Differentiation

Mathematics and Statistics

$$\int_M d\omega = \int_{\partial M} \omega$$

Mathematics 3A03
Real Analysis I

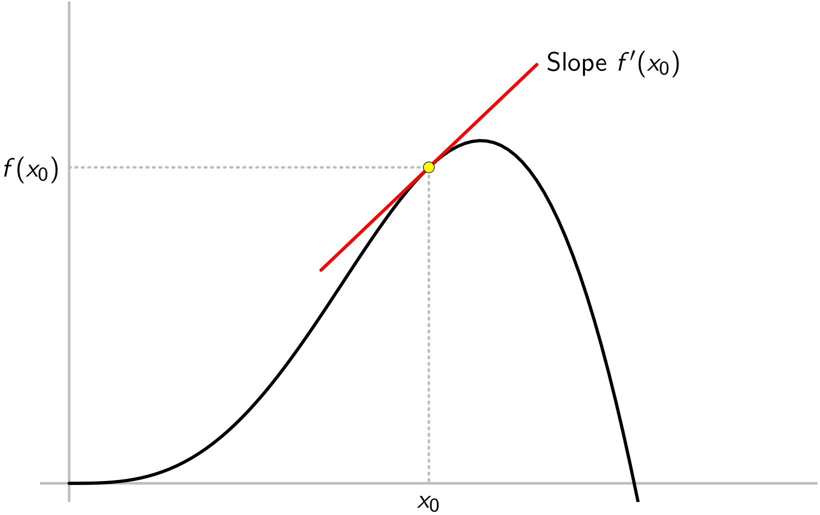
Instructor: David Earn

Lecture 2
Differentiation
Thursday 8 January 2026

Announcements

- Slides are posted on the course website:
  https://ms.mcmaster.ca/earn/3A03
- Lecture recordings are posted on Avenue-to-Learn

The Derivative



The Derivative

Definition (Derivative)

Let f be defined on an interval I and let $x_0 \in I$. The ***derivative*** of f at x_0, denoted by $f'(x_0)$, is defined as

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0},$$

provided either that this limit exists or is infinite. If $f'(x_0)$ is finite we say that f is ***differentiable*** at x_0. If f is differentiable at every point of a set $E \subseteq I$, we say that f is differentiable on E. If E is all of I, we simply say that f is a ***differentiable function***.

Note: "Differentiable" and "a derivative exists" always mean that the derivative is finite.

The Derivative

Example

$f(x) = x^2$. Find $f'(2)$.

$$f'(2) = \lim_{x\to 2} \frac{x^2-4}{x-2} = \lim_{x\to 2} \frac{(x+2)(x-2)}{x-2} = \lim_{x\to 2} x+2 = 4$$

Note:

- In the first two limits, we must have $x \neq 2$.
- But in the third limit, we just plug in $x = 2$.
- Two things are equal, but in one $x \neq 2$ and in the other $x = 2$.
- Good illustration of why it is important to define the meaning of limits rigorously.

Poll

- Go to
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- Click on Math 3A03
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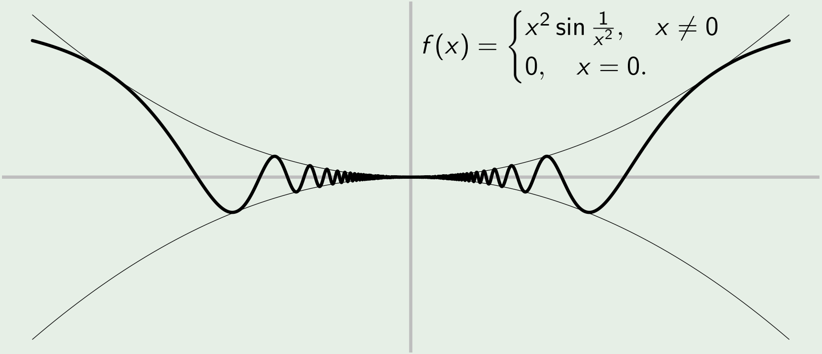
The Derivative

Example

Let f be defined in a neighbourhood I of 0, and suppose $|f(x)| \leq x^2$ for all $x \in I$. Is f necessarily differentiable at 0? *e.g.*,



$$f(x) = \begin{cases} x^2 \sin \frac{1}{x^2}, & x \neq 0 \\ 0, & x = 0. \end{cases}$$

The Derivative

Example (Trapping principle)

