Differentiation

McMaster University

Mathematics and Statistics

$$\int_M d\omega = \int_{\partial M} \omega$$

Mathematics 3A03
Real Analysis I

Instructor: David Earn

Lecture 2
Differentiation
Wednesday 8 January 2025

Survey to do right now

- Please go to
  https://www.childsmath.ca/childsa/forms/main_login.php
- Click on Math 3A03
- Click on Take Class Poll
- Fill in poll **Survey 2**
- `Submit`.

Announcements

- Results of Survey 1
- Results of Survey 2

Background / reminder

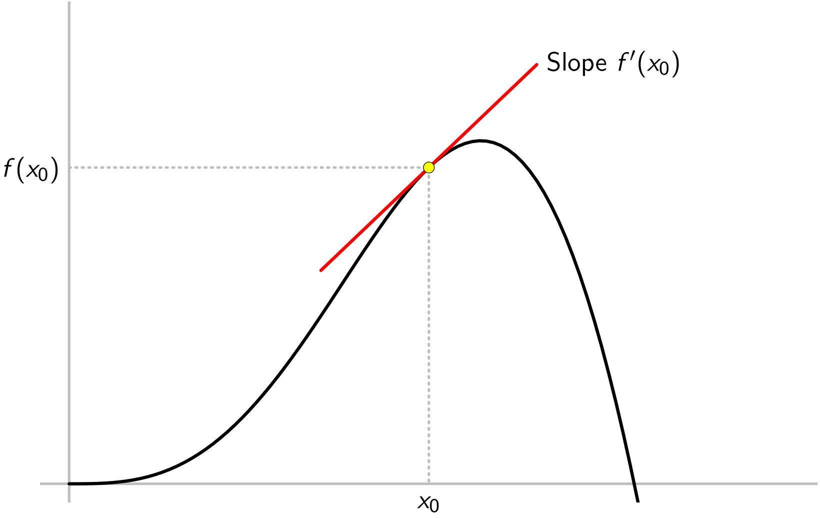
Definition (Cauchy sequence)

A sequence $\{s_n\}$ is said to be a ***Cauchy sequence*** iff for all $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that if $m \geq N$ and $n \geq N$ then $|s_n - s_m| < \varepsilon$.

Poll: another background check

- Go to
  https://www.childsmath.ca/childsa/forms/main_login.php
- Click on Math 3A03
- Click on Take Class Poll
- Fill in poll **Background: Cauchy sequences**
- `Submit`.

The Derivative



The Derivative

Definition (Derivative)

Let f be defined on an interval I and let $x_0 \in I$. The ***derivative*** of f at x_0, denoted by $f'(x_0)$, is defined as

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0},$$

provided either that this limit exists or is infinite. If $f'(x_0)$ is finite we say that f is ***differentiable*** at x_0. If f is differentiable at every point of a set $E \subseteq I$, we say that f is differentiable on E. If E is all of I, we simply say that f is a ***differentiable function***.

Note: "Differentiable" and "a derivative exists" always mean that the derivative is finite.

The Derivative

Example

$f(x) = x^2$. Find $f'(2)$.

$$f'(2) = \lim_{x \to 2} \frac{x^2 - 4}{x - 2} = \lim_{x \to 2} \frac{(x+2)(x-2)}{x - 2} = \lim_{x \to 2} x + 2 = 4$$

Note:

- In the first two limits, we must have $x \neq 2$.
- But in the third limit, we just plug in $x = 2$.
- Two things are equal, but in one $x \neq 2$ and in the other $x = 2$.
- Good illustration of why it is important to define the meaning of limits rigorously.

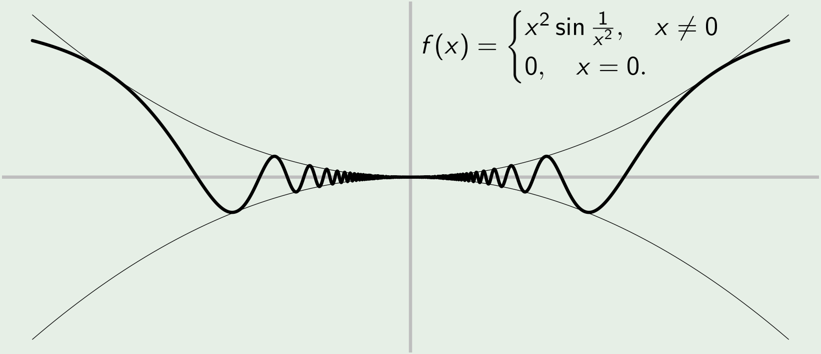
Poll

- Go to https://www.childsmath.ca/childsa/forms/main_login.php
- Click on Math 3A03
- Click on Take Class Poll
- Fill in poll **Derivatives: Differentiable at 0**
- `Submit`.

The Derivative

Example

Let f be defined in a neighbourhood I of 0, and suppose $|f(x)| \leq x^2$ for all $x \in I$. Is f necessarily differentiable at 0? *e.g.*,



$$f(x) = \begin{cases} x^2 \sin \frac{1}{x^2}, & x \neq 0 \\ 0, & x = 0. \end{cases}$$

The Derivative

Example (Trapping principle)

Suppose $f(x) = \begin{cases} x^2 \sin \frac{1}{x^2}, & x \neq 0 \\ 0, & x = 0. \end{cases}$ Then:

$$\forall x \neq 0 : \quad \left|\frac{f(x) - f(0)}{x - 0}\right| = \left|\frac{f(x)}{x}\right| = \left|\frac{x^2 \sin \frac{1}{x^2}}{x}\right| = \left|x \sin \frac{1}{x^2}\right| \leq |x|$$

Therefore:

$$|f'(0)| = \left|\lim_{x\to 0} \frac{f(x) - f(0)}{x - 0}\right| = \lim_{x\to 0} \left|\frac{f(x) - f(0)}{x - 0}\right| \leq \lim_{x\to 0} |x| = 0.$$

