

6 Integration

7 Integration II



Mathematics
and Statistics

$$\int_M d\omega = \int_{\partial M} \omega$$

Mathematics 3A03 Real Analysis I

Instructor: David Earn

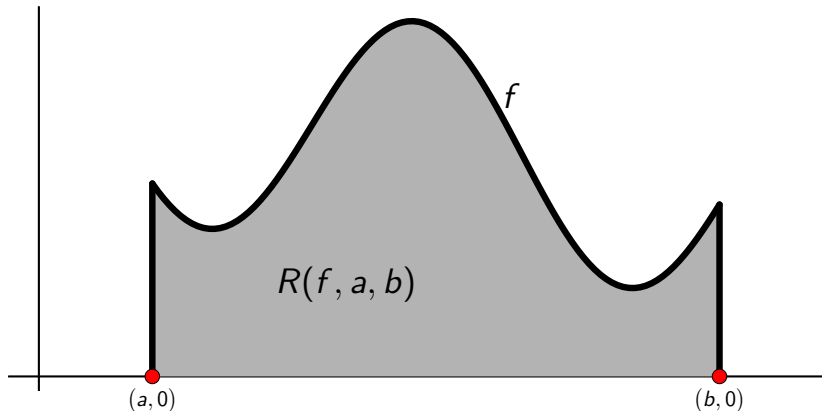
Lecture 6
Integration
Tuesday 20 January 2026

Announcements

- Participation deadline for [Assignment 1](#) was just before this class started.
- Solutions to [Assignment 1](#) will be posted today. Make sure to read the solutions even if you are confident your proofs are correct.
- I posted a [handout](#) addressing a student's question in the last class: Can one prove monotonicity via the Mean Value Theorem without Darboux's theorem?

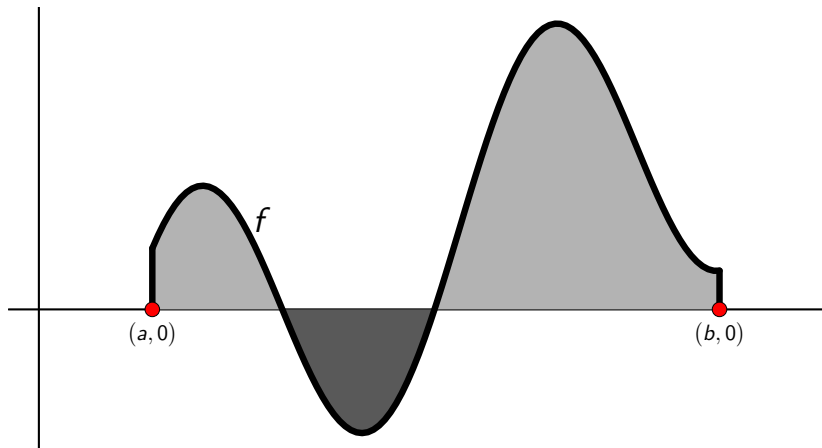
Integration

Integration



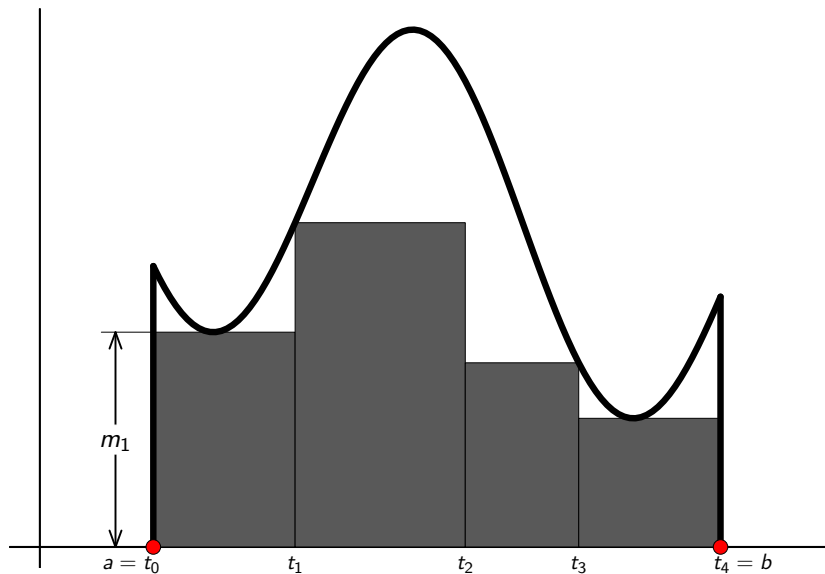
- “Area of region $R(f, a, b)$ ” is actually a very subtle concept.
- We will only scratch the surface of it (greater depth in Math 4A).
- Our treatment is similar to that in Michael Spivak’s “Calculus” (2008); BS refer to this approach as the Darboux integral (BS §7.4, p. 225).
- The Darboux and Riemann approaches to the integral are equivalent.