Suppose $f(x) = \begin{cases} x^2 \sin \frac{1}{x^2}, & x \neq 0 \\ 0, & x = 0. \end{cases}$ Then:

$$\forall x \neq 0 : \quad \left|\frac{f(x) - f(0)}{x - 0}\right| = \left|\frac{f(x)}{x}\right| = \left|\frac{x^2 \sin \frac{1}{x^2}}{x}\right| = \left|x \sin \frac{1}{x^2}\right| \leq |x|$$

Therefore:

$$|f'(0)| = \left|\lim_{x\to 0} \frac{f(x) - f(0)}{x - 0}\right| = \lim_{x\to 0} \left|\frac{f(x) - f(0)}{x - 0}\right| \leq \lim_{x\to 0} |x| = 0.$$

$\therefore$ f is differentiable at 0 and $f'(0) = 0$. □

The Derivative

Definition (One-sided derivatives)

Let f be defined on an interval I and let $x_0 \in I$. The ***right-hand derivative*** of f at x_0, denoted by $f'_+(x_0)$, is the limit

$$f'_+(x_0) = \lim_{x \to x_0^+} \frac{f(x) - f(x_0)}{x - x_0},$$

provided either that this one-sided limit exists or is infinite.
Similarly, the ***left-hand derivative*** of f at x_0, denoted by $f'_-(x_0)$, is the limit

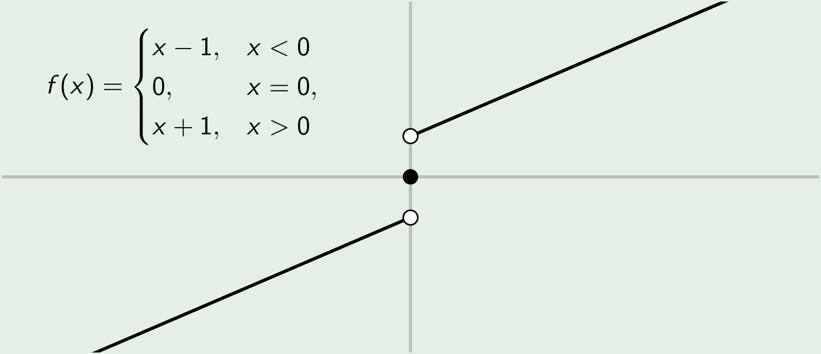
$$f'_-(x_0) = \lim_{x \to x_0^-} \frac{f(x) - f(x_0)}{x - x_0}.$$

Note: If x_0 is not an endpoint of the interval I then f is differentiable at x_0 iff $f'_+(x_0) = f'_-(x_0) \neq \pm\infty$.

The Derivative

Example

$$f(x) = \begin{cases} x-1, & x<0 \\ 0, & x=0, \\ x+1, & x>0 \end{cases}$$



- Same slope from left and right. Why isn't f differentiable???
- $\lim_{x\to 0^-} f'(x) = \lim_{x\to 0^+} f'(x) = \lim_{x\to 0} f'(x) = 1.$
- $f'_-(0) = f'_+(0) = f'(0) = \lim_{x\to 0} \frac{f(x)-f(0)}{x-0} = \infty.$

The Derivative

- Higher derivatives: we write
  - $f'' = (f')'$ if f' is differentiable;
  - $f^{(n+1)} = (f^{(n)})'$ if $f^{(n)}$ is differentiable.

- Other standard notation for derivatives:

$$\frac{df}{dx} = f'(x)$$

$$D = \frac{d}{dx}$$

$$D^n f(x) = \frac{d^n f}{dx^n} = f^{(n)}(x)$$

REMINDER: Algebra of limits

Theorem (Algebraic operations on limits of sequences)

Suppose $\{s_n\}$ *and* $\{t_n\}$ *are convergent sequences and* $C \in \mathbb{R}$.

1. $\lim_{n\to\infty} C\, s_n = C(\lim_{n\to\infty} s_n)$;
2. $\lim_{n\to\infty}(s_n + t_n) = (\lim_{n\to\infty} s_n) + (\lim_{n\to\infty} t_n)$;
3. $\lim_{n\to\infty}(s_n - t_n) = (\lim_{n\to\infty} s_n) - (\lim_{n\to\infty} t_n)$;
4. $\lim_{n\to\infty}(s_n t_n) = (\lim_{n\to\infty} s_n)(\lim_{n\to\infty} t_n)$;
5. *if* $t_n \neq 0$ *for all* n *and* $\lim_{n\to\infty} t_n \neq 0$ *then*
$$\lim_{n\to\infty}\left(\frac{s_n}{t_n}\right) = \frac{\lim_{n\to\infty} s_n}{\lim_{n\to\infty} t_n} .$$

(TBB §2.7, and problem 2.7.4)

REMINDER: Algebra of limits

Theorem (Algebraic operations on limits of functions)

Suppose $f, g : \mathbb{R} \to \mathbb{R}$, $x_0 \in \mathbb{R}$, *the limits as* $x \to x_0$ *of* $f(x)$ *and* $g(x)$ *both exist, and* $C \in \mathbb{R}$.