$\therefore$ f is differentiable at 0 and $f'(0) = 0$. □

The Derivative

Definition (One-sided derivatives)

Let f be defined on an interval I and let $x_0 \in I$. The ***right-hand derivative*** of f at x_0, denoted by $f'_+(x_0)$, is the limit

$$f'_+(x_0) = \lim_{x \to x_0^+} \frac{f(x) - f(x_0)}{x - x_0},$$

provided either that this one-sided limit exists or is infinite. Similarly, the ***left-hand derivative*** of f at x_0, denoted by $f'_-(x_0)$, is the limit

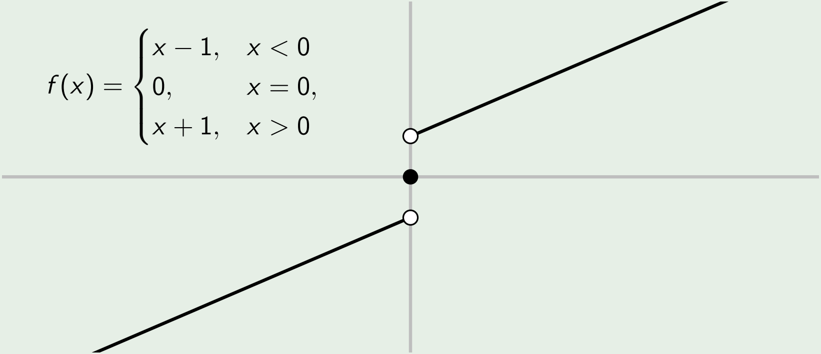
$$f'_-(x_0) = \lim_{x \to x_0^-} \frac{f(x) - f(x_0)}{x - x_0}.$$

Note: If x_0 is not an endpoint of the interval I then f is differentiable at x_0 iff $f'_+(x_0) = f'_-(x_0) \neq \pm\infty$.

The Derivative

Example

$$f(x) = \begin{cases} x-1, & x<0 \\ 0, & x=0, \\ x+1, & x>0 \end{cases}$$



- Same slope from left and right. Why isn't f differentiable???
- $\lim_{x\to 0^-} f'(x) = \lim_{x\to 0^+} f'(x) = \lim_{x\to 0} f'(x) = 1.$
- $f'_-(0) = f'_+(0) = f'(0) = \lim_{x\to 0} \frac{f(x)-f(0)}{x-0} = \infty.$

The Derivative

- Higher derivatives: we write
  - $f'' = (f')'$ if f' is differentiable;
  - $f^{(n+1)} = (f^{(n)})'$ if $f^{(n)}$ is differentiable.
- Other standard notation for derivatives:

$$\frac{df}{dx} = f'(x)$$

$$D = \frac{d}{dx}$$

$$D^n f(x) = \frac{d^n f}{dx^n} = f^{(n)}(x)$$

REMINDER: Algebra of limits

Theorem (Algebraic operations on limits of sequences)

Suppose $\{s_n\}$ *and* $\{t_n\}$ *are convergent sequences and* $C \in \mathbb{R}$.

1. $\lim\limits_{n\to\infty} C\, s_n = C\big(\lim\limits_{n\to\infty} s_n\big)$;
2. $\lim\limits_{n\to\infty} (s_n + t_n) = \big(\lim\limits_{n\to\infty} s_n\big) + \big(\lim\limits_{n\to\infty} t_n\big)$;
3. $\lim\limits_{n\to\infty} (s_n - t_n) = \big(\lim\limits_{n\to\infty} s_n\big) - \big(\lim\limits_{n\to\infty} t_n\big)$;
4. $\lim\limits_{n\to\infty} (s_n t_n) = \big(\lim\limits_{n\to\infty} s_n\big)\big(\lim\limits_{n\to\infty} t_n\big)$;
5. *if* $t_n \neq 0$ *for all* n *and* $\lim_{n\to\infty} t_n \neq 0$ *then*

$$\lim_{n\to\infty} \left(\frac{s_n}{t_n}\right) = \frac{\lim_{n\to\infty} s_n}{\lim_{n\to\infty} t_n} .$$

(TBB §2.7, and problem 2.7.4)

REMINDER: Algebra of limits

Theorem (Algebraic operations on limits of functions)

Suppose $f, g : \mathbb{R} \to \mathbb{R}$, $x_0 \in \mathbb{R}$, *the limits as* $x \to x_0$ *of* $f(x)$ *and* $g(x)$ *both exist, and* $C \in \mathbb{R}$.

1. $\lim_{x\to x_0} C\,f(x) = C\big(\lim_{x\to x_0} f(x)\big)$;
2. $\lim_{x\to x_0} (f(x) + g(x)) = \big(\lim_{x\to x_0} f(x)\big) + \big(\lim_{x\to x_0} g(x)\big)$;
3. $\lim_{x\to x_0} (f(x) - g(x)) = \big(\lim_{x\to x_0} f(x)\big) - \big(\lim_{x\to x_0} g(x)\big)$;
4. $\lim_{x\to x_0} (f(x)g(x)) = \big(\lim_{x\to x_0} f(x)\big)\big(\lim_{x\to x_0} g(x)\big)$;
5. *if* $g(x) \neq 0$ *for* $x \in (x_0 - \delta, x_0 + \delta)$ *for some* $\delta > 0$, *and* $\lim_{x\to x_0} g(x) \neq 0$ *then* $\lim_{x\to x_0} \left(\frac{f(x)}{g(x)}\right) = \frac{\lim_{x\to x_0} f(x)}{\lim_{x\to x_0} g(x)}$.

The Derivative

Theorem (Differentiable $\implies$ continuous)

If f is defined in a neighbourhood I of x_0 and f is differentiable at x_0 then f is continuous at x_0.

