McMaster University

Mathematics and Statistics

$$\int_M d\omega = \int_{\partial M} \omega$$

Mathematics 3A03
Real Analysis I

Instructor: David Earn

Lecture 6
Sequences
Friday 13 September 2019

Poll

- Go to https://www.childsmath.ca/childsa/forms/main_login.php
- Click on Math 3A03
- Click on Take Class Poll
- Fill in poll **Lecture 6: Sequence convergence**
- $\boxed{\texttt{Submit}}$.

Announcements

- Assignment 1 is due via crowdmark 5 minutes before class on Monday.
- Consider writing the Putnam competition.

Sequences

- A ***sequence*** is a list that goes on forever.
- There is a beginning (a "first term") but no end, *e.g.*,

$$\frac{1}{1},\ \frac{1}{2},\ \frac{1}{3},\ \frac{1}{4},\ \ldots,\ \frac{1}{n},\ \ldots$$

- We use the natural numbers $\mathbb{N}$ to label the terms of a sequence:

$$a_1,\ a_2,\ a_3,\ \ldots,\ a_n,\ \ldots$$

Formal definition of a sequence

Definition (Sequence of Real Numbers)

A ***sequence of real numbers*** is a function

$$f : \mathbb{N} \to \mathbb{R}\,.$$

A lot of different notation is common for sequences:

$$f(1),\ f(2),\ f(3),\ \ldots \qquad \{f(n)\}_{n=1}^{\infty}$$
$$f_1,\ f_2,\ f_3,\ \ldots \qquad \{f(n)\}$$
$$\{f(n) : n = 1, 2, 3, \ldots\} \qquad \{f_n\}_{n=1}^{\infty}$$
$$\{f(n) : n \in \mathbb{N}\} \qquad \{f_n\}$$

Specifying sequences

There are two main ways to specify a sequence:

1. Direct formula.

Specify $f(n)$ for each $n \in \mathbb{N}$. $\square$

Example (arithmetic progression with common difference d)

Sequence is:

$$c,\ c+d,\ c+2d,\ c+3d,\ \ldots$$

$$\therefore\ f(n) = c + (n-1)d, \qquad n \in \mathbb{N}$$

$$\text{i.e.,}\quad x_n = c + (n-1)d, \qquad n = 1, 2, 3, \ldots$$

Specifying sequences

2. Recursive formula.

Specify first term and function $f(x)$ to ***iterate***. □

i.e., Given x_1 and $f(x)$, we have $x_n = f(x_{n-1})$ for all $n > 1$.

$$x_2 = f(x_1), \quad x_3 = f(f(x_1)), \quad x_4 = f(f(f(x_1))), \quad \ldots$$

Example (arithmetic progression with common difference d)

$$x_1 = c, \quad f(x) = x + d$$

$$\therefore \; x_n = x_{n-1} + d, \qquad n = 2, 3, 4, \ldots$$

Note: f is the most typical function name for both the direct and recursive specifications. The correct interpretation of f should be clear from context.

Specifying sequences

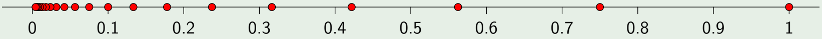
Example (geometric progression with common ratio r)

Sequence is: $c, \; cr, \; cr^2, \; cr^3, \; \ldots$

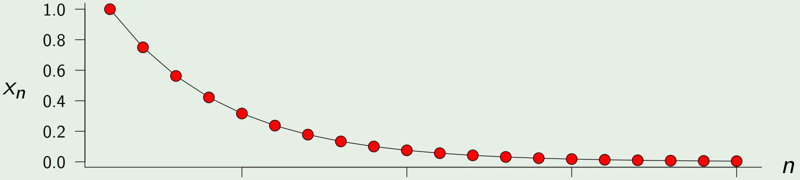
Direct formula: $x_n = f(n) = cr^{n-1}$, $n = 1, 2, 3, \ldots$

Recursive formula: $x_1 = c$, $f(x) = rx$, $x_n = f(x_{n-1})$

Number line representation of $\{x_n\}$ with $c = 1$ and $r = \frac{3}{4}$:



Graph of $f(n)$:



Specifying sequences

Example ($f(n) = 1 + \frac{1}{n^2}$)

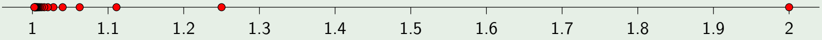
Sequence is: 2, $\frac{5}{4}$, $\frac{10}{9}$ $\frac{17}{16}$, ...

Direct formula: $x_n = f(n) = 1 + \frac{1}{n^2}$, $n = 1, 2, 3, \ldots$

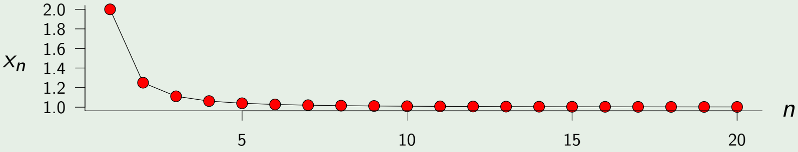
Recursive formula: $x_1 = 2, \quad f(x) = 1 + [1 + (x-1)^{-1/2}]^{-2}$

Get this formula by solving for n in terms of x in
$x = 1 + 1/(n-1)^2 \quad (= x_{n-1})$.
Such an inversion will NOT always be possible.

