

6 Integration



Mathematics
and Statistics

$$\int_M d\omega = \int_{\partial M} \omega$$

Mathematics 3A03 Real Analysis I

Instructor: David Earn

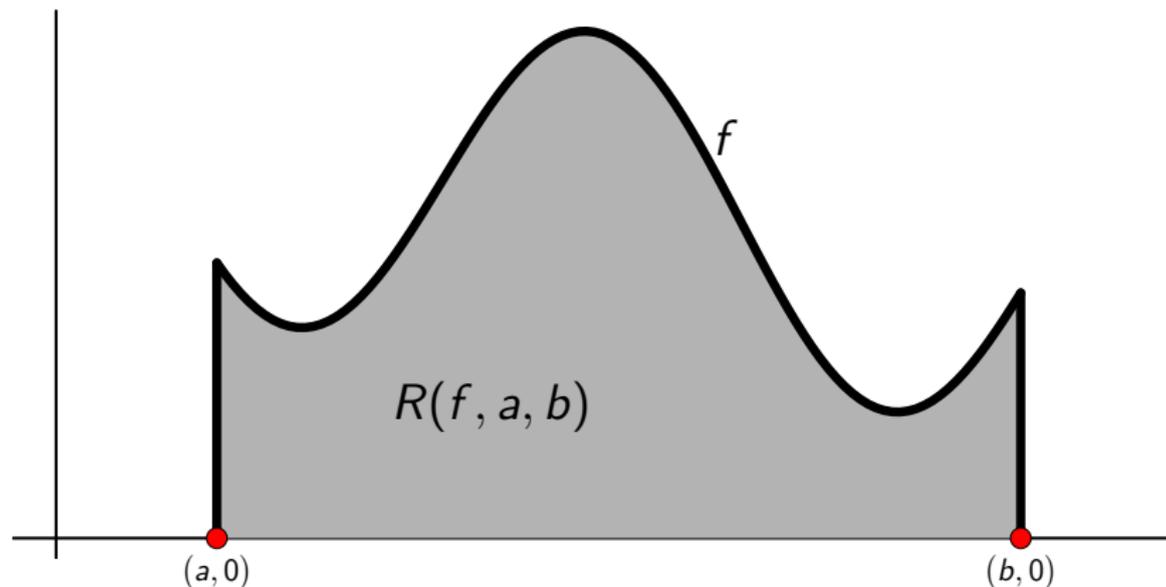
Lecture 6
Integration
Tuesday 20 January 2026

Announcements

- Participation deadline for [Assignment 1](#) was just before this class started.
- Solutions to [Assignment 1](#) will be posted today. Make sure to read the solutions even if you are confident your proofs are correct.
- I posted a [handout](#) addressing a student's question in the last class: Can one prove monotonicity via the Mean Value Theorem without Darboux's theorem?