Proof.

Must show $\lim_{x \to x_0} f(x) = f(x_0)$, *i.e.*, $\lim_{x \to x_0} (f(x) - f(x_0)) = 0$.

$$\begin{aligned}
\lim_{x \to x_0} (f(x) - f(x_0)) &= \lim_{x \to x_0} \left(\frac{f(x) - f(x_0)}{x - x_0} \times (x - x_0) \right) \\
&= \lim_{x \to x_0} \left(\frac{f(x) - f(x_0)}{x - x_0} \right) \times \lim_{x \to x_0} (x - x_0) \\
&= f'(x_0) \times 0 = 0 \,,
\end{aligned}$$

where we have used the theorem on the algebra of limits. □

McMaster University

Mathematics and Statistics

$$\int_M d\omega = \int_{\partial M} \omega$$

Mathematics 3A03
Real Analysis I

Instructor: David Earn

Lecture 3
Differentiation II
Friday 10 January 2025

Announcements

- Lectures are being live streamed and recorded as of today.
- Recordings will be available 24 hours after lectures.
  - Go to `https://echo360.ca`.
  - Sign in with your `macID@mcmaster.ca` e-mail address.
  - Click on the `Courses` tab.
  - Select "MATH 3A03 - WINTER 2025"
- Course evaluation scheme has changed (next two slides).
  - The course web site and online syllabus have not yet been updated to reflect these changes, but that will happen soon (hopefully over the weekend).

Course evaluation will be revised as follows:

- We will not have quizzes
- 5% for participating in at least 80% of in-class polls
- 15% for participating in assignments, based on multiple choice (MC) questions:

$$\text{assignment mark} = \frac{\text{number MC questions answered}}{\text{total number MC questions assigned}}$$

- 30% for midterm test on Thurs 27 Feb 2025
- 50% for final exam in April
- *Note:* If your final exam mark is better than your midterm test mark then the final exam mark will replace the midterm test mark.
- *Important:* Do NOT skip the midterm. Even if you don't feel well prepared, write it for practice so you are better prepared for writing the final exam.
- If you must miss the midterm (*e.g.*, illness or accepting a Nobel prize), your final exam mark will replace it.

Tentative plan for assignments

- There will be regular assignments.
- Each question will have a multiple choice component (probably on childsmath). Only participation counts for marks; you will get the same credit for correct and incorrect answers, or for selecting "I haven't had time to think about this yet".
- Optionally, full solutions/proofs can be written up and submitted on crowdmark. Feedback will be given, but no marks. The purpose is to help you prepare better for the test and exam.
- If you're not sure if your proof is complete, or you got stuck and don't know how to complete it, make that clear in the document that you submit on crowdmark, so the TA can focus on the help you need.
- Always try your best to solve problems on your own first. But if you used stackexchange or ChatGPT or whatever for help, provide a URL to your source if possible, so it is easier for the TA to provide the help you need.
- Make the best possible use of the TA's time: say what you think you do or don't understand.

Last time. . .

- Definition of the derivative.
  - Example: Trapping Principle
- Defined one-sided derivatives
  - Example
- Proved differentiable $\Longrightarrow$ continuous.

More on the derivative

Theorem (Algebra of derivatives)

Suppose f and g are defined on an interval I and $x_0 \in I$. If f and g are differentiable at x_0 then $f+g$ and fg are differentiable at x_0. If, in addition, $g(x_0) \neq 0$ then f/g is differentiable at x_0. Under these conditions:

1. $(cf)'(x_0) = cf'(x_0)$ *for all* $c \in \mathbb{R}$;
2. $(f+g)'(x_0) = (f'+g')(x_0)$;
3. $(fg)'(x_0) = (f'g + fg')(x_0)$;
4. $\left(\frac{f}{g}\right)'(x_0) = \left(\frac{gf' - fg'}{g^2}\right)(x_0) \qquad (g(x_0) \neq 0)$.

(TBB Theorem 7.7, p. 408)

The Derivative

Theorem (Chain rule)

Suppose f is defined in a neighbourhood U of x_0 and g is defined in a neighbourhood V of $f(x_0)$ such that $f(U) \subseteq V$. If f is differentiable at x_0 and g is differentiable at $f(x_0)$ then the composite function $h = g \circ f$ is differentiable at x_0 and

$$h'(x_0) = (g \circ f)'(x_0) = g'(f(x_0))f'(x_0)\,.$$

Informally, if $y = f(x)$ and $z = g(y)$ then $\frac{\mathrm{d}z}{\mathrm{d}x} = \frac{\mathrm{d}z}{\mathrm{d}y}\frac{\mathrm{d}y}{\mathrm{d}x}$.

(TBB §7.3.2, p. 411)

Why the chain rule is plausible

The derivative of $g \circ f$ at x_0 is the limit as $x \to x_0$ of the difference quotient

$$\frac{g(f(x)) - g(f(x_0))}{x - x_0} = \frac{g(f(x)) - g(f(x_0))}{f(x) - f(x_0)} \cdot \frac{f(x) - f(x_0)}{x - x_0} \qquad (\spadesuit)$$

Recall: $f'(x_0)$ exists $\implies$ f continuous at x_0
$\implies$ $f(x) \to f(x_0)$ as $x \to x_0$.

Can we take the limit as $x \to x_0$ and conclude that $(g \circ f)'(x_0) = g'(f(x_0)) \cdot f'(x_0)$?

- What if $f(x) = 0$ for all x?
- What if f is a constant function?
- What if $f(x) = f(x_0)$ for some $x \neq x_0$?
- Can we use $(\spadesuit)$ to prove the chain rule?

Poll

- Go to
  https://www.childsmath.ca/childsa/forms/main_login.php
- Click on Math 3A03
- Click on Take Class Poll
- Fill in poll **Derivatives: Chain Rule**
- Submit.