1. $\lim_{x \to x_0} C\, f(x) = C\big(\lim_{x \to x_0} f(x)\big)$;
2. $\lim_{x \to x_0} (f(x) + g(x)) = \big(\lim_{x \to x_0} f(x)\big) + \big(\lim_{x \to x_0} g(x)\big)$;
3. $\lim_{x \to x_0} (f(x) - g(x)) = \big(\lim_{x \to x_0} f(x)\big) - \big(\lim_{x \to x_0} g(x)\big)$;
4. $\lim_{x \to x_0} (f(x) g(x)) = \big(\lim_{x \to x_0} f(x)\big)\big(\lim_{x \to x_0} g(x)\big)$;
5. *if* $g(x) \neq 0$ *for* $x \in (x_0 - \delta, x_0 + \delta)$ *for some* $\delta > 0$, *and* $\lim_{x \to x_0} g(x) \neq 0$ *then* $\lim_{x \to x_0} \left(\frac{f(x)}{g(x)}\right) = \frac{\lim_{x \to x_0} f(x)}{\lim_{x \to x_0} g(x)}$.

The Derivative

Theorem (Differentiable $\implies$ continuous)

If f is defined in a neighbourhood I of x_0 and f is differentiable at x_0 then f is continuous at x_0.

Proof.

Must show $\lim_{x \to x_0} f(x) = f(x_0)$, *i.e.*, $\lim_{x \to x_0} (f(x) - f(x_0)) = 0$.

$$\begin{aligned}
\lim_{x \to x_0} (f(x) - f(x_0)) &= \lim_{x \to x_0} \left(\frac{f(x) - f(x_0)}{x - x_0} \times (x - x_0) \right) \\
&= \lim_{x \to x_0} \left(\frac{f(x) - f(x_0)}{x - x_0} \right) \times \lim_{x \to x_0} (x - x_0) \\
&= f'(x_0) \times 0 = 0 \,,
\end{aligned}$$

where we have used the theorem on the algebra of limits. □

McMaster University

Mathematics and Statistics

$$\int_M d\omega = \int_{\partial M} \omega$$

Mathematics 3A03
Real Analysis I

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Lecture 3
Differentiation II
Friday 9 January 2026

Last time...

- Definition of the derivative.
  - Example: Trapping Principle
- Defined one-sided derivatives
  - Example
- Proved differentiable $\implies$ continuous.

More on the derivative

Theorem (Algebra of derivatives)

Suppose f and g are defined on an interval I and $x_0 \in I$. If f and g are differentiable at x_0 then $f+g$ and fg are differentiable at x_0. If, in addition, $g(x_0) \neq 0$ then f/g is differentiable at x_0. Under these conditions:

1. $(cf)'(x_0) = cf'(x_0)$ *for all* $c \in \mathbb{R}$;
2. $(f+g)'(x_0) = (f'+g')(x_0)$;
3. $(fg)'(x_0) = (f'g + fg')(x_0)$;
4. $\left(\frac{f}{g}\right)'(x_0) = \left(\frac{gf' - fg'}{g^2}\right)(x_0) \qquad (g(x_0) \neq 0)$.

(TBB Theorem 7.7, p. 408)

The Derivative

Theorem (Chain rule)

Suppose f is defined in a neighbourhood U of x_0 and g is defined in a neighbourhood V of $f(x_0)$ such that $f(U) \subseteq V$. If f is differentiable at x_0 and g is differentiable at $f(x_0)$ then the composite function $h = g \circ f$ is differentiable at x_0 and

$$h'(x_0) = (g \circ f)'(x_0) = g'(f(x_0))f'(x_0)\,.$$

Informally, if $y = f(x)$ and $z = g(y)$ then $\frac{\mathrm{d}z}{\mathrm{d}x} = \frac{\mathrm{d}z}{\mathrm{d}y}\frac{\mathrm{d}y}{\mathrm{d}x}$.

(TBB §7.3.2, p. 411)

Why the chain rule is plausible

The derivative of $g \circ f$ at x_0 is the limit as $x \to x_0$ of the difference quotient

$$\frac{g(f(x)) - g(f(x_0))}{x - x_0} = \frac{g(f(x)) - g(f(x_0))}{f(x) - f(x_0)} \cdot \frac{f(x) - f(x_0)}{x - x_0} \qquad (\spadesuit)$$

Recall: $f'(x_0)$ exists $\implies$ f continuous at x_0
$\implies$ $f(x) \to f(x_0)$ as $x \to x_0$.

