7 Integration

Mathematics
and Statistics

$$\int_M d\omega = \int_{\partial M} \omega$$

Mathematics 3A03
Real Analysis I

Instructor: David Earn

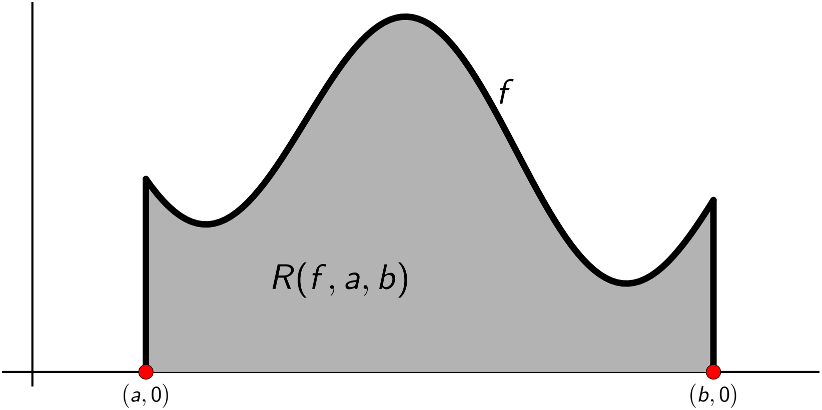
Lecture 7
Integration
Wednesday 22 January 2025

Announcements

- Solutions to Assignment 1 were posted last night.
- Kieran will have office hours tomorrow (Thursday) for two hours, 12:30–2:30 pm. (He will not have a Friday office hour this week.)

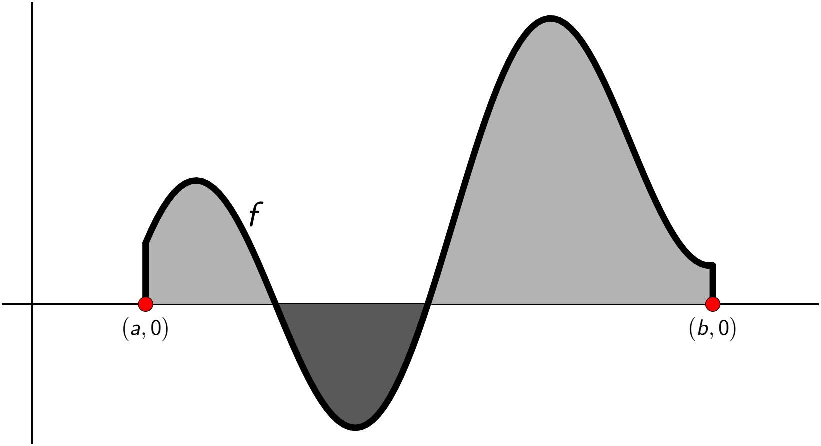
Integration

Integration



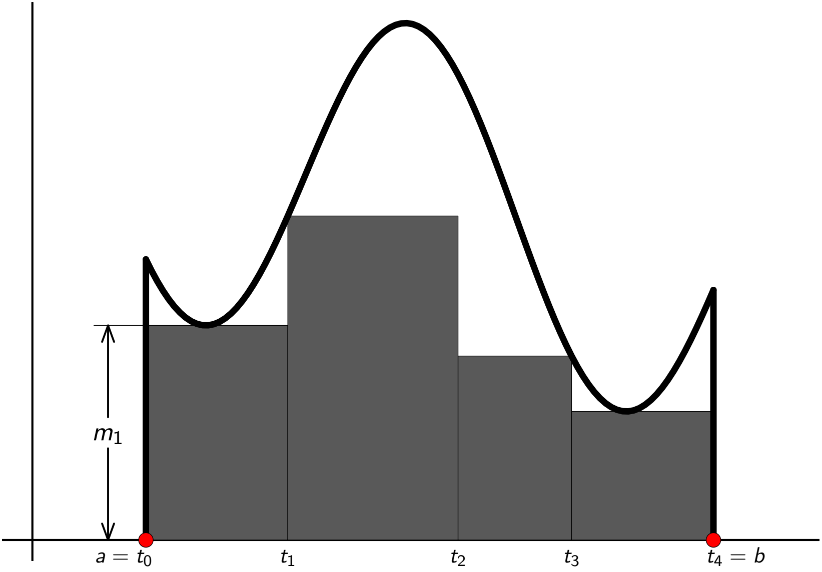
- "Area of region $R(f, a, b)$" is actually a very subtle concept.
- We will only scratch the surface of it (greater depth in Math 4A).
- Our treatment is similar to that in Michael Spivak's "Calculus" (2008); BS refer to this approach as the Darboux integral (BS §7.4, p. 225).
- The Darboux and Riemann approaches to the integral are equivalent.

