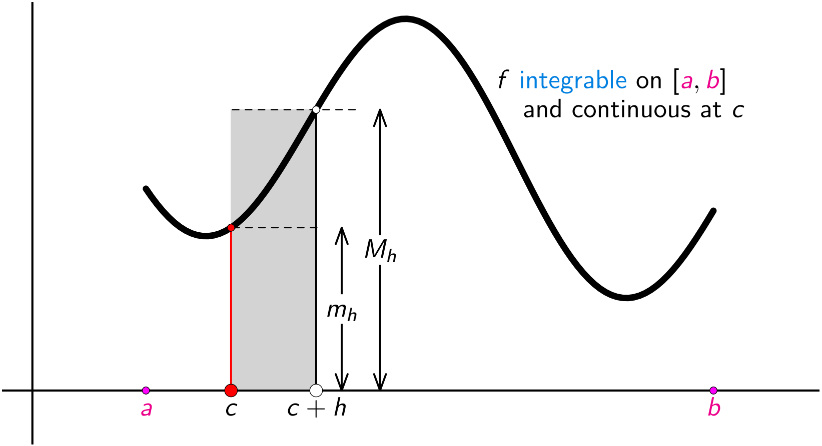
$$F(x) = \int_a^x f\,.$$

If f is continuous at $c \in [a, b]$, then F is differentiable at c, and

$$F'(c) = f(c)\,.$$

- If $c = a$ or $c = b$, then $F'(c)$ is understood to mean the right- or left-hand derivative of F.
- The "integrals are continuous" theorem implies that F is continuous on all of $[a, b]$. The FFTC says, in addition, that F is differentiable at the single point c.
- The FFTC implies that if f is continuous on all of $[a, b]$ then F is differentiable on all of $[a, b]$.

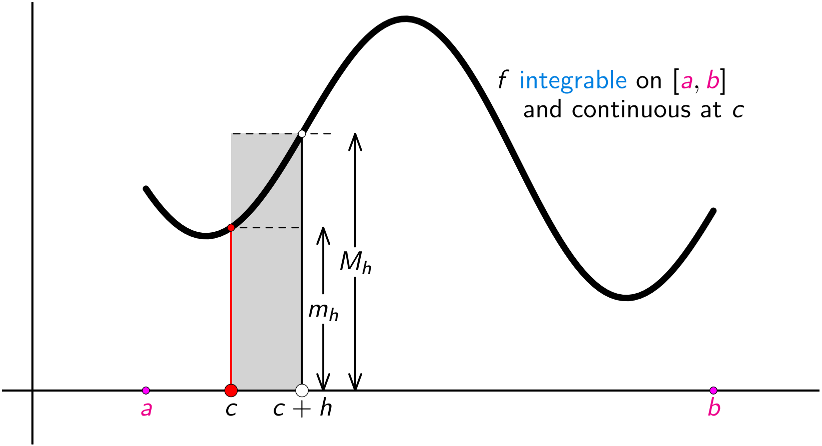
Fundamental Theorem of Calculus



$$F(c+h) - F(c) \simeq f(c+h) \cdot h$$

and $\lim_{h \to 0} f(c+h) = f(c)$ $\Longrightarrow$ $\lim_{h \to 0} \frac{F(c+h) - F(c)}{h} = f(c)$