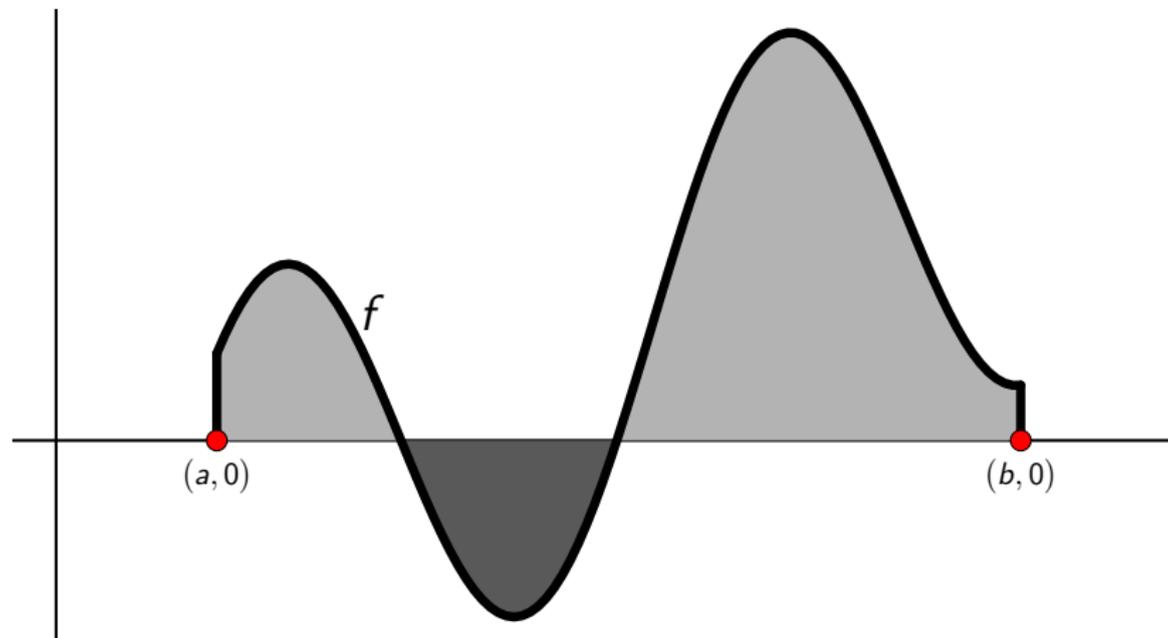
Integration

Integration



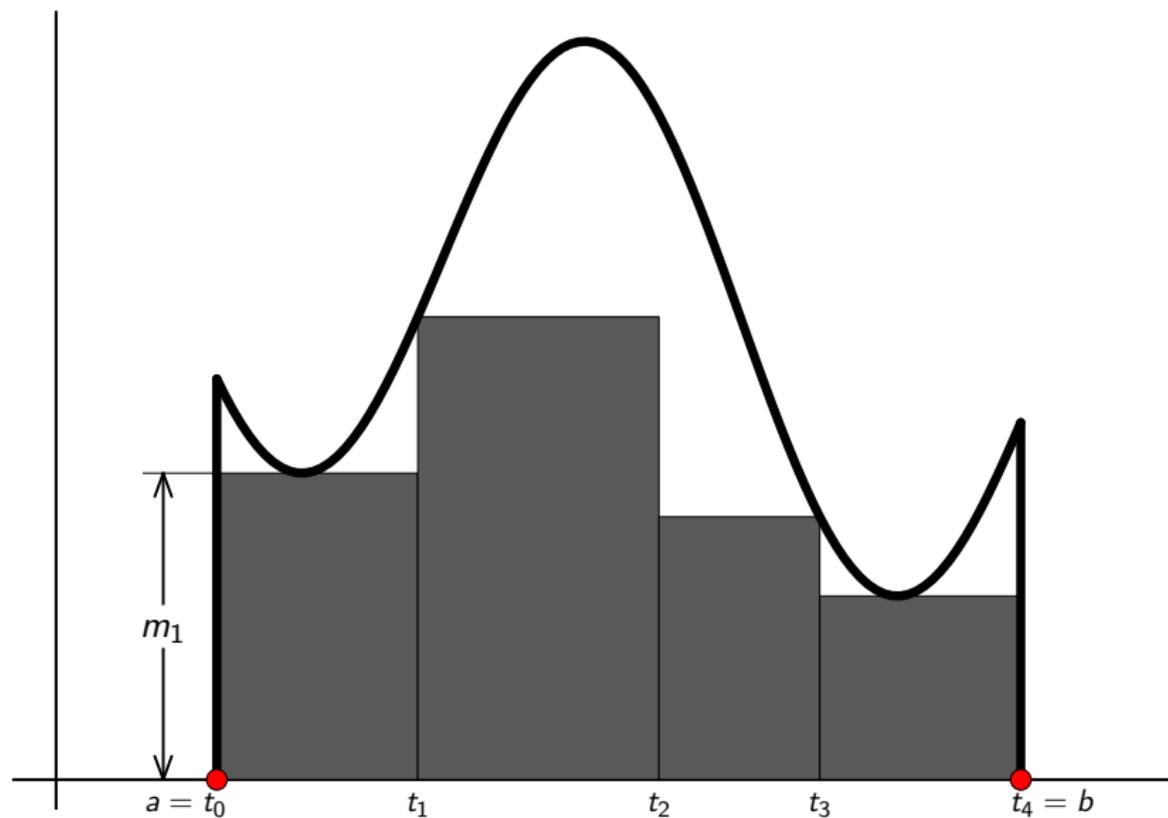
- “Area of region $R(f, a, b)$ ” is actually a very subtle concept.
- We will only scratch the surface of it (greater depth in Math 4A).
- Our treatment is similar to that in Michael Spivak’s “Calculus” (2008); BS refer to this approach as the Darboux integral (BS §7.4, p. 225).
- The Darboux and Riemann approaches to the integral are equivalent.