Integration

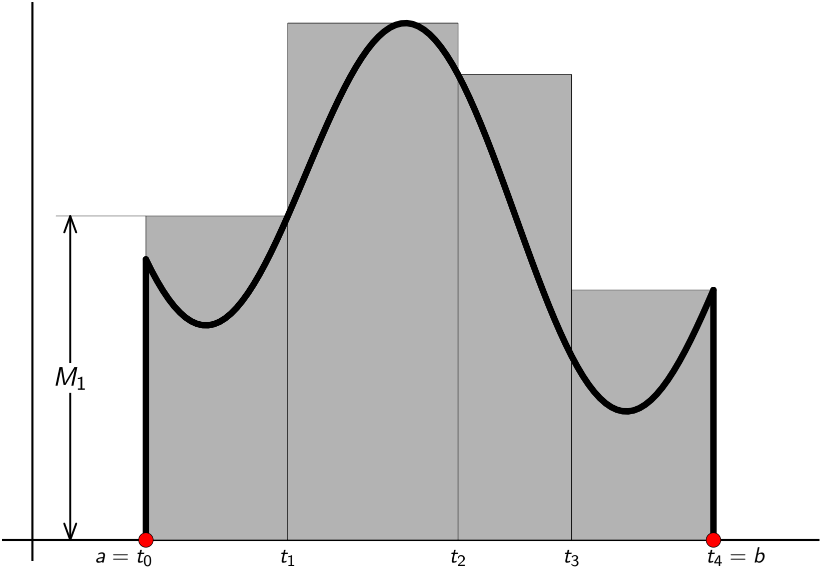


- Contribution to "area of $R(f, a, b)$" is positive or negative depending on whether f is positive or negative.