Fundamental Theorem of Calculus

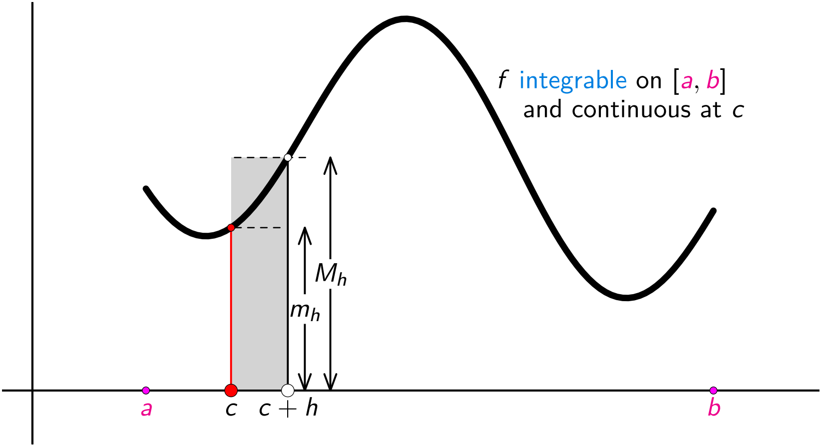


$$F(c+h) - F(c) \simeq f(c+h) \cdot h$$

and $\lim_{h \to 0} f(c+h) = f(c)$

$$\Longrightarrow \quad \lim_{h \to 0} \frac{F(c+h) - F(c)}{h} = f(c)$$

Fundamental Theorem of Calculus

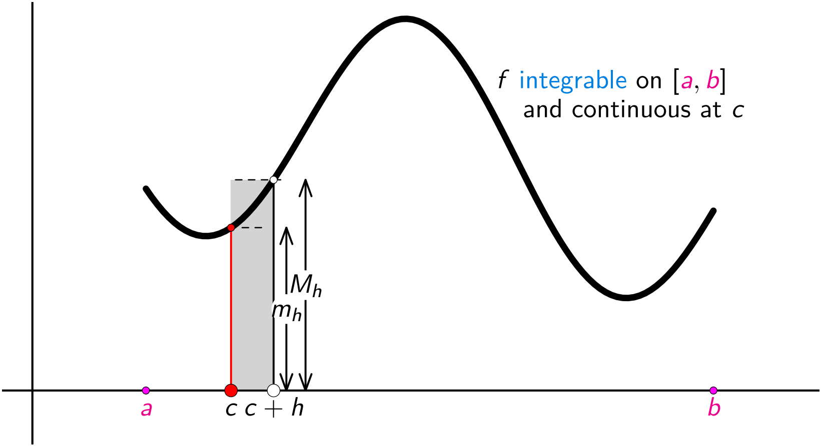


$$F(c+h) - F(c) \simeq f(c+h) \cdot h$$

and $\lim_{h \to 0} f(c+h) = f(c)$

$$\Longrightarrow \quad \lim_{h \to 0} \frac{F(c+h) - F(c)}{h} = f(c)$$

Fundamental Theorem of Calculus



$$F(c+h) - F(c) \simeq f(c+h) \cdot h \quad \text{and} \quad \lim_{h \to 0} f(c+h) = f(c) \quad \Longrightarrow \quad \lim_{h \to 0} \frac{F(c+h) - F(c)}{h} = f(c)$$

Fundamental Theorem of Calculus

Proof of First Fundamental Theorem of Calculus

Suppose $c \in [a, b)$, and $0 < h \leq b - c$. Then the integral segmentation theorem implies that

$$F(c+h) - F(c) \;=\; \int_a^{c+h} f - \int_a^c f \;=\; \int_c^{c+h} f\,.$$

Motivated by the sketch, define

$$m_h \;=\; \inf\{\, f(x) \,:\, x \in [c, c+h] \,\}\,,$$

$$M_h \;=\; \sup\{\, f(x) \,:\, x \in [c, c+h] \,\}\,.$$

Then the integral bounds lemma implies

$$m_h \cdot h \;\leq\; \int_c^{c+h} f \;\leq\; M_h \cdot h\,,$$

. . . continued. . .

Fundamental Theorem of Calculus

Proof of First Fundamental Theorem of Calculus (cont.)

and hence

$$m_h \leq \frac{F(c+h)-F(c)}{h} \leq M_h \,.$$

This inequality is true for any integrable function. However, because f is continuous at c, we have

$$\lim_{h\to 0^+} m_h = f(c) = \lim_{h\to 0^+} M_h \,,$$

so the squeeze theorem (BS Theorem 4.2.6, p. 114) implies

$$F'_+(c) = \lim_{h\to 0^+} \frac{F(c+h)-F(c)}{h} = f(c) \,.$$

A similar argument for $c \in (a,b]$ and $-(c-a) \leq h < 0$ yields $F'_-(c) = f(c)$. □

Fundamental Theorem of Calculus

Corollary

If f is continuous on $[a, b]$ and $f = g'$ for some function g, then

$$\int_a^b f \;=\; g(b) - g(a)\,.$$

Proof.