REMINDER: limits of functions

Theorem (Equivalence of ε-δ and sequence definitions of limits)

Let $a < x_0 < b$, $I = (a, b)$, and $f : I \setminus \{x_0\} \to \mathbb{R}$. Then the following two definitions of

$$\lim_{x \to x_0} f(x) = L$$

are equivalent:

1. *for all $\varepsilon > 0$ there exists $\delta > 0$ such that if $0 < |x - x_0| < \delta$ then $|f(x) - L| < \varepsilon$.*

2. *for every sequence $\{x_n\}$ of points in $I \setminus \{x_0\}$,*

$$\lim_{n \to \infty} x_n = x_0 \quad \Longrightarrow \quad \lim_{n \to \infty} f(x_n) = L\,.$$

Note: The deleted neighbourhood ($I \setminus \{x_0\}$) can be replaced by any set on which f is defined and x_0 is an accumulation point.

Proof of the chain rule.

1. Suppose there is an open interval I, with $x_0 \in I$, and $f(x) \neq f(x_0)$ for all $x \in I \setminus \{x_0\}$. Then we can take the limit $x \to x_0$ in (♠) and we get the chain rule.

2. Next suppose that no open interval like the one hypothesized above exists. Then, in any open interval containing x_0, there must be at least one point $x \neq x_0$ for which $f(x) = f(x_0)$. Therefore, we can construct a sequence of open intervals I_n, with lengths decreasing to 0, such that each I_n contains x_0 and a point $x_n \neq x_0$ with $f(x_n) = f(x_0)$. Therefore, since $f'(x_0)$ exists, and we recall the previous slide, we can compute $f'(x_0)$ via

$$f'(x_0) = \lim_{n\to\infty} \frac{f(x_n) - f(x_0)}{x_n - x_0} = \lim_{n\to\infty} \frac{0}{x_n - x_0} = 0.$$

We can also show that $(g \circ f)'(x_0) = 0$, using the sequence definition on the previous slide. *Try to fill in this last detail*, or look it up (TBB §7.3.2, p. 411).

Note: TBB's proof leaves out the proof that $f'(x_0) = 0$ in case 2 above. □

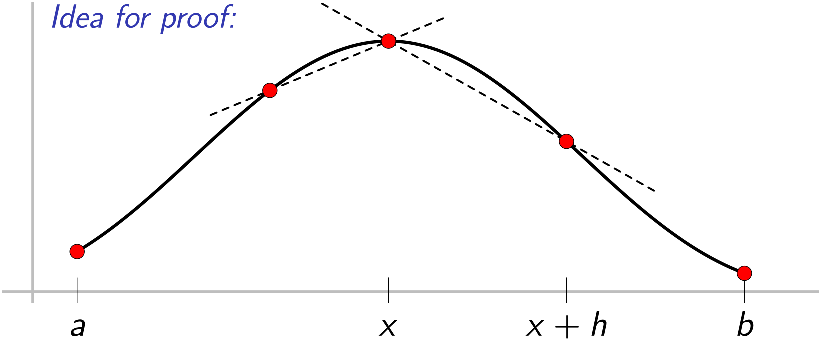
More on the derivative

Theorem (Derivative at local extrema)

Let $f : (a, b) \to \mathbb{R}$*. If* x *is a maximum or minimum point of* f *in* (a, b)*, and* f *is differentiable at* x*, then* $f'(x) = 0$.

Note: f need not be differentiable or even continuous at other points.

Idea for proof:



More on the derivative

Proof that the derivative vanishes at local extrema.

If f has a local maximum at $x \in (a, b)$, then for sufficiently small $h > 0$ we must have

$$\frac{f(x+h) - f(x)}{h} \leq 0 \leq \frac{f(x) - f(x-h)}{h}$$

Since f is differentiable at x, it is left and right differentiable at x, so we can evaluate the limits as $h \to 0$ to obtain

$$f'_{+}(x) \leq 0 \leq f'_{-}(x) \,.$$

But since f is differentiable at x, the left and right derivatives must be equal, hence $f'(x) = 0$. □

Mathematics
and Statistics

$$\int_M d\omega = \int_{\partial M} \omega$$

Mathematics 3A03
Real Analysis I

Instructor: David Earn

Lecture 4
Differentiation III
Monday 13 January 2025

Poll

- Go to
  https://www.childsmath.ca/childsa/forms/main_login.php
- Click on Math 3A03
- Click on Take Class Poll
- Fill in poll **Office hours**
- Submit.

Announcements

- In-Class polls: if you do not have a device that enables you to participate in polls in class, or if any other issue prevents you from participating in polls, please let me know by e-mail.
- This Friday (17 Jan 2025), I will be out of town and the class will be a Q&A session with the TA.

Announcements

- The online syllabus has been revised to account for the changes in how the course will be evaluated (see slides from Lecture 3). In particular, the statement about AI use has been changed to:

  Generative AI: Unrestricted Use
  Students may use generative AI throughout this course in whatever way enhances their learning; no special documentation or citation is required. Note that access to generative AI will **not** be available during tests or exams.

- The course web site has been updated to reflect the changes in how the course will be evaluated.

Assignment 1

- Assignment 1 has been posted on the course web site.

Last time. . .

- Discussed algebra of derivatives and chain rule.
- Proved the chain rule.
- Proved that that derivative is zero at extrema.

The Mean Value Theorem

Theorem (Rolle's theorem)

If f is continuous on $[a, b]$ and differentiable on (a, b), and $f(a) = f(b)$, then there exists $x \in (a, b)$ such that $f'(x) = 0$.