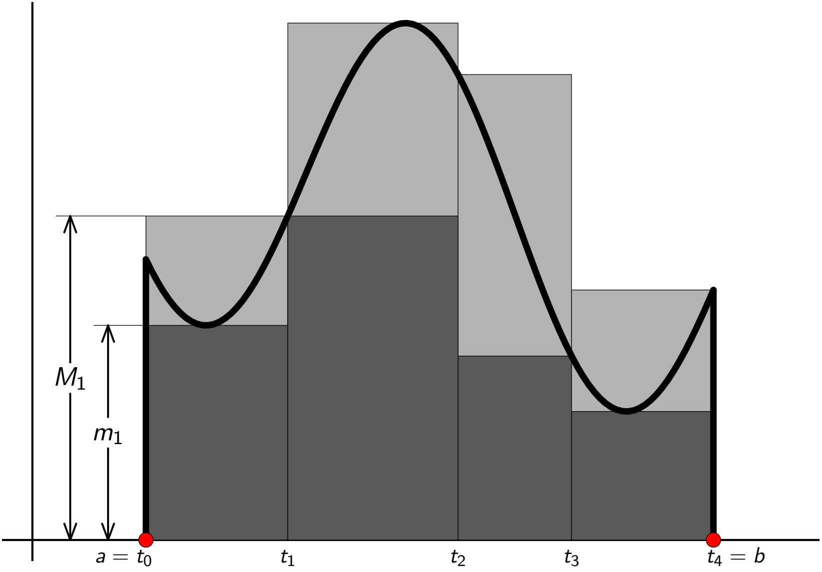
Lower sum



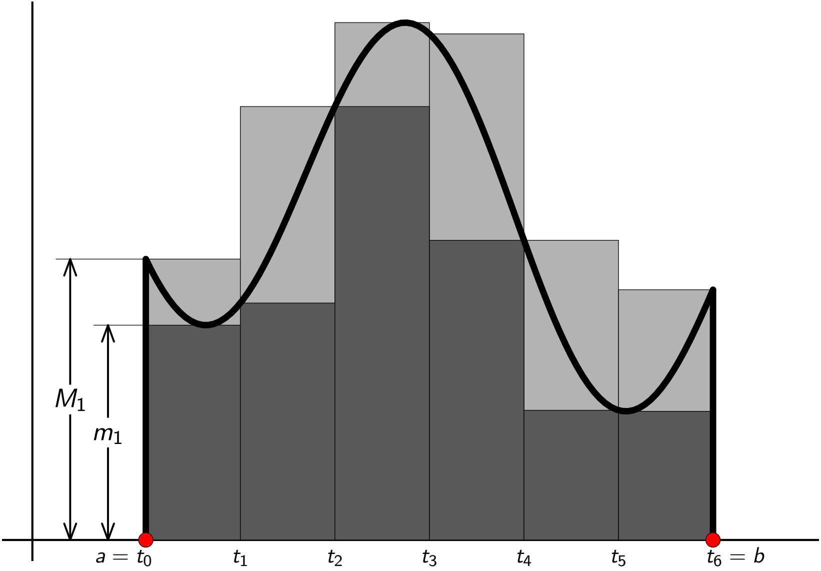
Upper sum



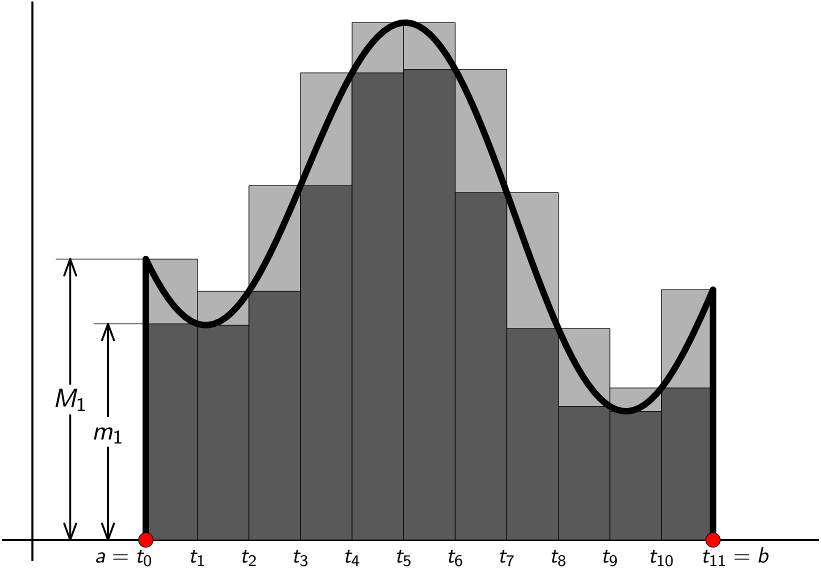
Lower and upper sums



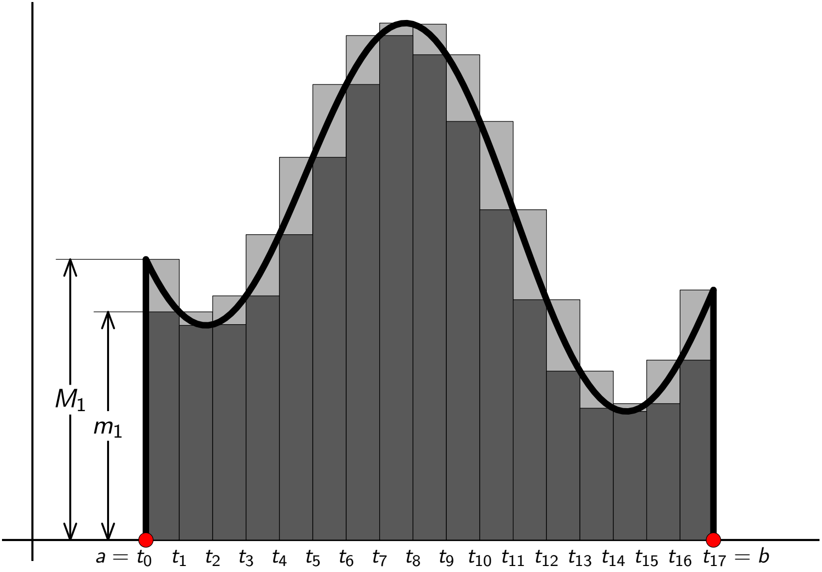
Lower and upper sums



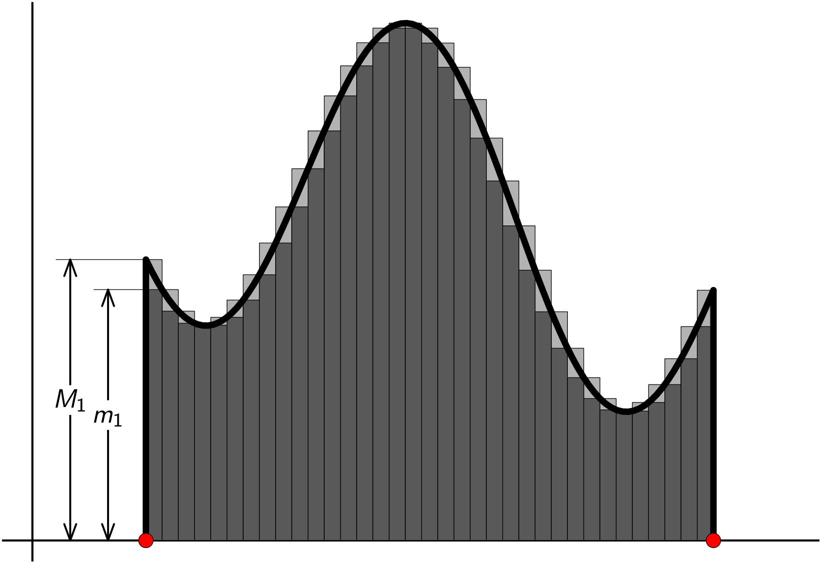
Lower and upper sums