Integration

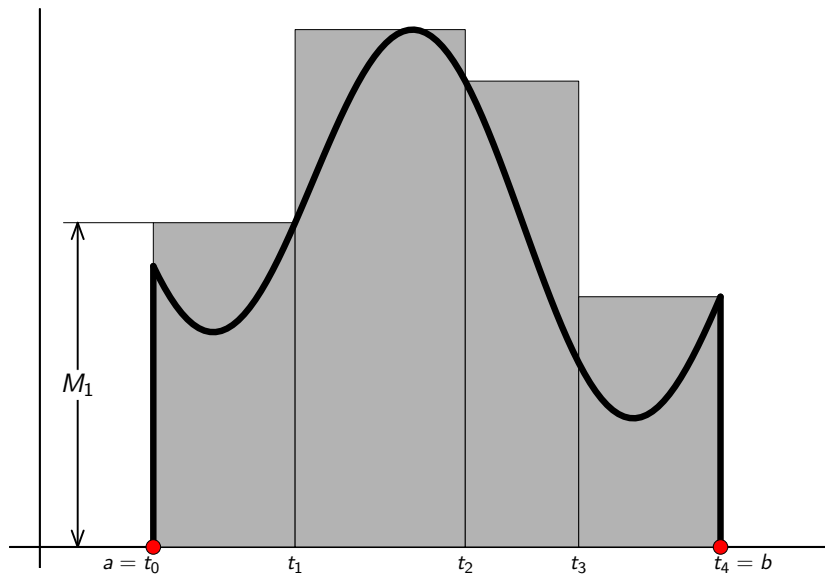


- Contribution to “area of $R(f, a, b)$ ” is positive or negative depending on whether f is positive or negative.