Number line representation of $\{x_n\}$:



Graph of $f(n)$:



Convergence of sequences

We know from previous experience that:

- $cr^{n-1} \to 0$ as $n \to \infty$ (if $|r| < 1$).
- $1 + \frac{1}{n^2} \to 1$ as $n \to \infty$.

How do we make our intuitive notion of ***convergence*** *mathematically rigorous*?

Informal definition: "$x_n \to L$ as $n \to \infty$" means "we can make the difference between x_n and L as small as we like by choosing n big enough".

More careful informal definition: "$x_n \to L$ as $n \to \infty$" means "given any *error tolerance*, say ε, we can make the distance between x_n and L smaller than ε by choosing n big enough".

Convergence of sequences

Definition (Limit of a sequence)

A sequence $\{s_n\}$ ***converges to*** L if, given any $\varepsilon > 0$ there is some integer N such that

$$\text{if } n \geq N \qquad \text{then} \qquad |s_n - L| < \varepsilon\,.$$

In this case, we write $\lim\limits_{n\to\infty} s_n = L$ or $s_n \to L$ as $n \to \infty$ and we say that L is the ***limit*** of the sequence $\{s_n\}$.

Note: To use this definition to prove that the limit of a sequence is L, we start by imagining that we are given some error tolerance $\varepsilon > 0$. Then we have to find a suitable N, which will depend on ε. This means that *the N that we find will be a function of ε*.

Shorthand:

$$\lim_{n\to\infty} s_n = L \quad \overset{\text{def}}{=} \quad \forall \varepsilon > 0 \quad \exists N \in \mathbb{N} \quad \ni \quad n \geq N \implies |s_n - L| < \varepsilon.$$

Convergence of sequences

Convergence terminology:

- A sequence that converges is said to be ***convergent***.
- A sequence that is <u>not convergent</u> is said to be ***divergent***.

Remark (Sequences in spaces other than $\mathbb{R}$)

The formal definition of a limit of a sequence works in any space where we have a notion of distance if we replace $|s_n - L|$ with $d(s_n, L)$.

Convergence of sequences

Example

Use the formal definition of a limit of a sequence to prove that

$$\frac{n^2+1}{n^2} \to 1 \quad \text{as} \quad n \to \infty\,.$$

(solution on board)

Note: Our strategy here was to solve for n in the inequality $|s_n - L| < \varepsilon$. From this we were able to infer how big N has to be in order to ensure that $|s_n - L| < \varepsilon$ for all $n \geq N$. That much was “rough work”. Only after this rough work did we have enough information to be able to write down a rigorous proof.

Convergence of sequences

Example

Use the formal definition of a limit of a sequence to prove that

$$\frac{n^5 - n^3 + 1}{n^8 - n^5 + n + 1} \to 0 \quad \text{as} \quad n \to \infty\,.$$

(solution on board)

Note: In this example, it was not possible to solve for n in the inequality $|s_n - L| < \varepsilon$. Instead, we first needed to bound $|s_n - L|$ by a much simpler expression that is always greater than $|s_n - L|$. If that bound is less than ε then so is $|s_n - L|$.

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Mathematics and Statistics

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Mathematics 3A03
Real Analysis I

Instructor: David Earn

Lecture 7
Sequences II
Tuesday 17 September 2019

Poll

- Go to https://www.childsmath.ca/childsa/forms/main_login.php
- Click on Math 3A03
- Click on Take Class Poll
- Fill in poll **Lecture 7: Sequence divergence**
- Submit.

Announcements

- If you are interested in becoming a volunteer notetaker to support students with disabilities, please go to `https://sas.mcmaster.ca/volunteer-notetaking/`.
- Solutions to Assignment 1 will be posted soon. **Study them!**
- Assignment 2 will be posted soon. Due in two weeks.
- No late submission of assignments. No exceptions. However, best 5 of 6 assignments will be counted. *Always due 5 minutes before class on the due date.*
- **Note as stated on course info sheet:** *Only a selection of problems on each assignment will be marked; your grade on each assignment will be based only on the problems selected for marking. Problems to be marked will be selected after the due date.*

Announcements continued...

- Remember that solutions to assignments and tests from previous years are available on the course web site. Take advantage of these problems and solutions. They provide many useful examples that should help you prepare for tests and the final exam. (However, note that while most of the content of the course is the same this year, there are some differences.)

Uniqueness of limits

Theorem (Uniqueness of limits)

If $\lim_{n\to\infty} s_n = L_1$ *and* $\lim_{n\to\infty} s_n = L_2$ *then* $L_1 = L_2$.

(solution on board)

So, we are justified in referring to "the" limit of a convergent sequence.