Let $F(x) = \displaystyle\int_a^x f$. Then $\forall x \in [a, b]$, $F'(x) = f(x)$ (by FFTC).

$$\implies F' \;=\; f = g'.$$

$\therefore\;\; \exists c \in \mathbb{R}$ such that $F = g + c$ (Assignment 1).

$\therefore\;\; F(a) = g(a) + c.$ But $F(a) = \int_a^a f = 0$, so $c = -g(a)$.

$\therefore\;\; F(x) = g(x) - g(a).$

This is true, in particular, for $x = b$, so $\displaystyle\int_a^b f = g(b) - g(a)$. □

Poll

- Go to https://www.childsmath.ca/childsa/forms/main_login.php
- Click on Math 3A03
- Click on Take Class Poll
- Fill in poll **Integrals: Fundamental Theorem of Calculus**
- `Submit`.

Fundamental Theorem of Calculus

Theorem (Second Fundamental Theorem of Calculus)

If f is integrable on $[a, b]$ and $f = g'$ for some function g, then

$$\int_a^b f = g(b) - g(a)\,.$$

Notes:

- This looks like the corollary to the first fundamental theorem, except that f is assumed only to be integrable, not continuous.
- Recall from Darboux's theorem that if $f = g'$ for some g then f has the intermediate value property, but f need not be continuous.
- g' exists on $[a, b] \implies$ applies to g.
- The proof of the second fundamental theorem is completely different from the corollary to the first, because we cannot use the first fundamental theorem (which assumed f is continuous).

Fundamental Theorem of Calculus

Proof of Second Fundamental Theorem of Calculus

Let $P = \{t_0, \ldots, t_n\}$ be any partition of $[a, b]$. By the Mean Value Theorem, for each $i = 1, \ldots, n$, $\exists\, x_i \in [t_{i-1}, t_i]$ such that

$$g(t_i) - g(t_{i-1}) \;=\; g'(x_i)(t_i - t_{i-1}) \;=\; f(x_i)(t_i - t_{i-1})\,.$$

Define m_i and M_i as usual. Then $m_i \le f(x_i) \le M_i \;\forall i$, so

$$m_i(t_i - t_{i-1}) \;\le\; f(x_i)(t_i - t_{i-1}) \;\le\; M_i(t_i - t_{i-1})\,,$$

$$\text{i.e.,}\quad m_i(t_i - t_{i-1}) \;\le\; g(t_i) - g(t_{i-1}) \;\le\; M_i(t_i - t_{i-1})\,.$$

$$\therefore\quad \sum_{i=1}^{n} m_i(t_i - t_{i-1}) \;\le\; \sum_{i=1}^{n} \Big(g(t_i) - g(t_{i-1})\Big) \;\le\; \sum_{i=1}^{n} M_i(t_i - t_{i-1})$$

$$\text{i.e.,}\qquad L(f,P) \;\le\; g(b) - g(a) \;\le\; U(f,P)$$

for <u>any</u> partition P. $\quad\therefore\quad g(b) - g(a) = \int_a^b f\,.$ □

What useful things can we do with integrals?

- Compute areas of complicated shapes: find anti-derivatives and use the second fundamental theorem of calculus.
- Define trigonometric functions (rigorously).
- Define logarithm and exponential functions (rigorously).

Mathematics
and Statistics

$$\int_M d\omega = \int_{\partial M} \omega$$

Mathematics 3A03
Real Analysis I

Instructor: David Earn

Lecture 11
Integration V
Monday 3 February 2025

Announcements

- The participation deadline for Assignment 2 is today, Monday 3 Feb 2025 @ 11:25am.
- If you haven't participated yet, do the poll **now**.