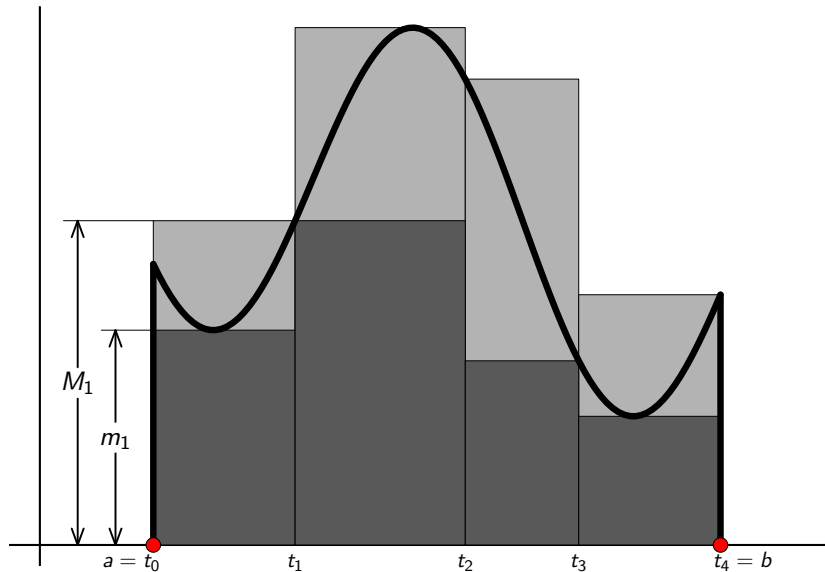
Lower sum



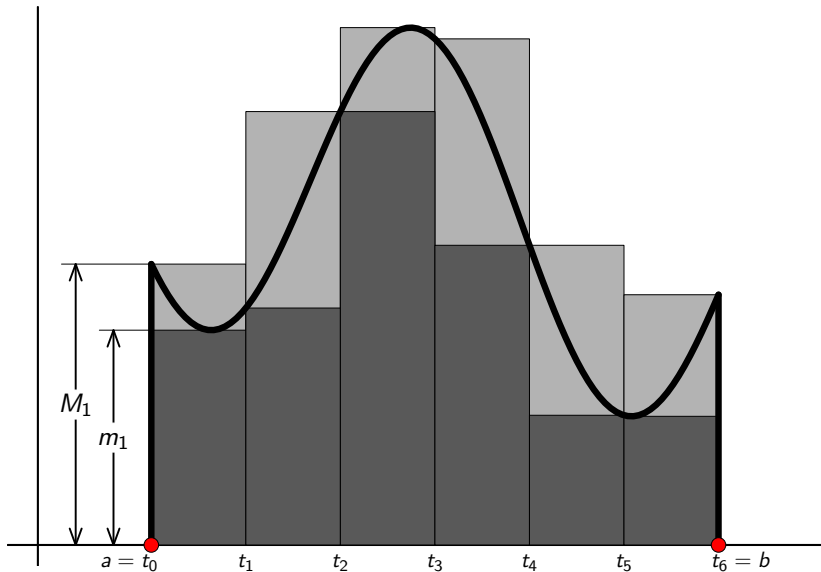
Upper sum



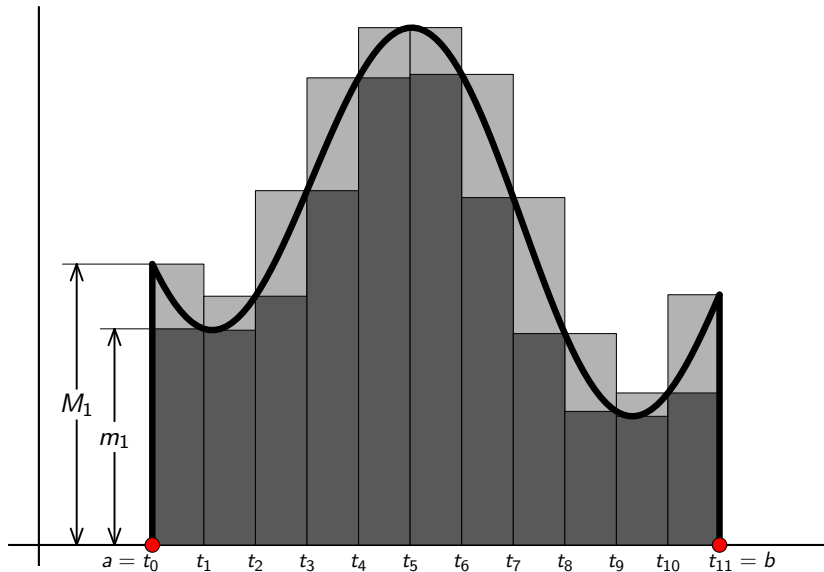
Lower and upper sums



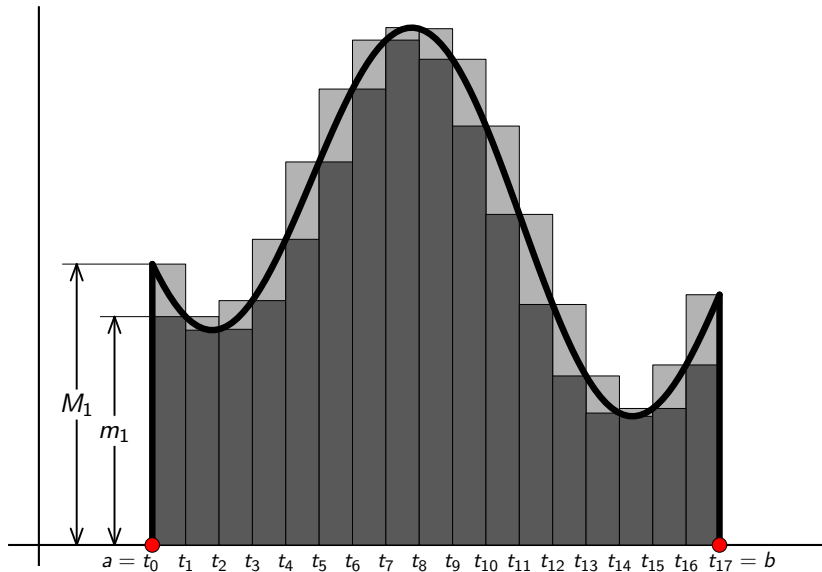
Lower and upper sums



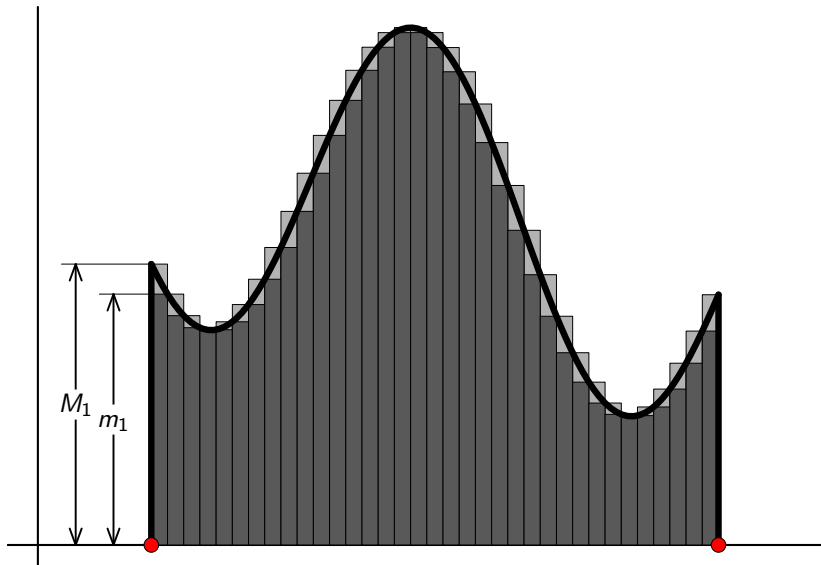
Lower and upper sums