Divergence of sequences

Divergence is the logical opposite (negation) of convergence.
We can infer the formal meaning of divergence by taking the *logical negation* of the formal definition of convergence.
Doing so, we find that the sequence $\{s_n\}$ diverges (*i.e.*, does not converge to any $L \in \mathbb{R}$) iff

$$\forall L \in \mathbb{R}, \;\; \exists \varepsilon > 0 \text{ such that: } \forall N \in \mathbb{N} \;\; \exists n \geq N \;\; \ni \;\; |s_n - L| \geq \varepsilon.$$

Notes:

- The n that exists will, in general, depend on L, ε and N.
- This is the meaning of not converging to any limit, but it does not tell us anything about what happens to the sequence $\{s_n\}$ as $n \to \infty$.

Divergence to $\pm\infty$

Definition (Divergence to ∞)

The sequence $\{s_n\}$ of real numbers ***diverges to*** ∞ if, for every real number M there is an integer N such that

$$n \geq N \quad \Longrightarrow \quad s_n \geq M\,,$$

in which case we write $s_n \to \infty$ as $n \to \infty$ or $\lim\limits_{n\to\infty} s_n = \infty$.

Definition (Divergence to $-\infty$)

The sequence $\{s_n\}$ of real numbers ***diverges to*** $-\infty$ if, for every real number M there is an integer N such that

$$n \geq N \quad \Longrightarrow \quad s_n \leq M\,.$$

Divergence to ∞

Example

Use the formal definition to prove that

$$\left\{ \frac{n^3 - 1}{n + 1} \right\} \quad \text{diverges to } \infty\,.$$

(solution on board)

Approach: Find a lower bound for the sequence that is a simple function of n and show that that can be made bigger than any given M.

Divergence to ∞

Example (from previous slide)

Use the formal definition to prove that $\left\{\dfrac{n^3-1}{n+1}\right\}$ diverges to ∞.

Clean proof.

Given $M \in \mathbb{R}^{>0}$, let $N = \lceil M \rceil + 1$. Then $N - 1 = \lceil M \rceil \geq M$.
$\therefore \forall n \geq N$, $n - 1 \geq M$. Now observe that

$$\forall n \in \mathbb{N}, \quad n - 1 = \frac{(n-1)(n+1)}{n+1} = \frac{n^2-1}{n+1} \leq \frac{n^3-1}{n+1}\,.$$

$\therefore \forall n \geq N$ we have

$$\frac{n^3-1}{n+1} \geq M\,,$$

as required. $\square$

Sequences of partial sums (a.k.a. Series)

Given a sequence $\{x_n\}$, we define the ***sequence of partial sums of*** $\{x_n\}$ to be $\{s_n\}$, where

$$s_n = \sum_{k=1}^{n} x_k = x_1 + x_2 + \cdots + x_n \,.$$

Note: We can start from any integer, not necessarily $k = 1$.

Boundedness of sequences

A sequence is said to be bounded if its range is a bounded set.

Definition (Bounded sequence)

A sequence $\{s_n\}$ is ***bounded*** if there is a real number M such that every term in the sequence satisfies $|s_n| \le M$.

Theorem (Every convergent sequence is bounded.)

$$L \in \mathbb{R} \ \wedge \ \lim_{n\to\infty} s_n = L \quad \Longrightarrow \quad \exists M > 0 \ \ni \ |s_n| \le M \ \forall n \in \mathbb{N}.$$

(solution on board)

Note: The converse is FALSE.
Proof? Find a counterexample, *e.g.*, $\{(-1)^n\}$.

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Mathematics and Statistics

$$\int_M d\omega = \int_{\partial M} \omega$$

Mathematics 3A03
Real Analysis I

Instructor: David Earn

Lecture 8
Sequences III
Thursday 19 September 2019

What we've done so far on sequences

- Definition of convergence.
- Definition of divergence.
- Definition of divergence to $\pm\infty$.
- Series: sequence of partial sums.
- Bounded sequences.
- Examples.

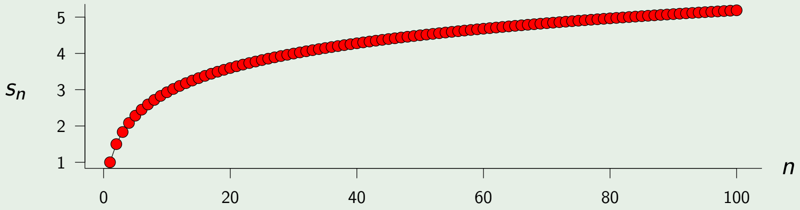
Boundedness of sequences

Corollary (Unbounded sequences diverge)

If $\{s_n\}$ is unbounded then $\{s_n\}$ diverges.

Example (The harmonic series diverges)

Consider the ***harmonic series*** $s_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$.



Prove that s_n diverges to ∞.

(solution on board)

Harmonic series – idea for proof of divergence

Approach: Group terms and use the corollary above.

$$\underbrace{\underbrace{\underbrace{\underbrace{\left(1+\frac{1}{2}\right)}_{>\ 1\times\frac{1}{2}}}_{s_2\ >\ 1\times\frac{1}{2}} + \underbrace{\left(\frac{1}{3}+\frac{1}{4}\right)}_{>\ 2\times\frac{1}{4}}}_{s_4\ >\ 2\times\frac{1}{2}} + \underbrace{\left(\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}\right)}_{>\ 4\times\frac{1}{8}}}_{s_8\ >\ 3\times\frac{1}{2}} + \cdots$$

$$\Longrightarrow \qquad s_{2^n} \ > \ n \times \frac{1}{2}$$

Note: These sorts calculations are just "rough work", not a formal proof.
A proof must be a clearly presented coherent argument from beginning to end.