Can we take the limit as $x \to x_0$ and conclude that $(g \circ f)'(x_0) = g'(f(x_0)) \cdot f'(x_0)$?

- What if $f(x) = 0$ for all x?
- What if f is a constant function?
- What if $f(x) = f(x_0)$ for some $x \neq x_0$?
- Can we use ($\spadesuit$) to prove the chain rule?

Poll

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- `Submit`.

REMINDER: limits of functions

Theorem (Equivalence of ε-δ and sequence definitions of limits)

Let $a < x_0 < b$, $I = (a,b)$, and $f : I \setminus \{x_0\} \to \mathbb{R}$. Then the following two definitions of

$$\lim_{x \to x_0} f(x) = L$$

are equivalent:

1. *for all $\varepsilon > 0$ there exists $\delta > 0$ such that if $0 < |x - x_0| < \delta$ then $|f(x) - L| < \varepsilon$.*

2. *for every sequence $\{x_n\}$ of points in $I \setminus \{x_0\}$,*

$$\lim_{n \to \infty} x_n = x_0 \quad \Longrightarrow \quad \lim_{n \to \infty} f(x_n) = L\,.$$

Note: The deleted neighbourhood ($I \setminus \{x_0\}$) can be replaced by any set on which f is defined and x_0 is an accumulation point.

Proof of the chain rule.

1. Suppose there is an open interval I, with $x_0 \in I$, and $f(x) \neq f(x_0)$ for all $x \in I \setminus \{x_0\}$. Then we can take the limit $x \to x_0$ in (♠) and we get the chain rule.

2. Next suppose that no open interval like the one hypothesized above exists. Then, in any open interval containing x_0, there must be at least one point $x \neq x_0$ for which $f(x) = f(x_0)$. Therefore, we can construct a sequence of open intervals I_n, with lengths decreasing to 0, such that each I_n contains x_0 and a point $x_n \neq x_0$ with $f(x_n) = f(x_0)$. Therefore, since $f'(x_0)$ exists, and we recall the previous slide, we can compute $f'(x_0)$ via

$$f'(x_0) = \lim_{n\to\infty} \frac{f(x_n) - f(x_0)}{x_n - x_0} = \lim_{n\to\infty} \frac{0}{x_n - x_0} = 0.$$

We can also show that $(g \circ f)'(x_0) = 0$, using the sequence definition on the previous slide. *Try to fill in this last detail*, or look it up (TBB §7.3.2, p. 411).

Note: TBB's proof leaves out the proof that $f'(x_0) = 0$ in case 2 above. □

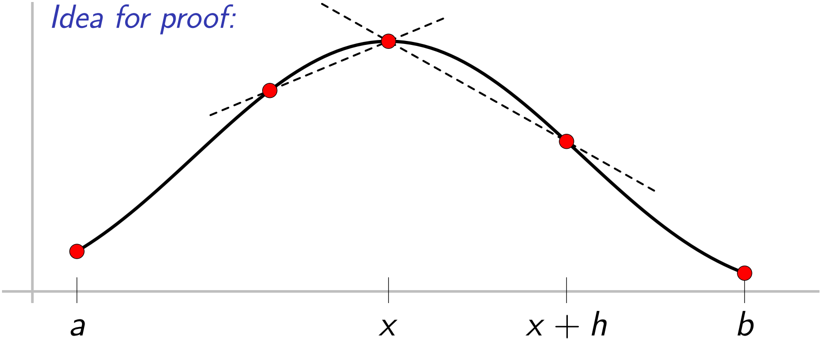
More on the derivative

Theorem (Derivative at local extrema)

Let $f : (a, b) \to \mathbb{R}$*. If* x *is a maximum or minimum point of* f *in* (a, b)*, and* f *is differentiable at* x*, then* $f'(x) = 0$.

Note: f need not be differentiable or even continuous at other points.

Idea for proof:



More on the derivative

Proof that the derivative vanishes at local extrema.

If f has a local maximum at $x \in (a, b)$, then for sufficiently small $h > 0$ we must have

$$\frac{f(x+h) - f(x)}{h} \leq 0 \leq \frac{f(x) - f(x-h)}{h}$$

Since f is differentiable at x, it is left and right differentiable at x, so we can evaluate the limits as $h \to 0$ to obtain

$$f'_+(x) \leq 0 \leq f'_-(x)\,.$$

But since f is differentiable at x, the left and right derivatives must be equal, hence $f'(x) = 0$. □