Proof.

f continuous on $[a, b] \implies f$ has a max and min value on $[a, b]$. If either a max or min occurs at $x \in (a, b)$ then $f'(x) = 0$. If no max or min occurs in (a, b) then they must both occur at the endpoints, a and b. But $f(a) = f(b)$, so f is constant. Hence $f'(x) = 0 \; \forall x \in (a, b)$. □

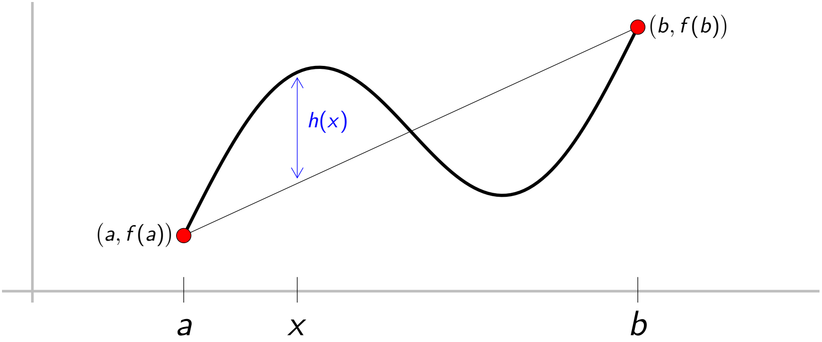
Theorem (Mean value theorem)

If f is continuous on $[a, b]$ and differentiable on (a, b) then there exists $x \in (a, b)$ such that

$$f'(x) = \frac{f(b) - f(a)}{b - a}\,.$$

The Mean Value Theorem

Idea for proof:



Proof.

Apply Rolle's theorem to

$$h(x) = f(x) - \left[f(a) + \left(\frac{f(b) - f(a)}{b - a}\right)(x - a)\right].$$

□

The Mean Value Theorem

Example

$f'(x) > 0$ on an interval $I \implies f$ strictly increasing on I.

Proof:

Suppose $x_1, x_2 \in I$ and $x_1 < x_2$. We must show $f(x_1) < f(x_2)$.

Since $f'(x)$ exists for all $x \in I$, f is certainly differentiable on the closed subinterval $[x_1, x_2]$.

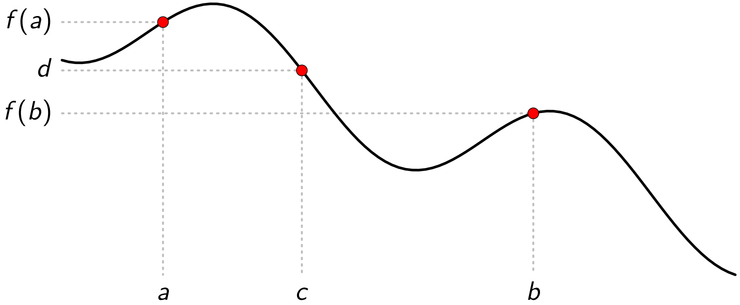
Hence by the Mean Value Theorem $\exists x_* \in (x_1, x_2)$ such that

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(x_*)\,.$$

But $x_2 - x_1 > 0$ and since $x_* \in I$, we know $f'(x_*) > 0$.

$\therefore\ f(x_2) - f(x_1) > 0, \qquad$ *i.e.,* $\quad f(x_1) < f(x_2)$. □

REMINDER: Intermediate Value Property



Definition (Intermediate Value Property (IVP))

A function f defined on an interval I is said to have the ***intermediate value property (IVP)*** on I iff for each $a, b \in I$ with $f(a) \neq f(b)$, and for each d between $f(a)$ and $f(b)$, there exists c between a and b for which $f(c) = d$.

REMINDER: Intermediate Value Property

Theorem (Intermediate Value Theorem (IVT))

If f is continuous on an interval I then f has the intermediate value property (IVP) on I.

Note: The interval I in the statement of the IVT does not have to be closed and it does not have to be bounded.

Unlike the extreme value theorem, the IVT is not a theorem about functions defined on closed and bounded intervals.

Poll

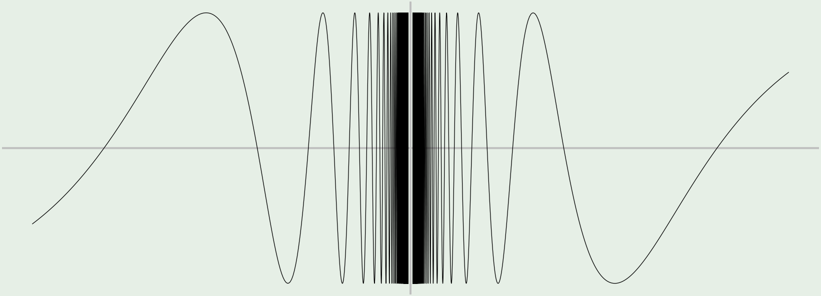
- Go to
  https://www.childsmath.ca/childsa/forms/main_login.php
- Click on Math 3A03
- Click on Take Class Poll
- Fill in poll **Derivatives: IVP and derivatives**
- Submit.

Intermediate Value Property

Question: If a function has the IVP on an interval I, must it be continuous on I?

Example

$$f(x) = \begin{cases} \sin\frac{1}{x} & x \neq 0, \\ 0 & x = 0. \end{cases}$$



Intermediate value property for derivatives

Theorem (Darboux's Theorem: IVP for derivatives)

If f is differentiable on an interval I then its derivative f' has the intermediate value property on I.

Notes:

- It is f', not f, that is claimed to have the intermediate value property in Darboux's theorem. This theorem does not follow from the standard intermediate value theorem because the derivative f' is not necessarily continuous.
- *Equivalent (contrapositive) statement of Darboux's theorem:* If a function does not have the intermediate value property on I then it is impossible that it is the derivative of any function on I.
- Darboux's theorem implies that a derivative cannot have jump or removable discontinities. Any discontinuity of a derivative must be essential. Recall example of a discontinuous function with IVP.