Harmonic series – clean proof of divergence

Proof.

Part (i). Prove (e.g., by induction) that $\quad s_{2^n} > n/2 \quad \forall n \in \mathbb{N}$.

Part (ii). Suppose we are given $M \in \mathbb{R}$.

- If $M \leq 0$ then note that $s_n > 0 \ \forall n \in \mathbb{N}$.
- If $M > 0$, let $\widetilde{N} = 2\lceil M \rceil$ and $N = 2^{\widetilde{N}}$. Then, $\forall n \geq N$, we have $s_n \geq s_N = s_{2^{\widetilde{N}}} > \widetilde{N}/2 = \lceil M \rceil \geq M$, as required.

$\square$

Poll

- Go to https://www.childsmath.ca/childsa/forms/main_login.php
- Click on Math 3A03
- Click on Take Class Poll
- Fill in poll **Lecture 8: Harmonic series of primes**
- $\boxed{\texttt{Submit}}$.

Algebra of limits

Theorem (Algebraic operations on limits)

Suppose $\{s_n\}$ and $\{t_n\}$ are convergent sequences and $C \in \mathbb{R}$.

1. $\lim\limits_{n\to\infty} C\, s_n = C\big(\lim\limits_{n\to\infty} s_n\big)$;
2. $\lim\limits_{n\to\infty} (s_n + t_n) = \big(\lim\limits_{n\to\infty} s_n\big) + \big(\lim\limits_{n\to\infty} t_n\big)$;
3. $\lim\limits_{n\to\infty} (s_n - t_n) = \big(\lim\limits_{n\to\infty} s_n\big) - \big(\lim\limits_{n\to\infty} t_n\big)$;
4. $\lim\limits_{n\to\infty} (s_n t_n) = \big(\lim\limits_{n\to\infty} s_n\big)\big(\lim\limits_{n\to\infty} t_n\big)$;
5. *if $t_n \neq 0$ for all n and $\lim_{n\to\infty} t_n \neq 0$ then*
   $$\lim_{n\to\infty} \left(\frac{s_n}{t_n}\right) = \frac{\lim_{n\to\infty} s_n}{\lim_{n\to\infty} t_n} .$$

(solution on board)

Revisit example

Example (previously proved directly from definition)

Use the algebraic properties of limits to prove that

$$\frac{n^5 - n^3 + 1}{n^8 - n^5 + n + 1} \to 0 \quad \text{as} \quad n \to \infty\,.$$

(solution on board)

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Mathematics and Statistics

$$\int_M d\omega = \int_{\partial M} \omega$$

Mathematics 3A03
Real Analysis I

Instructor: David Earn

Lecture 9
Sequences IV
Friday 20 September 2019

Announcements

- Assignment 2 is posted.
  Due 1 Oct 2019, at 2:25pm.

What we've done so far on sequences

- Definition of convergence.
- Definition of divergence.
- Definition of divergence to $\pm\infty$.
- Examples.
- Every convergent sequence is bounded.
- Harmonic series diverges.
- Algebra of limits (more today).

Product Rule for Limits

The 4th item in the algebra of limits theorem was:

Theorem (Product Rule for Limits)

If $s_n \to S$ and $t_n \to T$ as $n \to \infty$ then $s_n t_n \to ST$ as $n \to \infty$.

Proof.

For any $n \in \mathbb{N}$,
$$\begin{aligned} |s_n t_n - ST| &= |s_n t_n - ST + s_n T - s_n T| \\ &= |s_n(t_n - T) + T(s_n - S)| \\ &\le |s_n||t_n - T| + |T||s_n - S| \end{aligned}$$

Now, $\{s_n\}$ converges, so it is bounded by some $M > 0$, *i.e.*, $|s_n| \le M \; \forall n \in \mathbb{N}$. Therefore, given $\varepsilon > 0$, choose $N \in \mathbb{N}$ such that

$$|t_n - T| < \frac{\varepsilon}{2M} \qquad \text{and} \qquad |s_n - S| < \frac{\varepsilon}{2(1 + |T|)}\,.$$

Then $|s_n t_n - ST| < \varepsilon/2 + \varepsilon/2 = \varepsilon$, as required. □

Quotient Rule for Limits

Quotient Rule was the 5th item in the algebra of limits theorem.

Lemma (Reciprocal Rule for Limits)

If $t_n \neq 0\ \forall n$ and $t_n \to T \neq 0$ then $1/t_n \to 1/T$.

Proof.

For any $n \in \mathbb{N}$, $\left|\dfrac{1}{t_n} - \dfrac{1}{T}\right| = \left|\dfrac{t_n - T}{t_n T}\right| = |t_n - T| \cdot \dfrac{1}{|t_n|} \cdot \dfrac{1}{|T|}$.

Since $\{t_n\}$ converges, $\exists N_1 \in \mathbb{N}$ such that $\forall n \geq N_1$, $|t_n| > |T|/2$ (details on next slide) and hence $1/|t_n| < 2/|T|$.

