Differentiation

McMaster University

Mathematics and Statistics

$$\int_M d\omega = \int_{\partial M} \omega$$

Mathematics 3A03
Real Analysis I

Instructor: David Earn

Lecture 2
Differentiation
Wednesday 8 January 2025

Survey to do right now

- Please go to
  https://www.childsmath.ca/childsa/forms/main_login.php
- Click on Math 3A03
- Click on Take Class Poll
- Fill in poll **Survey 2**
- `Submit`.

Announcements

- Results of Survey 1
- Results of Survey 2

Background / reminder

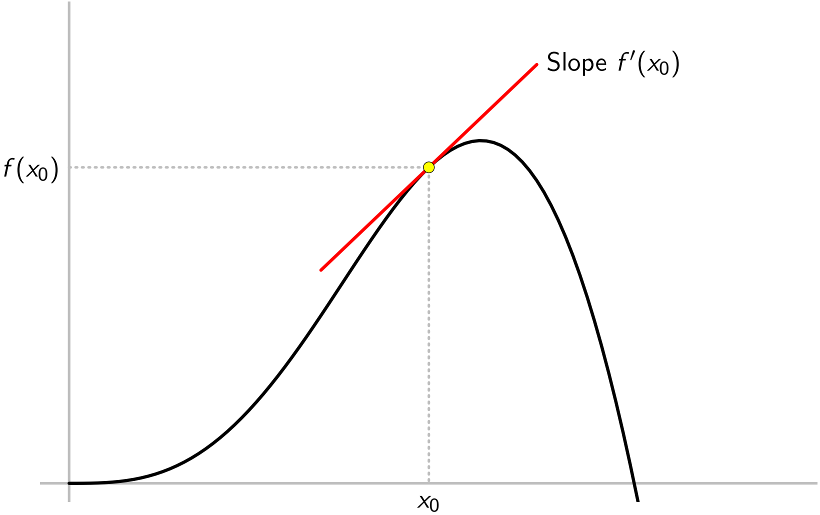
Definition (Cauchy sequence)

A sequence $\{s_n\}$ is said to be a ***Cauchy sequence*** iff for all $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that if $m \geq N$ and $n \geq N$ then $|s_n - s_m| < \varepsilon$.

Poll: another background check

- Go to
  https://www.childsmath.ca/childsa/forms/main_login.php
- Click on Math 3A03
- Click on Take Class Poll
- Fill in poll **Background: Cauchy sequences**
- `Submit`.

The Derivative



The Derivative

Definition (Derivative)

Let f be defined on an interval I and let $x_0 \in I$. The ***derivative*** of f at x_0, denoted by $f'(x_0)$, is defined as

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0},$$

provided either that this limit exists or is infinite. If $f'(x_0)$ is finite we say that f is ***differentiable*** at x_0. If f is differentiable at every point of a set $E \subseteq I$, we say that f is differentiable on E. If E is all of I, we simply say that f is a ***differentiable function***.

Note: "Differentiable" and "a derivative exists" always mean that the derivative is finite.

The Derivative

Example

$f(x) = x^2$. Find $f'(2)$.

$$f'(2) = \lim_{x\to 2} \frac{x^2-4}{x-2} = \lim_{x\to 2} \frac{(x+2)(x-2)}{x-2} = \lim_{x\to 2} x+2 = 4$$

Note:

- In the first two limits, we must have $x \neq 2$.
- But in the third limit, we just plug in $x = 2$.
- Two things are equal, but in one $x \neq 2$ and in the other $x = 2$.
- Good illustration of why it is important to define the meaning of limits rigorously.