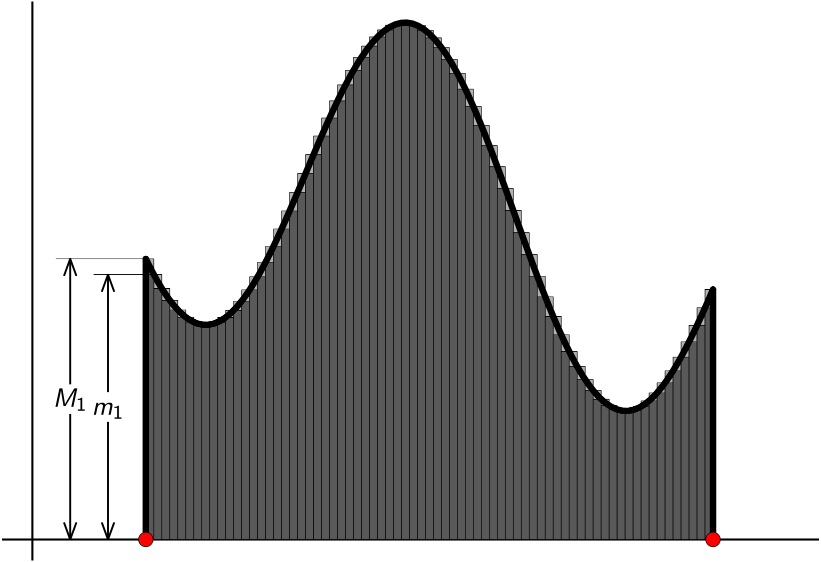
Lower and upper sums



Lower and upper sums



Lower and upper sums



Rigorous development of the integral

Definition (Partition)

Let $a < b$. A ***partition*** of the interval $[a, b]$ is a finite collection of points in $[a, b]$, one of which is a, and one of which is b.