Now choose $N \geq N_1$ such that $|t_n - T| < \varepsilon|T|^2/2$. Then

$$\left|\frac{1}{t_n} - \frac{1}{T}\right| = |t_n - T| \cdot \frac{1}{|t_n|} \cdot \frac{1}{|T|} < \frac{\varepsilon|T|^2}{2} \cdot \frac{2}{|T|} \cdot \frac{1}{|T|} = \varepsilon\,,$$

as required. $\square$

Quotient Rule for Limits

Details missing on previous slide: (consider $\varepsilon = \frac{|T|}{2}$)

Since $t_n \to T$, $\exists N_1 \in \mathbb{N}$ such that $\forall n \geq N_1$, $|t_n - T| < \dfrac{|T|}{2}$,

i.e., $-\dfrac{|T|}{2} < t_n - T < \dfrac{|T|}{2}$, i.e., $T - \dfrac{|T|}{2} < t_n < T + \dfrac{|T|}{2}$.

If $T > 0$ this says

$$0 < \frac{T}{2} < t_n < \frac{3T}{2},$$

whereas if $T < 0$ it says

$$\frac{3T}{2} < t_n < \frac{T}{2} < 0.$$

In either case, $\forall n \geq N_1$, we have $0 < \dfrac{|T|}{2} < |t_n|$.

Order properties of limits (§2.8)

Theorem (Limits retain order)

If $\{s_n\}$ *and* $\{t_n\}$ *are convergent sequences then*

$$s_n \le t_n \quad \forall n \in \mathbb{N} \qquad \Longrightarrow \qquad \lim_{n\to\infty} s_n \le \lim_{n\to\infty} t_n \,.$$

Proof.

Given $\varepsilon > 0$, choose $N \in \mathbb{N} \ni |s_n - S| < \frac{\varepsilon}{2}$ and $|t_n - T| < \frac{\varepsilon}{2}$. Then

$$\begin{aligned} S - T &= S - T + s_n - s_n + t_n - t_n \\ &= (S - s_n) + (t_n - T) + s_n - t_n \\ &\le (S - s_n) + (t_n - T) \qquad\qquad (\because s_n - t_n \le 0) \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \end{aligned}$$

Hence $S - T \le 0$, *i.e.*, $S \le T$. $\square$

Poll

- Go to https://www.childsmath.ca/childsa/forms/main_login.php
- Click on Math 3A03
- Click on Take Class Poll
- Fill in poll **Lecture 9: Order property of limits**
- $\boxed{\texttt{Submit}}$.

Order properties of limits (§2.8)

Question: If $s_n < t_n$ for all $n \in \mathbb{N}$, can we conclude that

$$\lim_{n\to\infty} s_n < \lim_{n\to\infty} t_n \qquad ?$$

No! No! No! No! No! No!! NO!!!!!!!!!!

Theorem (Limits retain bounds)

If $\{s_n\}$ is a convergent sequence then

$$\alpha \le s_n \le \beta \quad \forall n \in \mathbb{N} \qquad \Longrightarrow \qquad \alpha \le \lim_{n\to\infty} s_n \le \beta \,.$$

Proof.

Apply previous theorem with $\alpha_n = \alpha \ \forall n$ and $\beta_n = \beta \ \forall n$. $\square$

Order properties of limits (§2.8)

Theorem (Squeeze Theorem)

If $\{s_n\}$ and $\{t_n\}$ are convergent sequences such that

(i) $s_n \le x_n \le t_n \quad \forall n \in \mathbb{N}$, *($x_n$ is always between them)*

(ii) $\lim\limits_{n\to\infty} s_n = \lim\limits_{n\to\infty} t_n = L$. *(both approach the same limit)*

Then $\{x_n\}$ is convergent and $\lim\limits_{n\to\infty} x_n = L$.

Proof? (What's **WRONG**?).

$\{s_n\}$ and $\{t_n\}$ are both bounded since they both converge. $\{x_n\}$ is therefore bounded (by the lower bound of $\{s_n\}$ and the upper bound of $\{t_n\}$). $\{x_n\}$ therefore converges, say $x_n \to X$. Hence, by order retension, $L \le X \le L \implies X = L$. □

Mathematics
and Statistics

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Mathematics 3A03
Real Analysis I

Instructor: David Earn

Lecture 10
Sequences V
Tuesday 24 September 2019

Announcements

- Assignment 2 is posted.
  Due 1 Oct 2019, at 2:25pm.

What we've done so far on sequences

- Definition of convergence.
- Definition of divergence.
- Definition of divergence to $\pm\infty$.
- Every convergent sequence is bounded.
- Harmonic series diverges.
- Algebra of limits (sums, products, quotients).
- Order properties of limits; squeeze theorem

Today:

- Proof of Squeeze Theorem
- Absolute value and max/min of limits.
- Monotone convergence.

Order properties of limits (§2.8)

Theorem (Squeeze Theorem)

If $\{s_n\}$ and $\{t_n\}$ are convergent sequences such that

(i) $s_n \le x_n \le t_n \quad \forall n \in \mathbb{N}$, *($x_n$ is always between them)*

(ii) $\lim_{n\to\infty} s_n = \lim_{n\to\infty} t_n = L$. *(both approach the same limit)*

Then $\{x_n\}$ is convergent and $\lim_{n\to\infty} x_n = L$.