Integration

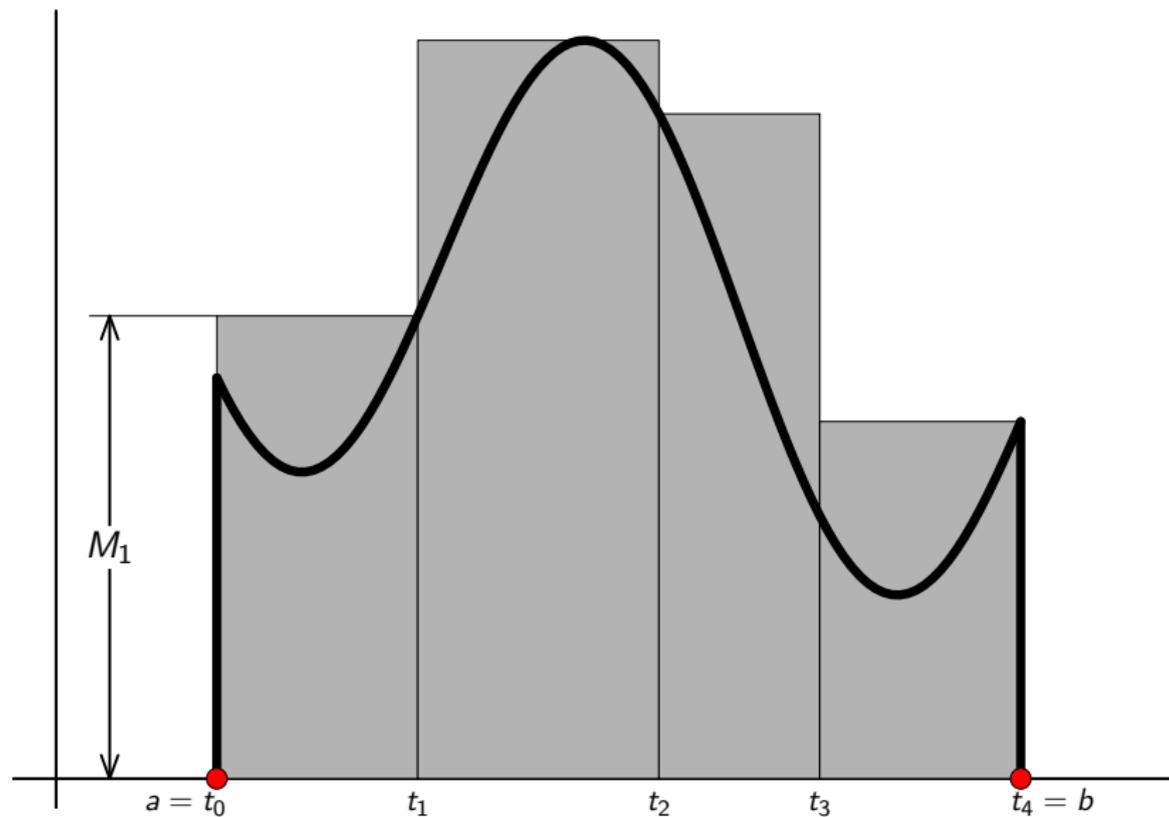


- Contribution to “area of $R(f, a, b)$ ” is positive or negative depending on whether f is positive or negative.