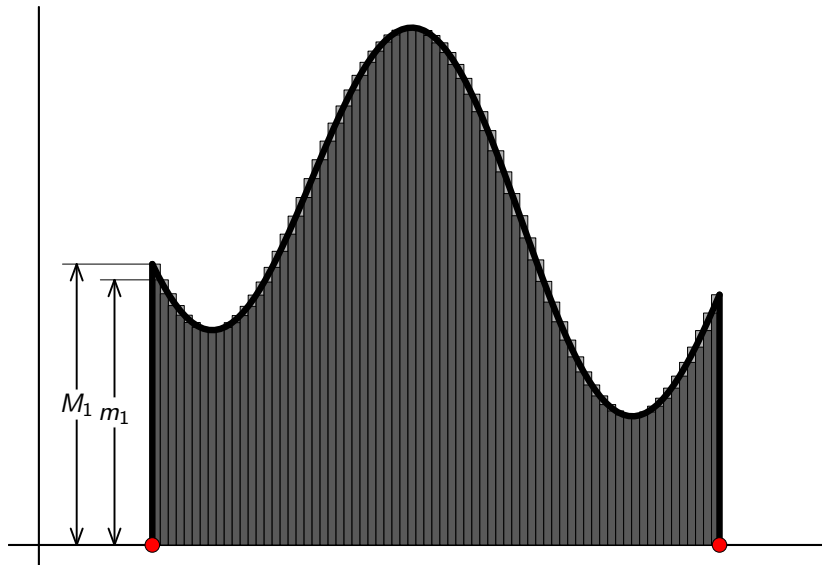
Lower and upper sums



Lower and upper sums



Lower and upper sums



Rigorous development of the integral

Definition (Partition)

Let $a < b$. A **partition** of the interval $[a, b]$ is a finite collection of points in $[a, b]$, one of which is a , and one of which is b .