Correct Proof.

Given $\varepsilon > 0$, find $N \ni \forall n \ge N$, $|s_n - L| < \varepsilon$ and $|t_n - L| < \varepsilon$, *i.e.*,

$$-\varepsilon < s_n - L < \varepsilon \qquad \text{and} \qquad -\varepsilon < t_n - L < \varepsilon.$$

But $s_n \le x_n \le t_n$

$$\begin{aligned} &\implies \quad s_n - L \le x_n - L \le t_n - L \\ &\implies -\varepsilon < s_n - L \le x_n - L \le t_n - L < \varepsilon \\ &\implies |x_n - L| < \varepsilon, \end{aligned}$$

as required. $\square$

Order properties of limits (§2.8)

Theorem (Limits of Absolute Values)

If $\{s_n\}$ converges then so does $\{|s_n|\}$, and

$$\lim_{n\to\infty} |s_n| = \left|\lim_{n\to\infty} s_n\right| .$$

Proof.

See Assignment 2! $\square$

Order properties of limits (§2.8)

Corollary (Max/Min of Limits)

If $\{s_n\}$ *and* $\{t_n\}$ *converge then* $\{\max\{s_n,t_n\}\}$ *and* $\{\min\{s_n,t_n\}\}$ *both converge and*

$$\lim_{n\to\infty} \max\{s_n,t_n\} = \max\left\{\lim_{n\to\infty} s_n,\ \lim_{n\to\infty} t_n\right\},$$

$$\lim_{n\to\infty} \min\{s_n,t_n\} = \min\left\{\lim_{n\to\infty} s_n,\ \lim_{n\to\infty} t_n\right\}.$$

Idea for proof:

$$\forall x,y\in\mathbb{R}\quad \max\{x,y\} = \frac{x+y}{2} + \frac{|x-y|}{2}$$

$$\forall x,y\in\mathbb{R}\quad \min\{x,y\} = \frac{x+y}{2} - \frac{|x-y|}{2}$$

Prove these facts, then use theorems on sums and absolute values of limits.

Monotone convergence (§2.9)

Definition (Monotonic sequence)

The sequence $\{s_n\}$ is ***monotonic*** iff it satisfies any of the following conditions:

(i) ***Increasing***: $s_1 < s_2 < s_3 < \cdots < s_n < s_{n+1} < \cdots$;

(ii) ***Decreasing***: $s_1 > s_2 > s_3 > \cdots > s_n > s_{n+1} > \cdots$;

(iii) ***Non-decreasing***: $s_1 \leq s_2 \leq s_3 \leq \cdots \leq s_n \leq s_{n+1} \leq \cdots$;

(iv) ***Non-increasing***: $s_1 \geq s_2 \geq s_3 \geq \cdots \geq s_n \geq s_{n+1} \geq \cdots$.

Poll

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Monotone convergence (§2.9)

Theorem (Monotone Convergence Theorem)

A monotonic sequence $\{s_n\}$ is convergent iff it is bounded. In particular,

(i) $\{s_n\}$ *non-decreasing and unbounded* $\implies s_n \to \infty$;

(ii) $\{s_n\}$ *non-decreasing and bounded* $\implies s_n \to \sup\{s_n\}$;

(iii) $\{s_n\}$ *non-increasing and unbounded* $\implies s_n \to -\infty$;

(iv) $\{s_n\}$ *non-increasing and bounded* $\implies s_n \to \inf\{s_n\}$.

Proof.

. . . next slide. . . $\square$

Proof of Monotone Convergence Theorem

Given a monotonic sequence $\{s_n\}$ we must show that

$$\{s_n\} \text{ converges } \iff \{s_n\} \text{ is bounded}$$

Proof of "$\Longrightarrow$" and part (ii).

$\Longrightarrow$ For <u>any</u> sequence (monotonic or not) convergent implies bounded.

$\Longleftarrow$ [part (ii)] Suppose $\{s_n\}$ is non-decreasing, *i.e.*, $s_n \leq s_{n+1}$ for all $n \in \mathbb{N}$. Since $\{s_n\}$ is bounded, it has a <u>least</u> upper bound, say $L = \sup\{s_n\}$. We will now show that $s_n \to L$, *i.e.*, $\forall \varepsilon > 0 \; \exists N \in \mathbb{N} \ni \forall n \geq N$, $|s_n - L| < \varepsilon$.

Before proceeding, note that since $L = \sup\{s_n\}$, it follows that $|s_n - L| < \varepsilon \iff L - s_n < \varepsilon \iff L - \varepsilon < s_n$.

Given $\varepsilon > 0$, choose $N \in \mathbb{N}$ such that $s_N > L - \varepsilon$ (which is possible $\because L$ is the <u>least</u> upper bound of $\{s_n\}$). But $\{s_n\}$ is non-decreasing, so $\forall n \geq N$ we have $s_N \leq s_n \implies -s_n \leq -s_N \implies L - s_n \leq L - s_N < \varepsilon$. □

Proof of Monotone Convergence Theorem

$$\text{Monotonic} \implies \left[\{s_n\} \text{ converges} \iff \{s_n\} \text{ is bounded}\right]$$

Proof of parts (i), (iii), (iv).