We normally label the points in a partition

$$a = t_0 < t_1 < \cdots < t_{n-1} < t_n = b\,,$$

so the i^{th} subinterval in the partition is

$$[t_{i-1}, t_i]\,.$$

Rigorous development of the integral

Definition (Lower and upper sums)

Suppose f is bounded on $[a, b]$ and $P = \{t_0, \dots, t_n\}$ is a partition of $[a, b]$. Recalling the motivating sketch, let

$$m_i = \inf \{ f(x) : x \in [t_{i-1}, t_i] \},$$

$$M_i = \sup \{ f(x) : x \in [t_{i-1}, t_i] \}.$$

The lower sum of f for P, denoted by $L(f, P)$, is defined as

$$L(f, P) = \sum_{i=1}^{n} m_i(t_i - t_{i-1}).$$

The upper sum of f for P, denoted by $U(f, P)$, is defined as

$$U(f, P) = \sum_{i=1}^{n} M_i(t_i - t_{i-1}).$$

Rigorous development of the integral

Relationship between motivating sketch and rigorous definition of lower and upper sums:

- The lower and upper sums correspond to the total areas of rectangles lying below and above the graph of f in our motivating sketch.
- However, these sums have been defined precisely without any appeal to a concept of "area".
- The requirement that f be bounded on $[a, b]$ is essential in order to be sure that all the m_i and M_i are well-defined.
- It is also essential that the m_i and M_i be defined as inf's and sup's (rather than maxima and minima) because f was not assumed to be continuous.

Rigorous development of the integral

Relationship between motivating sketch and rigorous definition of lower and upper sums:

- Since $m_i \leq M_i$ for each i, we have

$$m_i(t_i - t_{i-1}) \leq M_i(t_i - t_{i-1}), \qquad i = 1, \ldots, n.$$

$\therefore$ For <u>any</u> partition P of $[a, b]$ we have

$$L(f, P) \leq U(f, P),$$

because

$$L(f, P) = \sum_{i=1}^{n} m_i(t_i - t_{i-1}),$$

$$U(f, P) = \sum_{i=1}^{n} M_i(t_i - t_{i-1}).$$

Poll

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Rigorous development of the integral

Relationship between motivating sketch and rigorous definition of lower and upper sums:

- More generally, if P_1 and P_2 are any two partitions of $[a, b]$, it *ought* to be true that

$$L(f, P_1) \leq U(f, P_2),$$

  because $L(f, P_1)$ should be $\leq$ area of $R(f, a, b)$, and $U(f, P_2)$ should be $\geq$ area of $R(f, a, b)$.

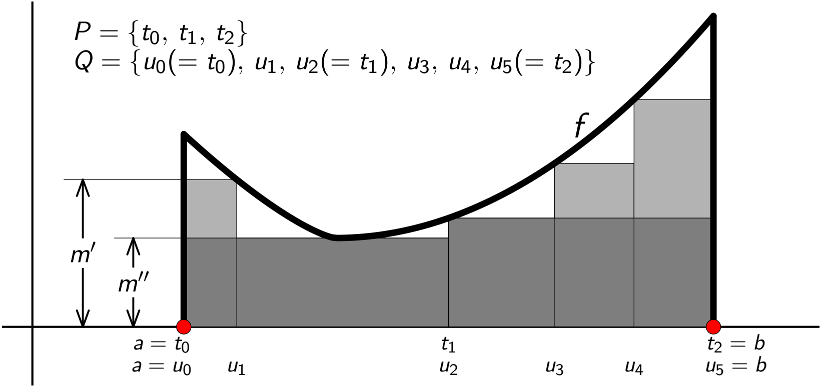
- But "ought to" and "should be" prove nothing, especially since we haven't yet even defined "area of $R(f, a, b)$".

- Before we can *define* "area of $R(f, a, b)$", we need to prove that $L(f, P_1) \leq U(f, P_2)$ for any partitions P_1, P_2 ...

Rigorous development of the integral

Lemma (Partition Lemma)

If partition $P \subseteq$ partition Q (i.e., if every point of P is also in Q), then $L(f,P) \le L(f,Q)$ *and* $U(f,P) \ge U(f,Q)$.