Last time. . .

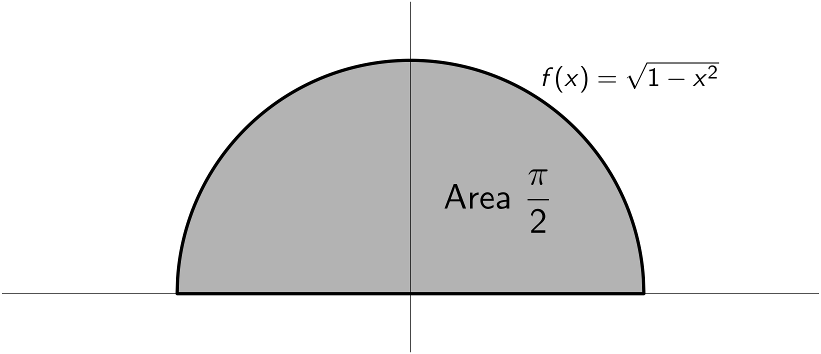
Rigorous development of the integral

- First Fundamental Theorem of Calculus.
- Corollary to FFTC.
- Second Fundamental Theorem of Calculus.
- What can we do with the integral?

Poll

- Go to https://www.childsmath.ca/childsa/forms/main_login.php
- Click on Math 3A03
- Click on Take Class Poll
- Fill in poll **What is π ?**
- Submit.

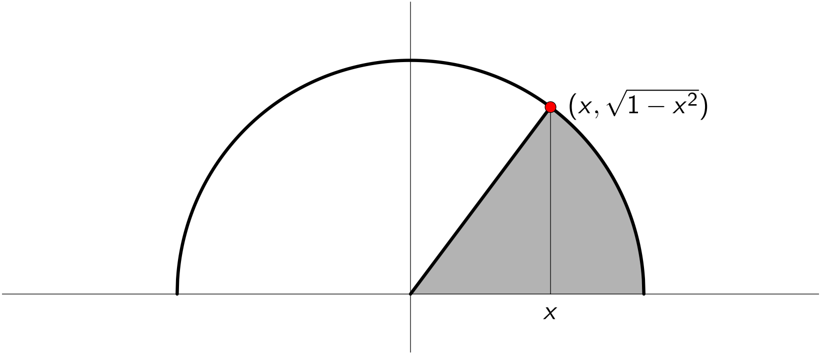
What is π ?



Definition

$$\pi \;\equiv\; 2\int_{-1}^{1} \sqrt{1 - x^2}\; dx\,.$$

What are cos and sin ?

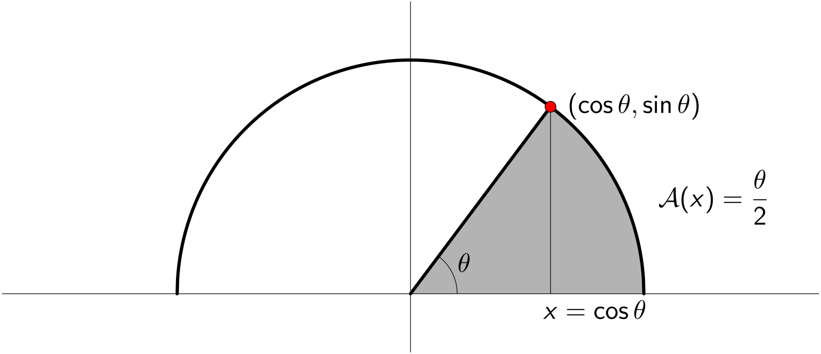


Definition (Sectoral area)

If $x \in [-1,1]$ then $\quad \mathcal{A}(x) = \dfrac{x\sqrt{1-x^2}}{2} + \displaystyle\int_x^1 \sqrt{1-t^2}\,dt\,.$

Note: $\mathcal{A}(-1) = \pi/2, \quad \mathcal{A}(1) = 0.$

What are cos and sin ?