[part (i)] Suppose $\{s_n\}$ is non-decreasing and *un*bounded. It follows that $\{s_n\}$ diverges, since convergent sequences are bounded. Since $\{s_n\}$ is non-decreasing, it is bounded below (by s_1, for example). Hence $\{s_n\}$ (which is unbounded) must not be bounded above. Consequently, given any $M \in \mathbb{R}$, $\exists N \in \mathbb{N}$ such that $s_N > M$. But $\{s_n\}$ is non-decreasing, so $s_n > M$ for all $n \geq N$, as required.

Proof of [part (iii)] is similar to [part (i)].

Proof of [part (iv)] is similar to [part (ii)]. $\square$

Subsequences

Definition (Subsequence)

Let $\{s_1, s_2, s_3, \ldots\}$ be a sequence. If $\{n_1, n_2, n_3, \ldots\}$ is an increasing sequence of natural numbers then $\{s_{n_1}, s_{n_2}, s_{n_3}, \ldots\}$ is a ***subsequence*** of $\{s_1, s_2, s_3, \ldots\}$.

Example (Subsequences)

Consider the sequence $\{s_n\}$ defined by $s_n = n^2$ for all $n \in \mathbb{N}$. What are the first few terms of these subsequences?

- $\{s_n : n \text{ even}\}$ $\quad \{2^2, 4^2, 6^2, \ldots\}$
- $\{s_n : n = 2k+1, \ \exists k \in \mathbb{N}\}$ $\quad \{3^2, 5^2, 7^2, \ldots\}$
- $\{s_{2n+1}\}$ Same as line above
- $\{s_{2^n}\}$ $\quad \{2^2, 4^2, 8^2, \ldots\}$
- $\{s_{n^2}\}$ $\quad \{1^2, 4^2, 9^2, \ldots\}$

Monotonic subsequences

Given any sequence $\{s_n\}$, can you always find a subsequence that is monotonic?

Theorem

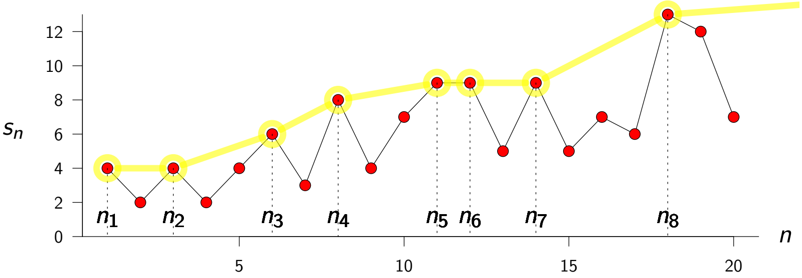
Every sequence contains a monotonic subsequence.

(Textbook (TBB) §2.11, Theorem 2.39, p. 79)

There are no pictures accompanying the proof in the textbook. So let's draw some pictures to help us visualize how we might construct a proof...

Idea for proof that every sequence contains a monotonic subsequence ("point of no return")

Given a sequence $\{s_1, s_2, s_3, \ldots\}$, try to build a subsequence $\{s_{n_1}, s_{n_2}, s_{n_3}, \ldots\}$ that is non-decreasing ($s_{n_1} \leq s_{n_2} \leq s_{n_3} \leq \cdots$) by discarding any terms that are less than the running maximum (the maximum of all previous terms):



If this works indefinitely then we have a non-decreasing subsequence. But if we can find only finitely many such terms then we're stuck because our subsequence is defined using earlier terms.

McMaster University

Mathematics and Statistics

$$\int_M d\omega = \int_{\partial M} \omega$$

Mathematics 3A03
Real Analysis I

Instructor: David Earn

Lecture 11
Sequences VI
Thursday 26 September 2019

Poll

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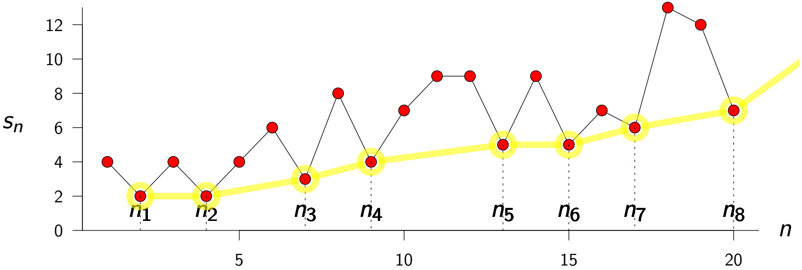
Announcements

- Please send me an e-mail ASAP if you have a conflict with either of the midterm tests.
- Plan for today:
  - Discuss correct "visual proof" that Every sequence contains a monotonic subsequence.
  - State and prove Bolzano-Weierstrass Theorem.
  - Cauchy sequences.