McMaster University

Mathematics and Statistics

$$\int_M d\omega = \int_{\partial M} \omega$$

Mathematics 3A03
Real Analysis I

Instructor: David Earn

Lecture 5
Differentiation IV
Wednesday 15 January 2025

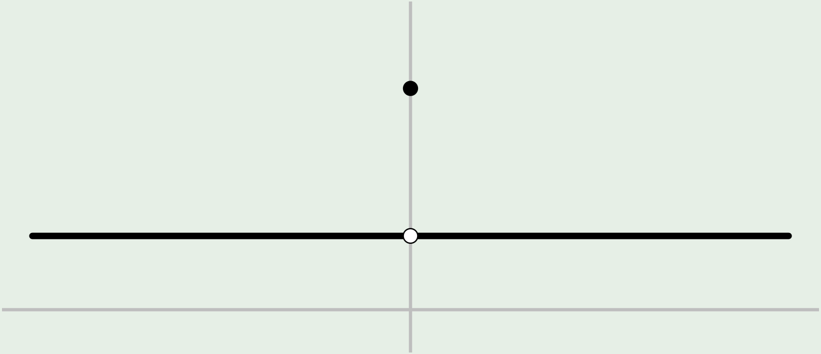
Announcements

- Kieran (the TA) has created a poll for his office hour time: https://forms.gle/WKZRvbVT4Q4wZrfaA. Please do this poll at your convenience. (It is a Google form, not a childsmath poll.)
- I will have an office hour today over zoom at 2:00pm. I will e-mail a link for the zoom room at 2:00pm.
- I will try to finalize a weekly office hour time next week.

Intermediate value property for derivatives

Example

What are the different types of discontinuities?



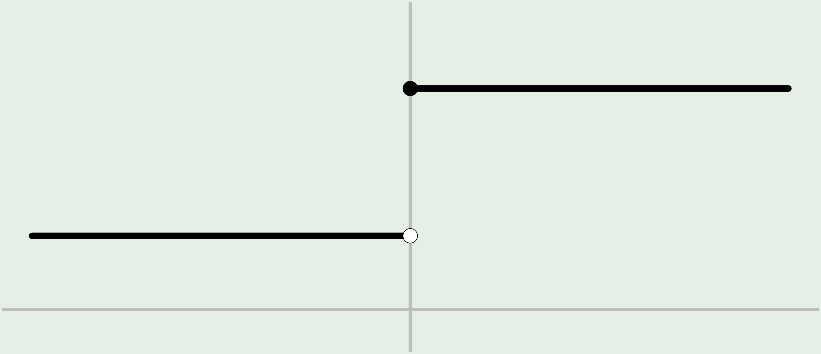
This is a ***removable discontinuity***. $\lim_{x\to 0} f(x)$ exists.

The discontinuity can be removed by redefining $f(0)$.

Intermediate value property for derivatives

Example

What are the different types of discontinuities?

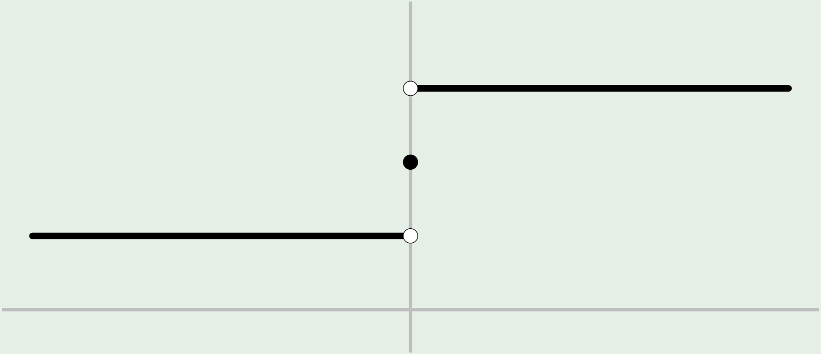


This is a ***jump discontinuity***. $\lim_{x\to 0^-} f(x) \neq f(0) = \lim_{x\to 0^+} f(x)$.

Intermediate value property for derivatives

Example

What are the different types of discontinuities?



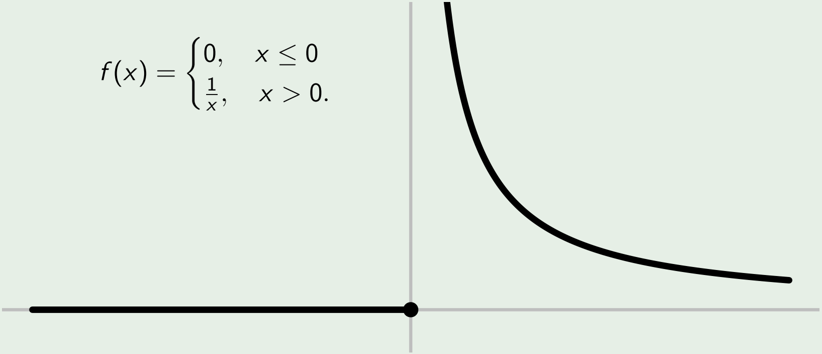
This is a ***jump discontinuity***. $\lim_{x\to 0^-} f(x) \neq f(0) \neq \lim_{x\to 0^+} f(x)$.