Rigorous development of the integral

Proof of Partition Lemma

As a first step, consider the special case in which the finer partition Q contains only one more point than P:

$$P = \{t_0, \dots, t_n\}\,,$$
$$Q = \{t_0, \dots, t_{k-1}, u, t_k, \dots, t_n\}\,,$$

where

$$a = t_0 < t_1 < \cdots < t_{k-1} < u < t_k < \cdots < t_n = b\,.$$

Because $[t_{k-1}, t_k]$ is split by u, we have two lower bounds:

$$m' = \inf\{\, f(x) \,:\, x \in [t_{k-1}, u] \,\}\,,$$
$$m'' = \inf\{\, f(x) \,:\, x \in [u, t_k] \,\}\,.$$

...continued...

Rigorous development of the integral

Proof of Partition Lemma (cont.)

Then $$L(f,P)=\sum_{i=1}^{n} m_i(t_i-t_{i-1})\,,$$

and $$L(f,Q)=\sum_{i=1}^{k-1} m_i(t_i-t_{i-1})+m'(u-t_{k-1}) + m''(t_k-u)+\sum_{i=k+1}^{n} m_i(t_i-t_{i-1})\,.$$

$\therefore$ To prove $L(f,P)\le L(f,Q)$, it is enough to show

$$m_k(t_k-t_{k-1})\le m'(u-t_{k-1})+m''(t_k-u)\,.$$

...continued...

Rigorous development of the integral

Proof of Partition Lemma (cont.)

Now note that since

$$\{\, f(x) \,:\, x \in [t_{k-1}, u] \,\} \quad \subseteq \quad \{\, f(x) \,:\, x \in [t_{k-1}, t_k] \,\},$$

the RHS might contain some additional *smaller* numbers, so we must have

$$\begin{aligned} m_k &= \inf\{\, f(x) \,:\, x \in [t_{k-1}, t_k] \,\} \\ &\le \inf\{\, f(x) \,:\, x \in [t_{k-1}, u] \,\} \quad = \quad m' . \end{aligned}$$

Thus, $m_k \le m'$, and, similarly, $m_k \le m''$.

$$\begin{aligned} \therefore \quad m_k(t_k - t_{k-1}) &= m_k(t_k - u + u - t_{k-1}) \\ &= m_k(u - t_{k-1}) + m_k(t_k - u) \\ &\le m'(u - t_{k-1}) + m''(t_k - u) , \end{aligned}$$

...continued...

Rigorous development of the integral

Proof of Partition Lemma (cont.)

which proves (in this special case where Q contains only one more point than P) that $L(f,P) \leq L(f,Q)$.

We can now prove the general case by adding one point at a time.

If Q contains ℓ more points than P, define a sequence of partitions

$$P = P_0 \subset P_1 \subset \cdots \subset P_\ell = Q$$

such that P_{j+1} contains exactly one more point than P_j. Then

$$L(f,P) = L(f,P_0) \leq L(f,P_1) \leq \cdots \leq L(f,P_\ell) = L(f,Q)\,,$$

so $L(f,P) \leq L(f,Q)$.

(Proving $U(f,P) \geq U(f,Q)$ is similar: check!) □

Rigorous development of the integral

Theorem (Partition Theorem)

Let P_1 and P_2 be any two partitions of $[a, b]$. If f is bounded on $[a, b]$ then

$$L(f, P_1) \le U(f, P_2)\,.$$

Proof.

This is a straightforward consequence of the partition lemma.

Let $P = P_1 \cup P_2$, *i.e.*, P is the partition obtained by combining all the points of P_1 and P_2.

Then

$$L(f, P_1) \le L(f, P) \le U(f, P) \le U(f, P_2)\,.$$

□

Rigorous development of the integral

Important inferences that follow from the partition theorem:

- For any partition P', the upper sum $U(f,P')$ is an upper bound for the set of all lower sums $L(f,P)$.

  $$\therefore \quad \sup\{L(f,P) : P \text{ a partition of } [a,b]\} \le U(f,P') \qquad \forall P'$$

  $$\therefore \quad \sup\{L(f,P)\} \le \inf\{U(f,P)\}$$

  $\therefore$ For any partition P',

  $$L(f,P') \le \sup\{L(f,P)\} \le \inf\{U(f,P)\} \le U(f,P')$$

- If $\sup\{L(f,P)\} = \inf\{U(f,P)\}$ then we can define "area of $R(f,a,b)$" to be this number.
  - Is it possible that $\sup\{L(f,P)\} < \inf\{U(f,P)\}$?

Poll

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