We normally label the points in a partition

$$a = t_0 < t_1 < \cdots < t_{n-1} < t_n = b,$$

so the i^{th} subinterval in the partition is

$$[t_{i-1}, t_i].$$

Rigorous development of the integral

Definition (Lower and upper sums)

Suppose f is bounded on $[a, b]$ and $P = \{t_0, \dots, t_n\}$ is a partition of $[a, b]$. Recalling the motivating sketch, let

$$m_i = \inf \{ f(x) : x \in [t_{i-1}, t_i] \},$$

$$M_i = \sup \{ f(x) : x \in [t_{i-1}, t_i] \}.$$

The lower sum of f for P , denoted by $L(f, P)$, is defined as

$$L(f, P) = \sum_{i=1}^n m_i(t_i - t_{i-1}).$$

The upper sum of f for P , denoted by $U(f, P)$, is defined as

$$U(f, P) = \sum_{i=1}^n M_i(t_i - t_{i-1}).$$

Rigorous development of the integral

Relationship between motivating sketch and rigorous definition of lower and upper sums:

- The lower and upper sums correspond to the total areas of rectangles lying below and above the graph of f in our motivating sketch.
- However, these sums have been defined precisely without any appeal to a concept of “area”.
- The requirement that f be bounded on $[a, b]$ is essential in order to be sure that all the m_i and M_i are well-defined.
- It is also essential that the m_i and M_i be defined as inf's and sup's (rather than maxima and minima) because f was not assumed to be continuous.

Rigorous development of the integral

Relationship between motivating sketch and rigorous definition of lower and upper sums:

- Since $m_i \leq M_i$ for each i , we have

$$m_i(t_i - t_{i-1}) \leq M_i(t_i - t_{i-1}), \quad i = 1, \dots, n.$$

$\therefore$ For any partition P of $[a, b]$ we have

$$L(f, P) \leq U(f, P),$$

because

$$L(f, P) = \sum_{i=1}^n m_i(t_i - t_{i-1}),$$
$$U(f, P) = \sum_{i=1}^n M_i(t_i - t_{i-1}).$$

Poll

- Go to
https://www.childsmath.ca/childsa/forms/main_login.php
- Click on **Math 3A03**
- Click on **Take Class Poll**
- Fill in poll **Integrals: Lower and Upper Sums**
- .

Rigorous development of the integral

Relationship between motivating sketch and rigorous definition of lower and upper sums:

- More generally, if P_1 and P_2 are any two partitions of $[a, b]$, it ought to be true that

$$L(f, P_1) \leq U(f, P_2),$$

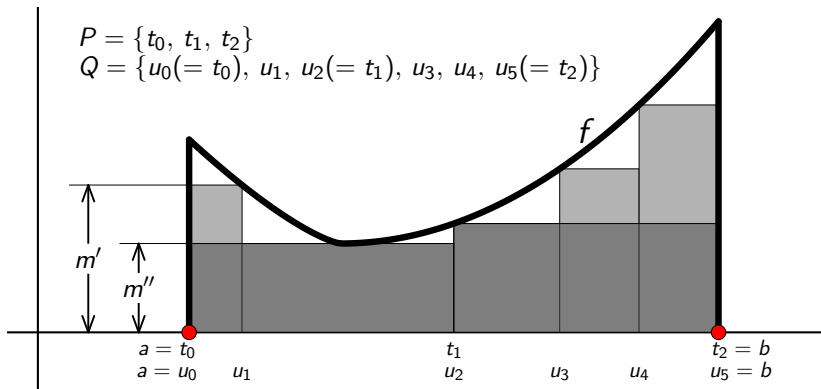
because $L(f, P_1)$ should be $\leq$ area of $R(f, a, b)$, and $U(f, P_2)$ should be $\geq$ area of $R(f, a, b)$.

- But “ought to” and “should be” prove nothing, especially since we haven’t yet even defined “area of $R(f, a, b)$ ”.
- Before we can *define* “area of $R(f, a, b)$ ”, we need to prove that $L(f, P_1) \leq U(f, P_2)$ for any partitions $P_1, P_2 \dots$

Rigorous development of the integral

Lemma (Partition Lemma)

If *partition* $P \subseteq$ *partition* Q (i.e., if every point of P is also in Q), then $L(f, P) \leq L(f, Q)$ and $U(f, P) \geq U(f, Q)$.