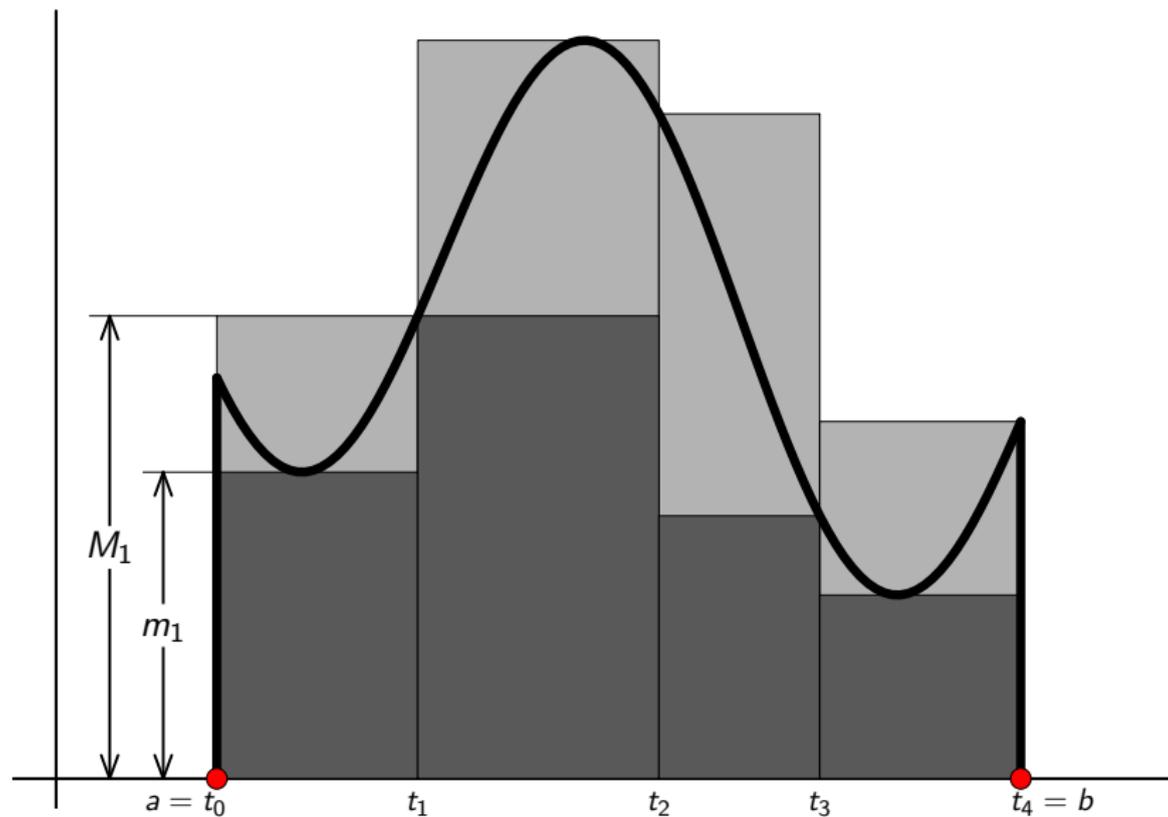
Lower sum



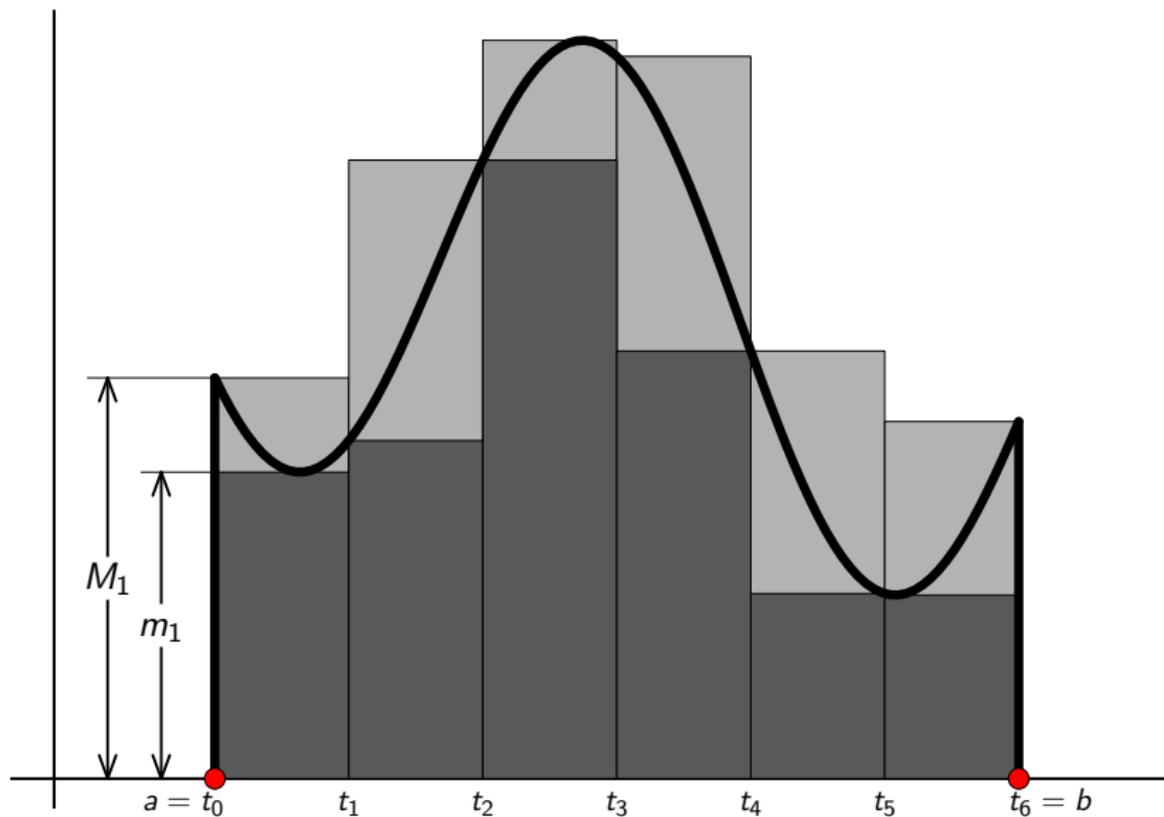
Upper sum



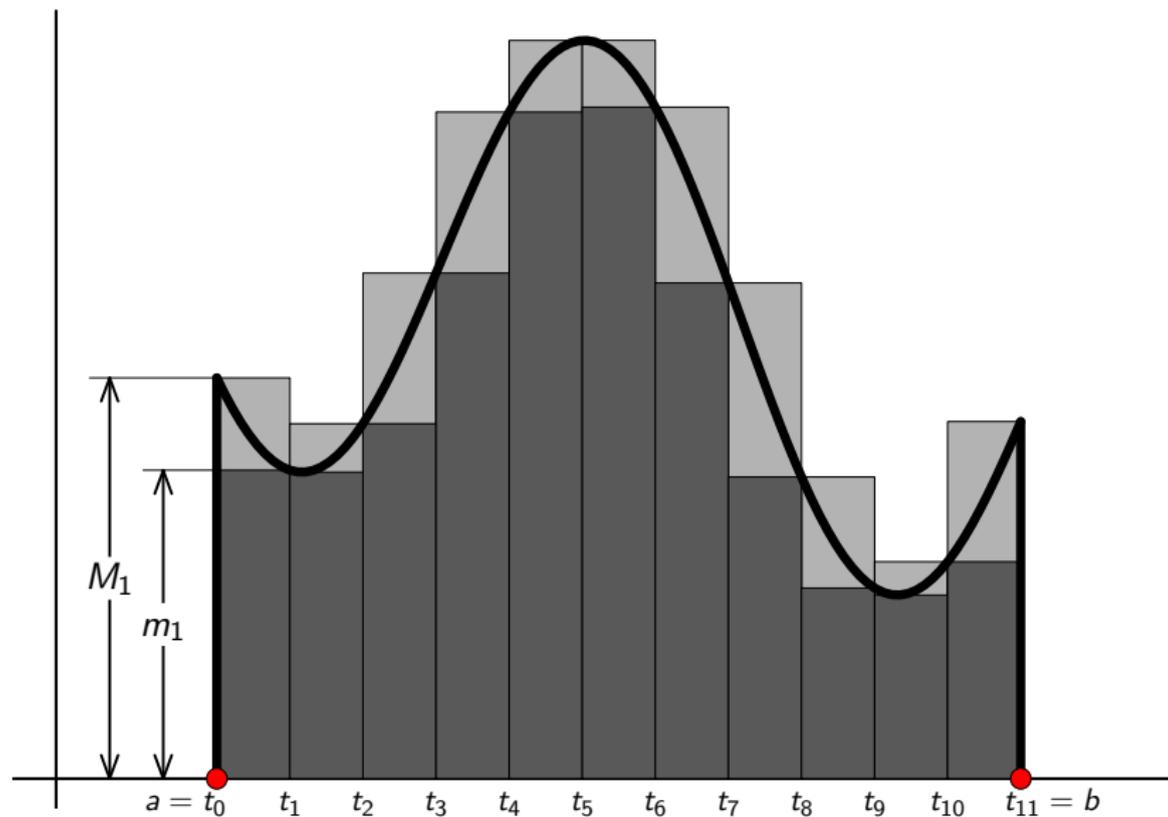
Lower and upper sums



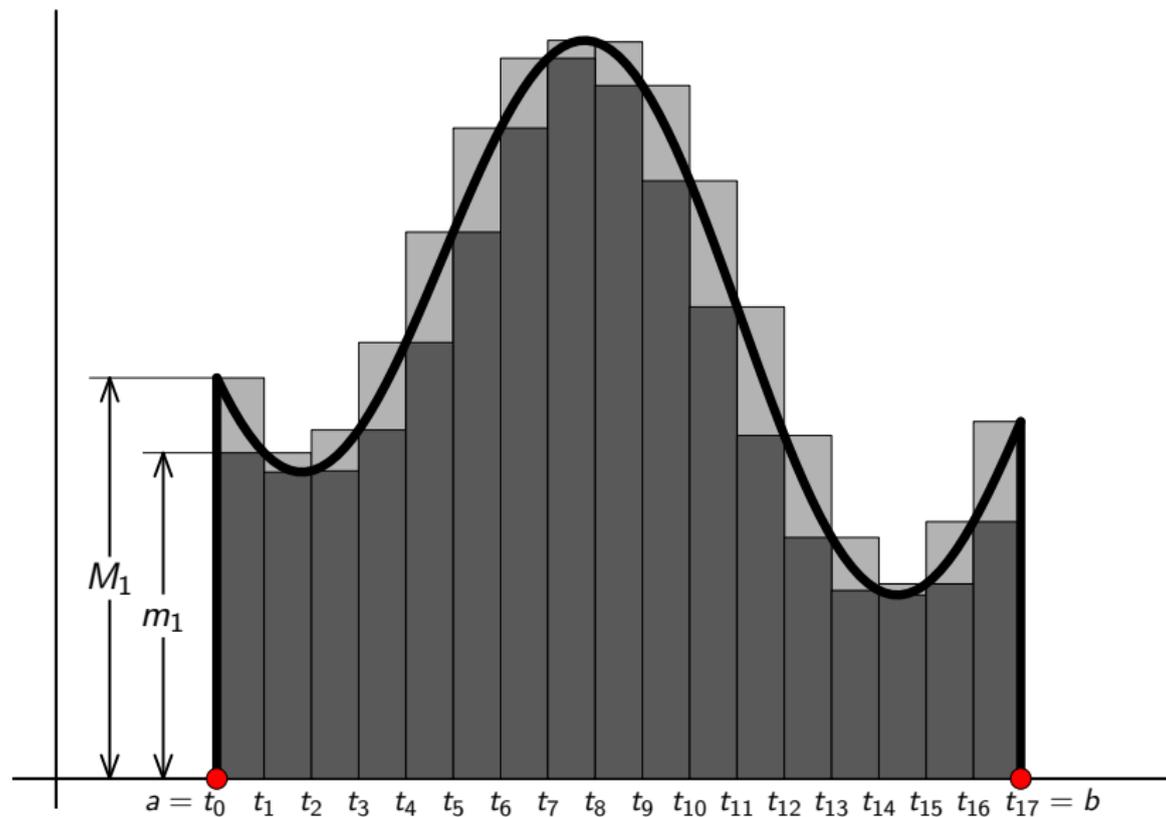