Intermediate value property for derivatives

Example

What are the different types of discontinuities?

$$f(x) = \begin{cases} 0, & x \leq 0 \\ \frac{1}{x}, & x > 0. \end{cases}$$



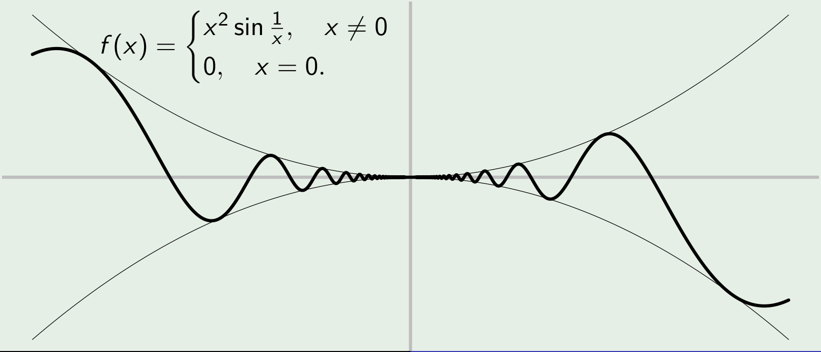
This is an ***essential discontinuity***. $\lim_{x \to 0^+} f(x)$ does not exist.

This function does not satisfy the intermediate value property, so it cannot be the derivative of any function.

Intermediate value property for derivatives

Example

What is an example of a function that is a derivative but is not continuous?
Consider the following differentiable function $f(x)$.
Is its derivative $f'(x)$ continuous?

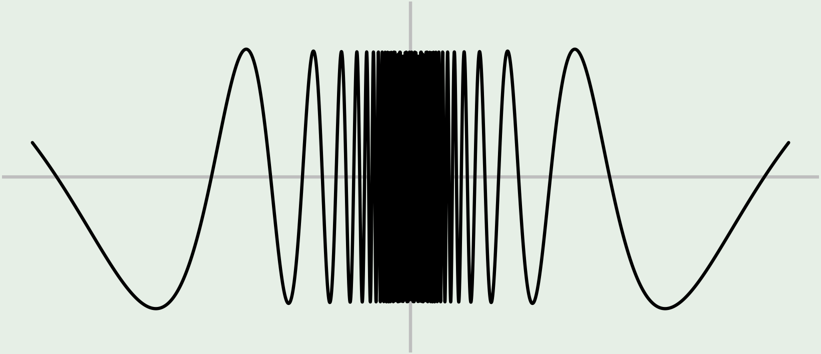


Intermediate value property for derivatives

Example

What is an example of a function that is a derivative but is not continuous? The derivative of $f(x)$ is

$$f'(x) = \begin{cases} 2x \sin \frac{1}{x} - \cos \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

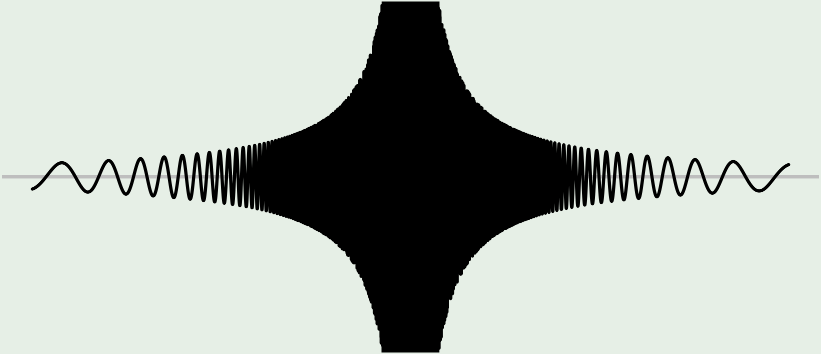


Intermediate value property for derivatives

Example

What about our original example $x^2 \sin \frac{1}{x^2}$? Its derivative is

$$f'(x) = \begin{cases} 2x \sin \frac{1}{x^2} - \frac{2}{x} \cos \frac{1}{x^2} & x \neq 0 \\ 0 & x = 0 \end{cases}$$



$f'(x)$ satisfies the IVP, even though it blows up near 0.

Intermediate value property for derivatives

Proof of Darboux's Theorem.

Consider $a, b \in I$ with $a < b$.

Suppose first that $f'(a) < 0 < f'(b)$. We will show $\exists x \in (a, b)$ such that $f'(x) = 0$. Since f' exists on $[a, b]$, we must have f continuous on $[a, b]$, so the Extreme Value Theorem implies that f attains its minimum at some point $x \in [a, b]$. This minimum point cannot be an endpoint of $[a, b]$ ($x \neq a$ because $f'(a) < 0$ and $x \neq b$ because $f'(b) > 0$). Therefore, $x \in (a, b)$. But f is differentiable everywhere in (a, b), so, by the theorem on the derivative at local extrema, we must have $f'(x) = 0$.

Now suppose more generally that $f'(a) < K < f'(b)$. Let $g(x) = f(x) - Kx$. Then g is differentiable on I and $g'(x) = f'(x) - K$ for all $x \in I$. In addition, $g'(a) = f'(a) - K < 0$ and $g'(b) = f'(b) - K > 0$, so by the argument above, $\exists x \in (a, b)$ such that $g'(x) = 0$, *i.e.*, $f'(x) - K = 0$, *i.e.*, $f'(x) = K$.

The case $f'(a) > K > f'(b)$ is similar. $\square$

Intermediate value property for derivatives

Example ($f'(x) \neq 0 \ \forall x \in I \implies f \nearrow$ or $\searrow$ on I)

If f is differentiable on an interval I and $f'(x) \neq 0$ for all $x \in I$ then f is either increasing or decreasing on the entire interval I.

Proof:

Suppose $\exists a, b \in I$ such that $f'(a) < 0$ and $f'(b) > 0$.

Then, from Darboux's theorem, $\exists c \in I$ such that $f'(c) = 0$. $\Rightarrow\Leftarrow$

$\therefore$ Either "$\exists a \in I \ni f'(a) < 0$" is $\texttt{FALSE}$
or "$\exists b \in I \ni f'(b) > 0$" is $\texttt{FALSE}$.