Rigorous development of the integral

Proof of Partition Lemma

As a first step, consider the special case in which the finer partition Q contains only one more point than P :

$$\begin{aligned}P &= \{t_0, \dots, t_n\}, \\Q &= \{t_0, \dots, t_{k-1}, u, t_k, \dots, t_n\},\end{aligned}$$

where

$$a = t_0 < t_1 < \dots < t_{k-1} < u < t_k < \dots < t_n = b.$$

Because $[t_{k-1}, t_k]$ is split by u , we have two lower bounds:

$$\begin{aligned}m' &= \inf \{ f(x) : x \in [t_{k-1}, u] \}, \\m'' &= \inf \{ f(x) : x \in [u, t_k] \}.\end{aligned}$$

... continued ...

Rigorous development of the integral

Proof of Partition Lemma (cont.)

Then
$$L(f, P) = \sum_{i=1}^n m_i(t_i - t_{i-1}),$$

and
$$\begin{aligned} L(f, Q) = & \sum_{i=1}^{k-1} m_i(t_i - t_{i-1}) + m'(u - t_{k-1}) \\ & + m''(t_k - u) + \sum_{i=k+1}^n m_i(t_i - t_{i-1}). \end{aligned}$$

$\therefore$ To prove $L(f, P) \leq L(f, Q)$, it is enough to show

$$m_k(t_k - t_{k-1}) \leq m'(u - t_{k-1}) + m''(t_k - u).$$

... continued ...

Rigorous development of the integral

Proof of Partition Lemma (cont.)

Now note that since

$$\{f(x) : x \in [t_{k-1}, u]\} \subseteq \{f(x) : x \in [t_{k-1}, t_k]\},$$

the RHS might contain some additional smaller numbers, so we must have

$$\begin{aligned} m_k &= \inf \{f(x) : x \in [t_{k-1}, t_k]\} \\ &\leq \inf \{f(x) : x \in [t_{k-1}, u]\} = m'. \end{aligned}$$

Thus, $m_k \leq m'$, and, similarly, $m_k \leq m''$.

$$\begin{aligned} \therefore m_k(t_k - t_{k-1}) &= m_k(t_k - u + u - t_{k-1}) \\ &= m_k(u - t_{k-1}) + m_k(t_k - u) \\ &\leq m'(u - t_{k-1}) + m''(t_k - u), \end{aligned}$$

... continued ...

Rigorous development of the integral

Proof of Partition Lemma (cont.)

which proves (in this special case where Q contains only one more point than P) that $L(f, P) \leq L(f, Q)$.

We can now prove the general case by adding one point at a time.

If Q contains ℓ more points than P , define a sequence of partitions

$$P = P_0 \subset P_1 \subset \cdots \subset P_\ell = Q$$

such that P_{j+1} contains exactly one more point than P_j . Then

$$L(f, P) = L(f, P_0) \leq L(f, P_1) \leq \cdots \leq L(f, P_\ell) = L(f, Q),$$

so $L(f, P) \leq L(f, Q)$.

(Proving $U(f, P) \geq U(f, Q)$ is similar: check!)



Rigorous development of the integral

Theorem (Partition Theorem)

Let P_1 and P_2 be any two partitions of $[a, b]$. If f is bounded on $[a, b]$ then

$$L(f, P_1) \leq U(f, P_2).$$

Proof.

This is a straightforward consequence of the [partition lemma](#).

Let $P = P_1 \cup P_2$, i.e., P is the partition obtained by combining all the points of P_1 and P_2 .

Then

$$L(f, P_1) \leq L(f, P) \leq U(f, P) \leq U(f, P_2).$$



Rigorous development of the integral

Important inferences that follow from the **partition theorem**:

- For any partition P' , the upper sum $U(f, P')$ is an upper bound for the set of all lower sums $L(f, P)$.

$$\therefore \sup \{L(f, P) : P \text{ a partition of } [a, b]\} \leq U(f, P') \quad \forall P'$$

$$\therefore \sup \{L(f, P)\} \leq \inf \{U(f, P)\}$$

$\therefore$ For any partition P' ,

$$L(f, P') \leq \sup \{L(f, P)\} \leq \inf \{U(f, P)\} \leq U(f, P')$$

- If $\sup \{L(f, P)\} = \inf \{U(f, P)\}$ then we can define “**area of $R(f, a, b)$** ” to be this number.

- Is it possible that $\sup \{L(f, P)\} < \inf \{U(f, P)\}$?

Poll

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- Fill in poll **Integrals:** $\sup \{L(f, P)\} < \inf \{U(f, P)\} ?$
- .