Length of circular arc swept out by angle θ: θ

Area of sectoral region swept out by angle θ: $\theta/2$

So, if $\theta \in [0, \pi]$ then we define $\cos\theta$ to be the unique number in $[-1, 1]$ such that $\mathcal{A}(\cos\theta) = \theta/2$, and we define $\sin\theta$ to be $\sqrt{1-(\cos\theta)^2}$.

We must prove: given $x \in [0, \pi]$ $\exists!\, y \in [-1, 1]$ such that $\mathcal{A}(y) = x/2$.

What are cos and sin ?

Proof that $\forall\, x \in [0, \pi]\ \exists!\, y \in [-1, 1]$ *such that* $\mathcal{A}(y) = x/2$*:*

Existence: $\mathcal{A}(1) = 0$, $\mathcal{A}(-1) = \pi/2$, and $\mathcal{A}$ is continuous. Hence by the intermediate value theorem $\exists\, y \in [-1, 1]$ such that $\mathcal{A}(y) = x/2$.

Uniqueness: $\mathcal{A}$ is differentiable on $(-1, 1)$ and $\mathcal{A}'(x) < 0$ on $(-1, 1)$.
$\therefore$ On $(-1, 1)$, $\mathcal{A}$ is decreasing, and hence one-to-one.

Definition (cos and sin)

If $x \in [0, \pi]$ then $\cos x$ is the unique number in $[-1, 1]$ such that $\mathcal{A}(\cos x) = x/2$, and $\sin x = \sqrt{1 - (\cos x)^2}$.

These definitions are easily extended to all of $\mathbb{R}$:

- For $x \in [\pi, 2\pi]$, define $\cos x = \cos(2\pi - x)$ and $\sin x = -\sin(2\pi - x)$.
- Then, for $x \in \mathbb{R} \setminus [0, 2\pi]$ define $\cos x = \cos(x \mod 2\pi)$ and $\sin x = \sin(x \mod 2\pi)$.

Trigonometric theorems

Given the rigorous definition of cos and sin, we can prove:

1. cos and sin are differentiable on $\mathbb{R}$. Moreover, $\cos' = -\sin$ and $\sin' = \cos$.
2. sec, tan, csc and cot can all be defined in the usual way and have all the usual properties.
3. The inverse function theorem allows us to define, and compute the derivatives of, all the inverse trigonometric functions.
4. If f is twice differentiable on $\mathbb{R}$, $f'' + f = 0$, $f(0) = a$ and $f'(0) = b$, then $f = a\cos + b\sin$.
5. For all $x, y \in \mathbb{R}$,

$$\sin(x+y) = \sin x \cos y + \cos x \sin y\,,$$
$$\cos(x+y) = \cos x \cos y - \sin x \sin y\,.$$

Something deep that you know enough to prove

Extra Challenge Problem:

Prove that π is irrational.

Hint: Suppose $\pi^2 = \frac{a}{b}$, for $a, b \in \mathbb{N}$.
Show that the smallest positive root of sin is irrational.

What are log and exp ?

Consider the function

$$f(x) = 10^x .$$

What exactly is this function?

In our mathematically naïve previous life, we just assumed that $f(x)$ is well-defined $\forall\, x \in \mathbb{R}$, and that f has a well-defined inverse function,

$$f^{-1}(x) = \log_{10}(x) .$$

But how are 10^x and $\log_{10}(x)$ defined for irrational x ?

Let's review what we know. . .

What are log and exp ?

$$n \in \mathbb{N} \quad \Longrightarrow \quad 10^n = \underbrace{10 \cdots 10}_{n \text{ times}}$$

$$n, m \in \mathbb{N} \quad \Longrightarrow \quad 10^n \cdot 10^m = 10^{n+m}$$

When we extend 10^x to $x \in \mathbb{Q}$, we want this product rule to be preserved:

$$10^0 \cdot 10^n = 10^{0+n} = 10^n \quad \Longrightarrow \quad 10^0 = 1$$

$$10^{-n} \cdot 10^n = 10^0 = 1 \quad \Longrightarrow \quad 10^{-n} = \frac{1}{10^n}$$

$$\underbrace{10^{1/n} \cdots 10^{1/n}}_{n \text{ times}} = 10^{\overbrace{}^{}}\underbrace{^{1/n \cdots 1/n}}_{n \text{ times}} = 10^1 = 10 \quad \Longrightarrow \quad 10^{1/n} = \sqrt[n]{10}$$

What are log and exp ?