Lower and upper sums



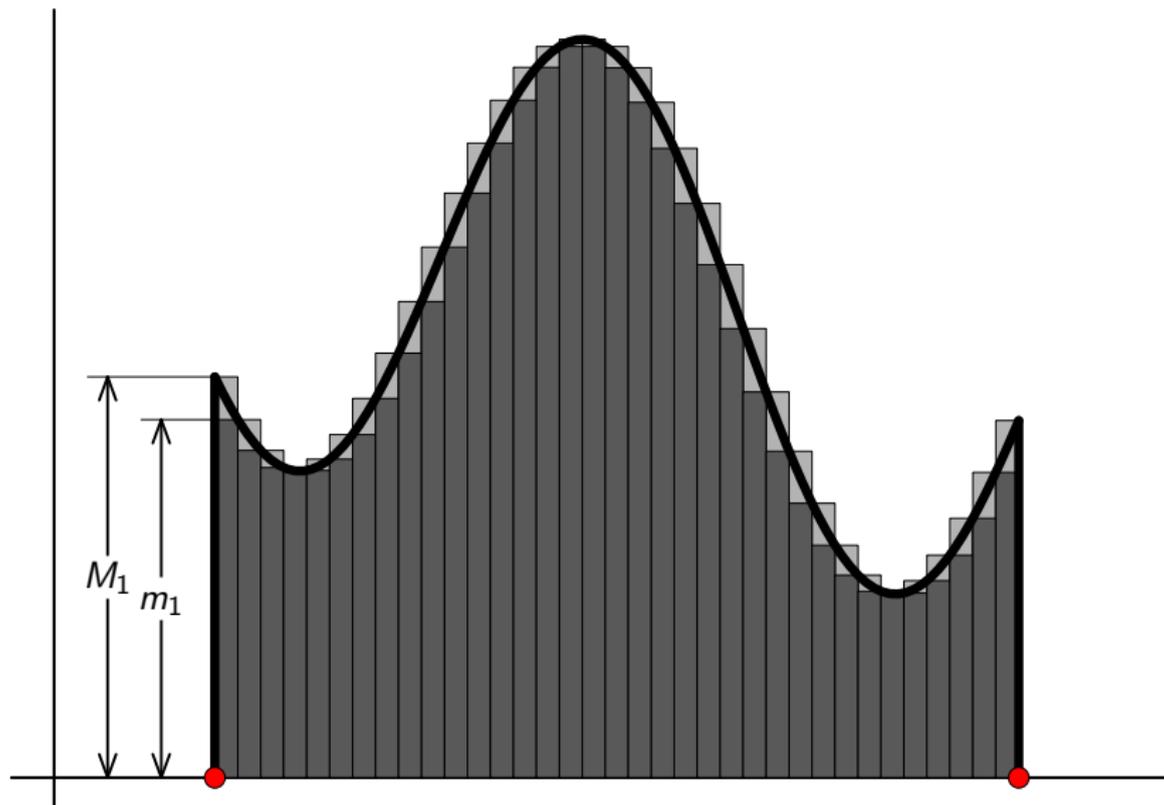
Lower and upper sums



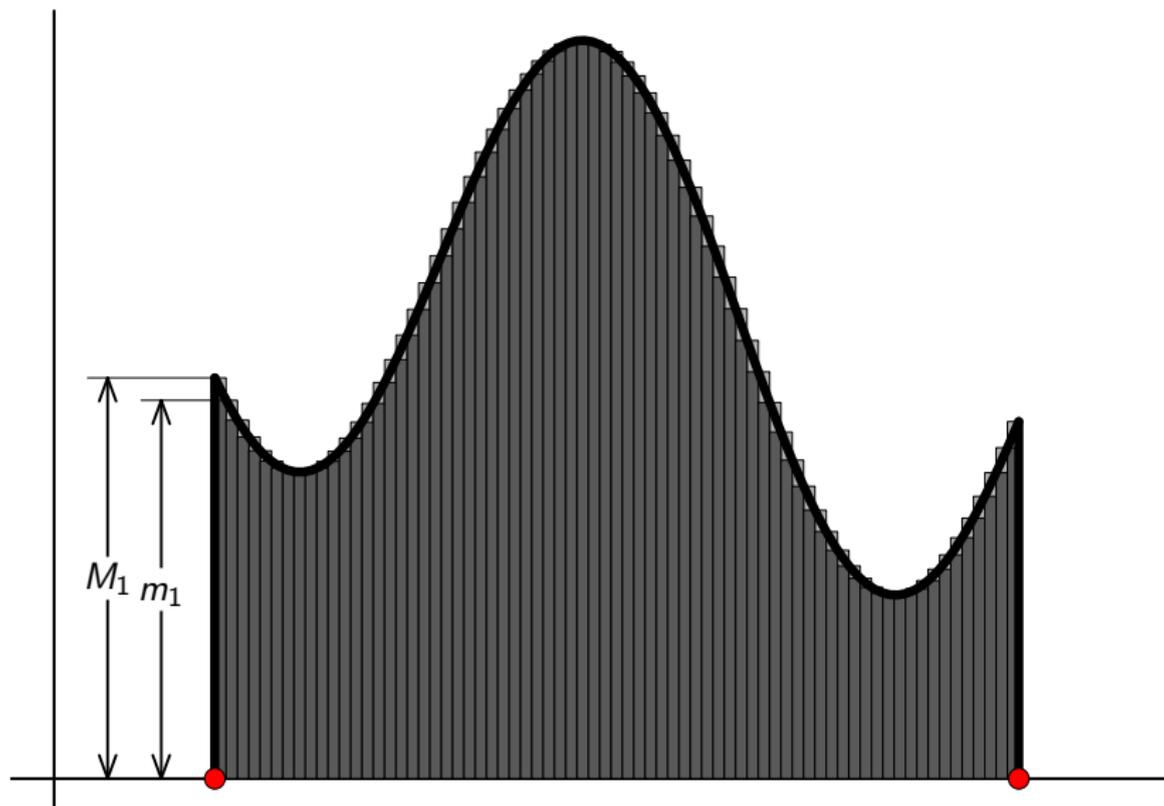
Lower and upper sums



Lower and upper sums



Lower and upper sums



Rigorous development of the integral

Definition (Partition)

Let $a < b$. A **partition** of the interval $[a, b]$ is a finite collection of points in $[a, b]$, one of which is a , and one of which is b .