Mathematics
and Statistics

$$\int_M d\omega = \int_{\partial M} \omega$$

Mathematics 3A03 Real Analysis I

Instructor: David Earn

Lecture 7
Integration II
Thursday 22 January 2026

Announcements

- [Assignment 2](#) has been posted on the course website.
 - The participation deadline is Tuesday 3 Feb 2026 @ 2:25pm.

Rigorous development of the integral

Example

$\exists? f : [a, b] \rightarrow \mathbb{R}$ (bounded) $\nmid \sup \{L(f, P)\} < \inf \{U(f, P)\}$

$$\text{Let } f(x) = \begin{cases} 1 & x \in \mathbb{Q} \cap [a, b], \\ 0 & x \in \mathbb{Q}^c \cap [a, b]. \end{cases}$$

Consider any partition P of $[a, b]$.

If $P = \{t_0, \dots, t_n\}$ then $m_i = 0 \forall i$ ($\because [t_{i-1}, t_i] \cap \mathbb{Q}^c \neq \emptyset$),
and $M_i = 1 \forall i$ ($\because [t_{i-1}, t_i] \cap \mathbb{Q} \neq \emptyset$).

$\therefore L(f, P) = 0$ and $U(f, P) = b - a$ for any partition P .

$\therefore \sup \{L(f, P)\} = 0 < b - a = \inf \{U(f, P)\}.$ □

Can we define “area of $R(f, a, b)$ ” for such a weird function?

Yes, but not in this course!

Rigorous development of the integral

Definition (Integrable)

A function $f : [a, b] \rightarrow \mathbb{R}$ is said to be *integrable* on $[a, b]$ if it is bounded on $[a, b]$ and

$$\begin{aligned} & \sup\{L(f, P) : P \text{ a partition of } [a, b]\} \\ &= \inf\{U(f, P) : P \text{ a partition of } [a, b]\}. \end{aligned}$$

In this case, this common number is called the *integral* of f on $[a, b]$ and is denoted

$$\int_a^b f$$

Note: If f is integrable then for any partition P we have

$$L(f, P) \leq \int_a^b f \leq U(f, P),$$

and $\int_a^b f$ is the unique number with this property.

Rigorous development of the integral

■ *Notation:*

$$\int_a^b f(x) dx \quad \text{means precisely the same as} \quad \int_a^b f.$$

- The symbol “ dx ” has no meaning in isolation just as “ $x \rightarrow$ ” has no meaning except in $\lim_{x \rightarrow a} f(x)$.
- It is not clear from the definition which functions are **integrable**.
- The definition of the **integral** does not itself indicate how to compute the integral of any given **integrable** function. So far, without a lot more effort, we can't say much more than these two things:
 - 1 If $f(x) \equiv c$ then f is **integrable** on $[a, b]$ and $\int_a^b f = c \cdot (b - a)$.
 - 2 The **weird example** function is not **integrable**.

Rigorous development of the integral

- Bartle and Sherbert refer to functions that are **integrable** according to our definition as ***Darboux integrable*** (BS §7.4, p. 225).
- BS develop the integral using one value of the function within each subinterval of a partition, rather than starting with upper and lower sums. They refer to functions that are integrable in this sense as ***Riemann integrable***.
- BS also prove (BS Theorem 7.4.11, p. 232) that a function is Riemann integrable if and only if it is Darboux integrable. So the two definitions are, in fact, equivalent.
- In Math 4A03 you will define ***Lebesgue integrable***, a more subtle concept that makes it possible to attach meaning to “area of $R(f, a, b)$ ” for the **weird example** function (among others), and to precisely characterize functions that are Riemann integrable.

Rigorous development of the integral

Theorem (Equivalent “ ε - P ” criterion for integrability)

A bounded function $f : [a, b] \rightarrow \mathbb{R}$ is *integrable* on $[a, b]$ iff for all $\varepsilon > 0$ there is a partition P of $[a, b]$ such that

$$U(f, P) - L(f, P) < \varepsilon.$$

(BS Theorem 7.4.8, p. 229)

Note: This theorem is just a restatement of the definition of *integrability*. It is often more convenient to work with $\varepsilon > 0$ than with sup's and inf's.