Better idea for proof that every sequence contains a monotonic subsequence ("turn-back point")

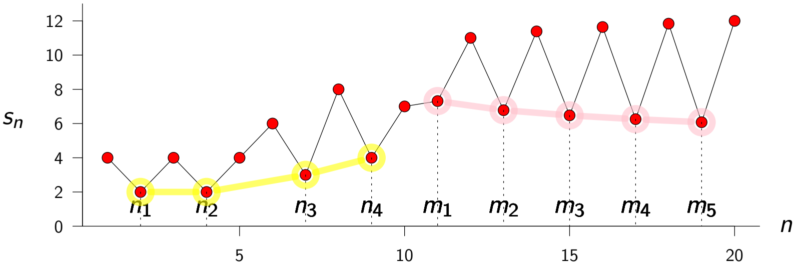
Given a sequence $\{s_1, s_2, s_3, \ldots\}$, try to build a subsequence $\{s_{n_1}, s_{n_2}, s_{n_3}, \ldots\}$ that is non-decreasing ($s_{n_1} \leq s_{n_2} \leq s_{n_3} \leq \cdots$) by identifying terms that are less than or equal to all later terms.



If this works indefinitely then we have a non-decreasing subsequence. What if there are only finitely many such terms? (There might not be any at all!)

Better idea for proof that every sequence contains a monotonic subsequence ("turn-back point")

If there are only finitely many s_{n_i} such that $s_{n_i} \leq s_n \ \forall n > n_i \ldots$



... then after the last "turn-back point" (s_{n_4} above) there must be some $m_1 > n_4$ such that s_{m_1} is **<u>not</u>** $\leq$ all later terms, *i.e.*, $\exists m_2 > m_1$ with $s_{m_2} < s_{m_1}$, and similarly for m_2, so there must be a decreasing subsequence $\quad s_{m_1} > s_{m_2} > s_{m_3} > \cdots$

Convergent subsequences

Theorem (Bolzano-Weierstrass theorem)

Every bounded sequence of real numbers contains a convergent subsequence.

Proof.

Suppose $\{x_n\}$ is a bounded sequence. It follows from the previous theorem that $\{x_n\}$ contains a subsequence $\{x_{m_k}\}$ that is monotonic. Since $\{x_n\}$ is bounded, the subsequence $\{x_{m_k}\}$ is bounded as well (by the same bound). Thus, $\{x_{m_k}\}$ is a subsequence of $\{x_n\}$ that is both bounded and monotonic. Hence, it converges by the Monotone Convergence Theorem. □

Cauchy sequences

Definition (Cauchy sequence)

A sequence $\{s_n\}$ is said to be a ***Cauchy sequence*** iff for all $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that if $m \geq N$ and $n \geq N$ then $|s_n - s_m| < \varepsilon$.

Theorem (Cauchy criterion)

A sequence of real numbers $\{s_n\}$ is convergent iff it is a Cauchy sequence.

Remark: The proof of the "only if" direction is easy. The proof of the "if" direction contains only one tricky feature: showing that every Cauchy sequence $\{s_n\}$ is bounded.

Cauchy sequences

Proof of Cauchy criterion

"only if": If $\{s_n\}$ converges then, given $\varepsilon > 0$, choose $N \in \mathbb{N}$ such that for all $n \geq N$, $|s_n - L| < \varepsilon/2$. Then, for any $m, n > N$ we have $|s_m - s_n| = |s_m - L + L - s_n| \leq |s_m - L| + |s_n - L| < (\varepsilon/2) + (\varepsilon/2) = \varepsilon$, as required.

"if": If we take $\varepsilon = 1$ in the definition of a Cauchy sequence, we find that there is some N such that $|s_m - s_n| < 1$ for all $m, n > N$. In particular, this means that $|s_m - s_{N+1}| < 1$ for all $m > N$, *i.e.,*

$$s_{N+1} - 1 < s_m < s_{N+1} + 1 \qquad \forall m > N.$$

Thus $\{s_m : m > N\}$ is bounded; moreover, since there are only finitely many other s_i's, the whole sequence $\{s_n\}$ is bounded. Hence, by the Bolzano-Weierstrass theorem, some subsequence of s_n converges; let's write this subsequence as $\{s_{m_k}\}$, and its limit as L.

. . . continued on next slide. . .

Cauchy sequences

Proof of Cauchy criterion (cont'd).

We will show that $\{s_n\}$ converges to L. To prove this, consider any $\varepsilon > 0$. Since the sequence $\{s_n\}$ is Cauchy, there is some $N \in \mathbb{N}$ such that

$$|s_n - s_m| < \frac{\varepsilon}{2} \qquad \text{for all } n, m \geq N.$$

Since the subsequence $\{s_{m_k}\}$ converges to L, there is some N' so that

$$|s_{m_k} - L| < \frac{\varepsilon}{2} \qquad \text{for all } k \geq N'.$$

Now fix an integer k satisfying $k \geq N'$ and $m_k \geq N$. Then $\forall n \geq N$,

$$\begin{aligned} |s_n - L| &= |s_n - s_{m_k} + s_{m_k} - L| \\ &\leq |s_n - s_{m_k}| + |s_{m_k} - L| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, \end{aligned}$$

as required. □

Cauchy sequences

Notes:

- The Cauchy criterion is sometimes easier to use in proofs than the original definition of convergence.
- Its significance is more evident in spaces other than $\mathbb{R}$, where Cauchy sequences do not necessarily converge.