Finally, to define 10^q for all $q \in \mathbb{Q}$, note that we must have

$$\left(10^{\frac{1}{n}}\right)^m = \underbrace{10^{\frac{1}{n}} \cdots 10^{\frac{1}{n}}}_{m \text{ times}} = 10^{\overbrace{\frac{1}{n}+\cdots+\frac{1}{n}}^{}}_{\,m \text{ times}} = 10^{\frac{m}{n}} \quad \Longrightarrow \quad 10^{\frac{m}{n}} \stackrel{\textbf{def}}{=} (\sqrt[n]{10})^m$$

Now we're stuck. *How do we extend this scheme to irrational x?*

We need a more sophisticated idea.

Let's try to find a function on all of $\mathbb{R}$ that satisfies

$$f(x+y) = f(x) \cdot f(y)\,, \qquad \forall x, y \in \mathbb{R},$$
$$\text{and} \qquad f(1) = 10\,.$$

It then follows that $f(0) = 1$ and, $\forall\, x \in \mathbb{Q}$, $f(x) = [f(1)]^x$.

What additional properties can we impose on $f(x)$ that will lead us to a sensible definition of $f(x)$ for all $x \in \mathbb{R}$?

Mathematics and Statistics

$$\int_M d\omega = \int_{\partial M} \omega$$

Mathematics 3A03
Real Analysis I

Instructor: David Earn

Lecture 12
Integration VI
Wednesday 5 February 2025

Announcements

- Assignment 2 solutions are posted.

Last time. . .

- Rigorous definition of trig functions.
- Working towards rigorous definition of 10^x for $x \in \mathbb{R}$.

What are log and exp ?

One approach is to insist that f is *differentiable*.

Then we can compute

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{f(x)\cdot f(h)-f(x)}{h}$$
$$= f(x)\cdot \lim_{h\to 0} \frac{f(h)-1}{h} \quad = \quad f(x)\cdot f'(0) \quad \equiv \quad \alpha\, f(x)$$

So $f'(x) = \alpha\, f(x)$, *i.e.*, we have f' in terms of unknowns f and α.
So what?!?

Let's look at the inverse function, f^{-1} (think "$\log_{10}$"):

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))} = \frac{1}{\alpha\, f(f^{-1}(x))} = \frac{1}{\alpha\, x}$$

Holy $#@%! We have a simple formula for the derivative of f^{-1}!

What are log and exp ?

Since we want $\log_{10} 1 = 0$, we should define $\log_{10} x$ as $(1/\alpha) \int_1^x t^{-1}\, dt$. *Great idea, but we don't know what α is.*

So, let's ignore α ...

(and hope that what we end up with is log to some "natural" base).

Definition (Logarithm function)

If $x > 0$ then

$$\log x = \int_1^x \frac{1}{t}\, dt\,.$$

This function is strictly increasing ($\log'(x) > 0$ for all $x > 0$) and hence one-to-one, so we can now define:

Definition (Exponential function)

$$\exp = \log^{-1}\,.$$

What are log and exp ?

With these rigorous defintions of log and exp, we can prove the following as theorems:

1. If $x, y > 0$ then $\log(xy) = \log x + \log y$.
2. If $x, y > 0$ then $\log(x/y) = \log x - \log y$.
3. If $n \in \mathbb{N}$ and $x > 0$ then $\log(x^n) = n \log x$.
4. For all $x \in \mathbb{R}$, $\exp'(x) = \exp(x)$.
5. For all $x, y \in \mathbb{R}$, $\exp(x + y) = \exp(x) \cdot \exp(y)$.
6. For all $x \in \mathbb{Q}$, $\exp(x) = [\exp(1)]^x$.