We normally label the points in a partition

$$a = t_0 < t_1 < \cdots < t_{n-1} < t_n = b,$$

so the i^{th} subinterval in the partition is

$$[t_{i-1}, t_i].$$

Rigorous development of the integral

Definition (Lower and upper sums)

Suppose f is bounded on $[a, b]$ and $P = \{t_0, \dots, t_n\}$ is a **partition** of $[a, b]$. Recalling the **motivating sketch**, let

$$m_i = \inf \{ f(x) : x \in [t_{i-1}, t_i] \},$$

$$M_i = \sup \{ f(x) : x \in [t_{i-1}, t_i] \}.$$

The **lower sum** of f for P , denoted by $L(f, P)$, is defined as

$$L(f, P) = \sum_{i=1}^n m_i(t_i - t_{i-1}).$$

The **upper sum** of f for P , denoted by $U(f, P)$, is defined as

$$U(f, P) = \sum_{i=1}^n M_i(t_i - t_{i-1}).$$

Rigorous development of the integral

Relationship between motivating sketch and rigorous definition of lower and upper sums:

- The **lower and upper sums** correspond to the total areas of rectangles lying below and above the graph of f in our **motivating sketch**.
- However, these sums have been defined precisely without any appeal to a concept of “area”.
- The requirement that f be bounded on $[a, b]$ is essential in order to be sure that all the m_i and M_i are well-defined.
- It is also essential that the m_i and M_i be defined as inf's and sup's (rather than maxima and minima) because f was not assumed to be continuous.

Rigorous development of the integral

Relationship between motivating sketch and rigorous definition of lower and upper sums:

- Since $m_i \leq M_i$ for each i , we have

$$m_i(t_i - t_{i-1}) \leq M_i(t_i - t_{i-1}), \quad i = 1, \dots, n.$$

∴ For any partition P of $[a, b]$ we have

$$L(f, P) \leq U(f, P),$$

because

$$L(f, P) = \sum_{i=1}^n m_i(t_i - t_{i-1}),$$
$$U(f, P) = \sum_{i=1}^n M_i(t_i - t_{i-1}).$$

Poll

- Go to
https://www.childsmath.ca/childsa/forms/main_login.php
- Click on **Math 3A03**
- Click on **Take Class Poll**
- Fill in poll **Integrals: Lower and Upper Sums**
- .

Rigorous development of the integral

Relationship between motivating sketch and rigorous definition of lower and upper sums:

- More generally, if P_1 and P_2 are any two partitions of $[a, b]$, it ought to be true that

$$L(f, P_1) \leq U(f, P_2),$$

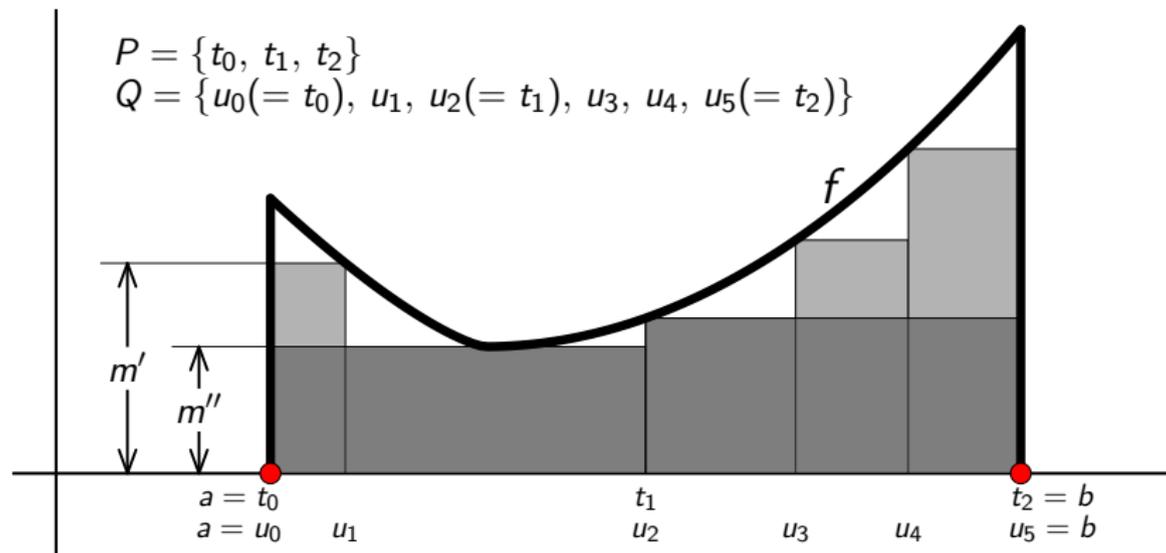
because $L(f, P_1)$ should be \leq area of $R(f, a, b)$, and $U(f, P_2)$ should be \geq area of $R(f, a, b)$.