Rigorous development of the integral

Proof of equivalence of "sup = inf" and " ε - P " definitions of integrability.

($\Rightarrow$) Suppose the bounded function f is integrable, i.e.,

$$\begin{aligned} \sup\{L(f, P) : P \text{ a partition of } [a, b]\} \\ = \inf\{U(f, P) : P \text{ a partition of } [a, b]\} = \int_a^b f \end{aligned}$$

Given $\varepsilon > 0$, since $\int_a^b f$ is the least upper bound of the lower sums, there is a partition P_1 such that

$$\int_a^b f = \sup_{P'}\{L(f, P')\} < L(f, P_1) + \frac{\varepsilon}{2},$$

$$\text{i.e., such that} \quad -L(f, P_1) < -\int_a^b f + \frac{\varepsilon}{2}. \quad (\heartsuit)$$

... continued...

Rigorous development of the integral

Proof of equivalence of “ $\sup = \inf$ ” and “ ε - P ” definitions of integrability.

Similarly, there is a partition P_2 such that

$$U(f, P_2) < \inf_{P'} \{U(f, P')\} + \frac{\varepsilon}{2} = \int_a^b f + \frac{\varepsilon}{2}. \quad (\diamond)$$

Therefore, putting together inequalities $(\diamond)$ and $(\heartsuit)$, we have

$$U(f, P_2) - L(f, P_1) < \int_a^b f + \frac{\varepsilon}{2} - \int_a^b f + \frac{\varepsilon}{2} = \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

But that's not quite **what we need**. We need, for a *single* partition P ,

$$U(f, P) - L(f, P) < \varepsilon.$$

How should we proceed?

Hint: Recall the **partition lemma** ...

... continued...

Rigorous development of the integral

Proof of equivalence of “ $\sup = \inf$ ” and “ ε - P ” definitions of integrability.

Let $P = P_1 \cup P_2$. Then the **partition lemma** implies that $L(f, P) \geq L(f, P_1)$, and $U(f, P) \leq U(f, P_2)$, so

$$\begin{aligned} U(f, P) - L(f, P) &\leq U(f, P_2) - L(f, P_1) \\ &< \int_a^b f + \frac{\varepsilon}{2} - \int_a^b f + \frac{\varepsilon}{2} = \varepsilon, \end{aligned}$$

which completes the proof that $\sup = \inf \implies \varepsilon$ - P .

($\Leftarrow$) We now need to show that if a bounded function f satisfies the ε - P definition of integrability then it also satisfies the $\sup = \inf$ definition of integrability.

Given $\varepsilon > 0$, we can choose a partition P (depending on ε) such that

$$U(f, P) - L(f, P) < \varepsilon.$$

... continued ...

Rigorous development of the integral

Proof of equivalence of “ $\sup = \inf$ ” and “ ε - P ” definitions of integrability.

Now, for any partition, and in particular for P , we have

$$L(f, P) \leq \sup_{P'} \{L(f, P')\} \leq \inf_{P'} \{U(f, P')\} \leq U(f, P),$$

We can temporarily write this more simply as

$$L \leq S \leq I \leq U$$

Subtracting S from this chain of inequalities implies

$$L - S \leq 0 \leq I - S \leq U - S$$

Now note that $L \leq S$ implies $U - S \leq U - L$, so we have

$$0 \leq I - S \leq U - L$$

i.e., $0 \leq \inf_{P'} \{U(f, P')\} - \sup_{P'} \{L(f, P')\} \leq U(f, P) - L(f, P) < \varepsilon$.

But by hypothesis, such a partition P can be found for any given $\varepsilon > 0$.

Therefore, $\inf_{P'} \{U(f, P')\} = \sup_{P'} \{L(f, P')\}$. □

Rigorous development of the integral

Example

Suppose $b > 0$ and $f(x) = x$ for all $x \in \mathbb{R}$. Prove, using only the definition of the integral via $\sup = \inf$ or ε - P , that

$$\int_0^b f = \frac{b^2}{2}.$$

(This exercise, which is on [Assignment 2](#), should help you appreciate the Fundamental Theorem of Calculus.)

Note: If working through the above example doesn't convince you of the power of the Fundamental Theorem of Calculus, try computing $\int_0^b x^2 dx$ directly from the definition of the integral.