The last theorem above motivates:

Definition

$$e \equiv \exp(1)\,,$$
$$e^x \equiv \exp(x) \qquad \textbf{for all } x \in \mathbb{R}.$$

What are log and exp ?

We can now give a rigorous definition of 10^x for any $x \in \mathbb{R}$.
In fact, we can do this for any $a > 0$.

Definition (a^x)

If $a > 0$ and x is any real number then

$$a^x \equiv e^{x \log a} .$$

We then have the following theorems for any $a > 0$:

1. $(a^x)^y = a^{xy}$ for all $x, y \in \mathbb{R}$;
2. $a^0 = 1; \quad a^1 = a$;
3. $a^{x+y} = a^x \cdot a^y$ for all $x, y \in \mathbb{R}$;
4. $a^{-x} = 1/a^x$ for all $x \in \mathbb{R}$;
5. if $a > 1$ then a^x is increasing on $\mathbb{R}$;
6. if $0 < a < 1$ then a^x is decreasing on $\mathbb{R}$.

Using the integral to define useful functions rigorously

- Just as we defined 10^x via the definition of $\log x = \int_1^x \frac{1}{t}\,dt$, we could have defined the trigonometric functions starting from

$$\arcsin x = \int_0^x \frac{1}{\sqrt{1-t^2}}\,dt\,, \qquad -1 < x < 1,$$

  rather than the definition of cos via $\mathcal{A}(x)$. Many common functions are defined as integrals of rational functions of square roots.

- Any compositions of trig functions, log, exp, rational functions and radicals, are called ***elementary functions***.

- Most functions that turn up a lot in applications can be defined rigorously via integrals of elementary functions. Such functions are collectively called ***special functions***.

Poll

- Go to https://www.childsmath.ca/childsa/forms/main_login.php
- Click on Math 3A03
- Click on Take Class Poll
- Fill in poll **What kind of number is e ?**
- Submit.

Approximation by Polynomial Functions

Definition (Taylor polynomial)

If f is n times differentiable at a then the ***Taylor polynomial of degree*** n ***for*** f ***at*** a is

$$P_{n,a}(x) = f(a) + f'(a)(x-a) + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n .$$

Theorem (Taylor's theorem)

Suppose $f', \dots, f^{(n+1)}$ *are defined on* $[a, x]$*, and that* $R_{n,a}(x)$ *is defined by* $f(x) = P_{n,a}(x) + R_{n,a}(x)$. *Then*

$$R_{n,a}(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-a)^{n+1}, \qquad \text{for some } \xi \in (a, x) . \qquad (\heartsuit)$$

Note: The form of the remainder term here is known as the ***Lagrange form*** of the remainder.

Approximation by Polynomial Functions

Proof of Taylor's Theorem.

Let's prove this by induction, starting from the base case, $n = 0$.
For $n = 0$, the statement of Taylor's theorem is:

> *Suppose f' is defined on $[a, x]$, and that $R_{0,a}(x)$ is defined by* $f(x) = P_{0,a}(x) + R_{0,a}(x)$. *Then*
>
> $$R_{0,a}(x) = f'(\xi)(x - a), \qquad \textit{for some } \xi \in (a, x).$$

But $P_{0,a}(x) = f(a)$, so the claim for $n = 0$ is that

$$f(x) = f(a) + f'(\xi)(x - a), \qquad \text{for some } \xi \in (a, x).$$

Thus, for $n = 0$, Taylor's Theorem reduces to the Mean Value Theorem! So the base case ($n = 0$) is true.

. . . continued. . .

Approximation by Polynomial Functions

Proof of Taylor's Theorem.