- But “ought to” and “should be” prove nothing, especially since we haven’t yet even defined “area of $R(f, a, b)$ ”.
- Before we can *define* “area of $R(f, a, b)$ ”, we need to prove that $L(f, P_1) \leq U(f, P_2)$ for any partitions $P_1, P_2 \dots$

Rigorous development of the integral

Lemma (Partition Lemma)

If *partition* $P \subseteq$ *partition* Q (i.e., if every point of P is also in Q), then $L(f, P) \leq L(f, Q)$ and $U(f, P) \geq U(f, Q)$.



Rigorous development of the integral

Proof of Partition Lemma

As a first step, consider the special case in which the finer partition Q contains only one more point than P :

$$P = \{t_0, \dots, t_n\},$$

$$Q = \{t_0, \dots, t_{k-1}, u, t_k, \dots, t_n\},$$

where

$$a = t_0 < t_1 < \dots < t_{k-1} < u < t_k < \dots < t_n = b.$$

Because $[t_{k-1}, t_k]$ is split by u , we have two lower bounds:

$$m' = \inf \{ f(x) : x \in [t_{k-1}, u] \},$$

$$m'' = \inf \{ f(x) : x \in [u, t_k] \}.$$

... continued ...

Rigorous development of the integral

Proof of Partition Lemma (cont.)

Then
$$L(f, P) = \sum_{i=1}^n m_i(t_i - t_{i-1}),$$

and
$$L(f, Q) = \sum_{i=1}^{k-1} m_i(t_i - t_{i-1}) + m'(u - t_{k-1}) \\ + m''(t_k - u) + \sum_{i=k+1}^n m_i(t_i - t_{i-1}).$$

\therefore To prove $L(f, P) \leq L(f, Q)$, it is enough to show

$$m_k(t_k - t_{k-1}) \leq m'(u - t_{k-1}) + m''(t_k - u).$$

... continued ...

Rigorous development of the integral

Proof of Partition Lemma (cont.)

Now note that since

$$\{f(x) : x \in [t_{k-1}, u]\} \subseteq \{f(x) : x \in [t_{k-1}, t_k]\},$$

the RHS might contain some additional smaller numbers, so we must have

$$\begin{aligned} m_k &= \inf \{f(x) : x \in [t_{k-1}, t_k]\} \\ &\leq \inf \{f(x) : x \in [t_{k-1}, u]\} = m'. \end{aligned}$$

Thus, $m_k \leq m'$, and, similarly, $m_k \leq m''$.

$$\begin{aligned} \therefore m_k(t_k - t_{k-1}) &= m_k(t_k - u + u - t_{k-1}) \\ &= m_k(u - t_{k-1}) + m_k(t_k - u) \\ &\leq m'(u - t_{k-1}) + m''(t_k - u), \end{aligned}$$

... continued ...

Rigorous development of the integral

Proof of Partition Lemma (cont.)

which proves (in this special case where Q contains only one more point than P) that $L(f, P) \leq L(f, Q)$.

We can now prove the general case by adding one point at a time.

If Q contains ℓ more points than P , define a sequence of partitions

$$P = P_0 \subset P_1 \subset \cdots \subset P_\ell = Q$$

such that P_{j+1} contains exactly one more point than P_j . Then

$$L(f, P) = L(f, P_0) \leq L(f, P_1) \leq \cdots \leq L(f, P_\ell) = L(f, Q),$$

so $L(f, P) \leq L(f, Q)$.

(Proving $U(f, P) \geq U(f, Q)$ is similar: check!)



Rigorous development of the integral

Theorem (Partition Theorem)

Let P_1 and P_2 be any two partitions of $[a, b]$. If f is bounded on $[a, b]$ then

$$L(f, P_1) \leq U(f, P_2).$$

Proof.

This is a straightforward consequence of the [partition lemma](#).

Let $P = P_1 \cup P_2$, i.e., P is the partition obtained by combining all the points of P_1 and P_2 .

Then

$$L(f, P_1) \leq L(f, P) \leq U(f, P) \leq U(f, P_2).$$



Rigorous development of the integral

Important inferences that follow from the **partition theorem**:

- For any partition P' , the upper sum $U(f, P')$ is an upper bound for the set of all lower sums $L(f, P)$.

$$\therefore \sup \{L(f, P) : P \text{ a partition of } [a, b]\} \leq U(f, P') \quad \forall P'$$

$$\therefore \sup \{L(f, P)\} \leq \inf \{U(f, P)\}$$

\therefore For any partition P' ,

$$L(f, P') \leq \sup \{L(f, P)\} \leq \inf \{U(f, P)\} \leq U(f, P')$$

- If $\sup \{L(f, P)\} = \inf \{U(f, P)\}$ then we can define “**area of $R(f, a, b)$** ” to be this number.
 - Is it possible that $\sup \{L(f, P)\} < \inf \{U(f, P)\}$?

Poll

- Go to https://www.childsmath.ca/childsa/forms/main_login.php
- Click on **Math 3A03**
- Click on **Take Class Poll**
- Fill in poll **Integrals: $\sup \{L(f, P)\} < \inf \{U(f, P)\} ?$**
- .