$\therefore$ Since we know $f'(x) \neq 0 \ \forall x \in I$, it must be that
either $f'(x) > 0 \ \forall x \in I$ or $f'(x) < 0 \ \forall x \in I$,
i.e., either f is increasing on I or decreasing on I. □

Assignment 1
Participation deadline: Monday 20 Jan 2025 @ 11:25am

- Go to
  https://www.childsmath.ca/childsa/forms/main_login.php
- Click on Math 3A03
- Click on Take Class Poll
- Fill in poll **Assignment 1: The Derivative**
- `Submit`.

McMaster University

Mathematics and Statistics

$$\int_M d\omega = \int_{\partial M} \omega$$

Mathematics 3A03
Real Analysis I

Instructor: David Earn

Lecture 6
Differentiation V
Monday 20 January 2025

Announcements

- Until further notice, my office hours will be Wednesdays 1:30-2:30pm in Hamilton Hall 317.
- Assignment 1 results
- *Point of clarification:* On the test and exam, you will be assumed to be familiar with whatever has been discussed in class, and can make use of results proved in class unless the question is to prove the result proved in class. You cannot just quote theorems from the textbooks (or elsewhere) that we didn't discuss in detail in class.

The Derivative of an Inverse

Example (Sufficient condition for *non*-differentiable inverse)

Suppose f is continuous and one-to-one on an interval I. If $x \in I$, f is differentiable at x, and $f'(x) = 0$ then f^{-1} is not differentiable at $y = f(x)$.

Proof: By definition, the inverse function satisfies

$$f(f^{-1}(y)) = y\,.$$

Suppose that f^{-1} is differentiable at y. Then, by the Chain Rule,

$$f'(f^{-1}(y)) \cdot (f^{-1})'(y) = 1\,.$$

But $f^{-1}(y) = x$, and $f'(x) = 0$, so

$$0 \cdot (f^{-1})'(y) = 1\,,$$

which is impossible! $\Rightarrow\Leftarrow$. Therefore, $(f^{-1})'(f(x))$ does not exist. □

The Derivative of an Inverse

Theorem (Inverse function theorem)

If f is differentiable on an interval I and $f'(x) \neq 0\ \forall x \in I$, then

1. *f is one-to-one on I (so f^{-1} exists on $J = f(I)$);*
2. *f^{-1} is differentiable on $J = f(I)$;*
3. $(f^{-1})'(f(x)) = \dfrac{1}{f'(x)}$ *for all $x \in I$,*

   i.e., $(f^{-1})'(y) = \dfrac{1}{f'((f^{-1}(y))}$ *for all $y \in J$.*

(TBB Theorem 7.32, p. 445)

The Derivative of an Inverse

Proof of the Inverse Function Theorem.

1. f is one-to-one on I (so f^{-1} exists on $J = f(I)$);

By hypothesis, f is differentiable on I and $f'(x) \neq 0$ on I. We proved in class previously (and Assignment 1) that if $f'(x) \neq 0$ $\forall x \in I$ then

$$\text{Darboux's theorem} \implies f \nearrow \text{ or } f \searrow \text{ on } I.$$

It follows that f is $1:1$ on I. *Why?*

Proof that $f \nearrow$ or $f \searrow$ on $I \implies f$ $1:1$ on I

$1:1$ on I means $(\forall x_1, x_2 \in I)$ $f(x_1) = f(x_2) \implies x_1 = x_2$. Equivalently, $(\forall x_1, x_2 \in I)$ $x_1 \neq x_2 \implies f(x_1) \neq f(x_2)$. But $x_1 \neq x_2 \implies$ either $x_1 < x_2$ or $x_1 > x_2$. In either case, since $f \nearrow$ or $\searrow$ on I, either $f(x_1) < f(x_2)$ or $f(x_1) > f(x_2)$, *i.e.*, in either case, $f(x_1) \neq f(x_2)$. So f is $1:1$ on I.

The Derivative of an Inverse

Proof of the Inverse Function Theorem.

2. f^{-1} is differentiable on $J = f(I)$;

First, recall standard results related to *continuity* of inverse functions:

> Preservation of Intervals Theorem (BS Theorem 5.3.10, p. 140)
> Let I be an interval and let $f : I \to \mathbb{R}$ be continuous on I. Then the set $J = f(I)$ is an interval.

Preservation of intervals is used to prove:

> Continuous Inverse Theorem (BS Theorem 5.6.5, p. 156)
> Let $I \subset \mathbb{R}$ be an interval and let $f : I \to \mathbb{R}$ be strictly monotone and continuous on I. Then the function g inverse to f is strictly monotone and continuous on $J = f(I)$.

Thus, f *continuous* on I implies f^{-1} *continuous* on $J = f(I)$.

The Derivative of an Inverse

Proof of the Inverse Function Theorem.

2 f^{-1} is differentiable on $J = f(I)$;

We must show that $\lim\limits_{y \to y_0} \frac{f^{-1}(y) - f^{-1}(y_0)}{y - y_0}$ exists for all $y_0 \in J$.

Consider any $y_0 \in J$. Since f^{-1} is continuous, if $y \to y_0$, then $f^{-1}(y) \to f^{-1}(y_0)$, *i.e.*, $y \to y_0 \implies x \to x_0$. Therefore,

$$\lim_{y \to y_0} \frac{f^{-1}(y) - f^{-1}(y_0)}{y - y_0} = \lim_{x \to x_0} \frac{x - x_0}{f(x) - f(x_0)} = \lim_{x \to x_0} \frac{1}{\frac{f(x) - f(x_0)}{x - x_0}}$$

$$= \frac{1}{\lim\limits_{x \to x_0} \dfrac{f(x) - f(x_0)}{x - x_0}} = \frac{1}{f'(x_0)} = \frac{1}{f'\left(f^{-1}(y_0)\right)}$$

Since y_0 was an arbitrary point in J, it follows that:

3 $(f^{-1})'(y) = \dfrac{1}{f'\left((f^{-1}(y)\right)}$ for all $y \in J$.

□