Now suppose $n \geq 1$. By the induction hypothesis, we have

$$R_{n-1,a}(x) = \frac{f^{(n)}(\xi)}{n!}(x-a)^n\,, \qquad \text{for some } \xi \in (a,x)\,.$$

From this, how can we infer something related to $(\heartsuit)$? By definition,

$$\begin{aligned}
f(x) &= P_{n,a}(x) + R_{n,a}(x) = P_{n-1,a}(x) + \frac{f^{(n)}(a)}{n!}(x-a)^n + R_{n,a}(x) \\
&= \Big[P_{n-1,a}(x) + R_{n-1,a}(x)\Big] + \frac{f^{(n)}(a)}{n!}(x-a)^n + \Big[R_{n,a}(x) - R_{n-1,a}(x)\Big]
\end{aligned}$$

$$\begin{aligned}
\therefore\ \ 0 &= \frac{f^{(n)}(a)}{n!}(x-a)^n + R_{n,a}(x) - R_{n-1,a}(x) \\
&= \frac{f^{(n)}(a)}{n!}(x-a)^n + R_{n,a}(x) - \frac{f^{(n)}(\xi)}{n!}(x-a)^n\,, \quad \text{for some } \xi \in (a,x)\,.
\end{aligned}$$

Thus, $R_{n,a}(x) = \left[\frac{f^{(n)}(\xi)}{n!} - \frac{f^{(n)}(a)}{n!}\right](x-a)^n$, so $R_{n,a}(a) = 0$.

In fact, $R_{n,a}^{(k)}(a) = 0 \ \ \forall k = 0, 1, \ldots, n-1$. *...continued...*

Approximation by Polynomial Functions

Proof of Taylor's Theorem.

Now, since $a < x$, proving $(\heartsuit)$ is equivalent to proving

$$\frac{R_{n,a}(x)}{(x-a)^{n+1}} = \frac{f^{(n+1)}(\xi)}{(n+1)!}\,, \qquad \text{for some } \xi \in (a,x)\,. \qquad (\spadesuit)$$

To make the notation less cumbersome, write $G(x) = (x-a)^n$ (and note that $G^{(k)}(a) = 0 \ \forall k = 0, 1, \ldots, n-1$).

Then, for any $x > a$, we have

$$\frac{R_{n,a}(x)}{G(x)} = \frac{R_{n,a}(x) - R_{n,a}(a)}{G(x) - G(a)} = \frac{R'_{n,a}(\xi_1)}{G'(\xi_1)} \qquad \exists \xi_1 \in (a,x)$$

by Cauchy MVT (proved in Assignment 1). Similarly,

$$\frac{R'_{n,a}(\xi_1)}{G'(\xi_1)} = \frac{R'_{n,a}(\xi_1) - R'_{n,a}(a)}{G'(\xi_1) - G'(a)} = \frac{R''_{n,a}(\xi_2)}{G''(\xi_2)} \qquad \exists \xi_2 \in (a,\xi_1) \subset (a,x)$$

$$= \cdots = \frac{R^{(n+1)}_{n,a}(\xi_{n+1})}{G^{(n+1)}(\xi_{n+1})} \qquad \exists \xi_{n+1} \in (a,\xi_n) \subset (a,x)$$

Approximation by Polynomial Functions

Proof of Taylor's Theorem.

But

$$R_{n,a}^{(n+1)}(x) = \frac{\mathrm{d}}{\mathrm{d}x^{n+1}}\Big(R_{n,a}(x)\Big) = \frac{\mathrm{d}}{\mathrm{d}x^{n+1}}\Big(f(x)-P_{n,a}(x)\Big) = \Big(f^{(n+1)}(x)-0\Big)$$

and

$$G^{(n+1)}(x) = \frac{\mathrm{d}}{\mathrm{d}x^{n+1}}\Big((x-a)^{n+1}\Big) = (n+1)!$$

Therefore,

$$\frac{R_{n,a}(x)}{(x-a)^{n+1}} = \frac{R_{n,a}^{(n+1)}(\xi)}{G^{(n+1)}(\xi)} = \frac{f^{(n+1)}(\xi)}{(n+1)!} \qquad \exists \xi \in (a,x)\,,$$

which verifies (♡), as required. □

Note: From Taylor's theorem with $a=0$ and $f=\exp$, it follows that $e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + R_n(x)$, where $R_n(x) = \dfrac{e^t}{(n+1)!}$ for some $t \in (0,x)$.