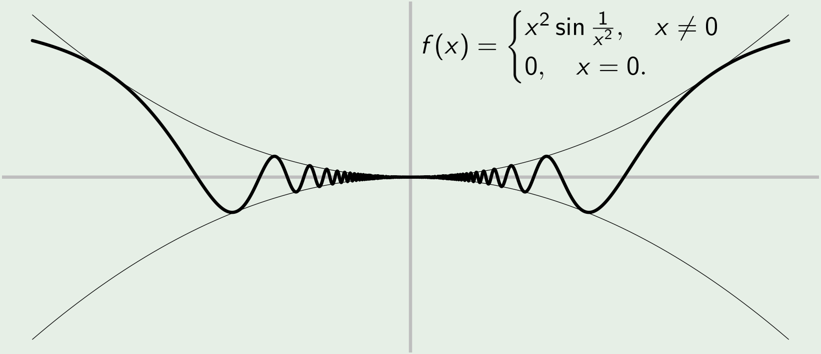
Poll

- Go to
  https://www.childsmath.ca/childsa/forms/main_login.php
- Click on Math 3A03
- Click on Take Class Poll
- Fill in poll **Derivatives: Differentiable at 0**
- Submit.

The Derivative

Example

Let f be defined in a neighbourhood I of 0, and suppose $|f(x)| \leq x^2$ for all $x \in I$. Is f necessarily differentiable at 0? *e.g.*,



$$f(x) = \begin{cases} x^2 \sin \frac{1}{x^2}, & x \neq 0 \\ 0, & x = 0. \end{cases}$$

The Derivative

Example (Trapping principle)

Suppose $f(x) = \begin{cases} x^2 \sin \frac{1}{x^2}, & x \neq 0 \\ 0, & x = 0. \end{cases}$ Then:

$$\forall x \neq 0: \quad \left|\frac{f(x) - f(0)}{x - 0}\right| = \left|\frac{f(x)}{x}\right| = \left|\frac{x^2 \sin \frac{1}{x^2}}{x}\right| = \left|x \sin \frac{1}{x^2}\right| \leq |x|$$

Therefore:

$$|f'(0)| = \left|\lim_{x \to 0} \frac{f(x) - f(0)}{x - 0}\right| = \lim_{x \to 0} \left|\frac{f(x) - f(0)}{x - 0}\right| \leq \lim_{x \to 0} |x| = 0.$$

$\therefore$ f is differentiable at 0 and $f'(0) = 0$. □

The Derivative

Definition (One-sided derivatives)

Let f be defined on an interval I and let $x_0 \in I$. The ***right-hand derivative*** of f at x_0, denoted by $f'_+(x_0)$, is the limit

$$f'_+(x_0) = \lim_{x \to x_0^+} \frac{f(x) - f(x_0)}{x - x_0},$$

provided either that this one-sided limit exists or is infinite. Similarly, the ***left-hand derivative*** of f at x_0, denoted by $f'_-(x_0)$, is the limit

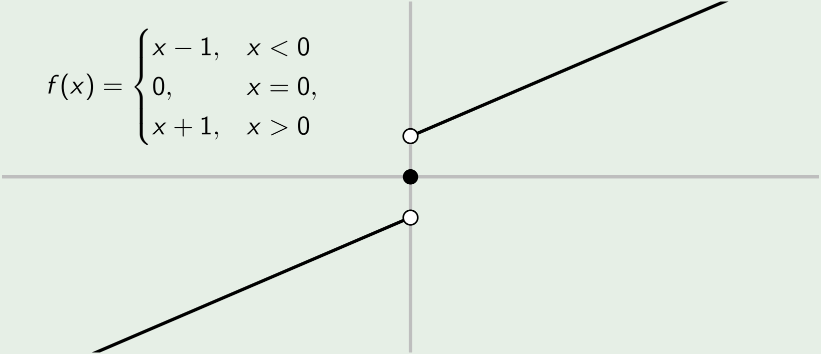
$$f'_-(x_0) = \lim_{x \to x_0^-} \frac{f(x) - f(x_0)}{x - x_0}.$$

Note: If x_0 is not an endpoint of the interval I then f is differentiable at x_0 iff $f'_+(x_0) = f'_-(x_0) \neq \pm\infty$.

The Derivative

Example

$$f(x) = \begin{cases} x-1, & x<0 \\ 0, & x=0, \\ x+1, & x>0 \end{cases}$$



- Same slope from left and right. Why isn't f differentiable???
- $\lim_{x\to 0^-} f'(x) = \lim_{x\to 0^+} f'(x) = \lim_{x\to 0} f'(x) = 1.$
- $f'_-(0) = f'_+(0) = f'(0) = \lim_{x\to 0} \frac{f(x)-f(0)}{x-0} = \infty.$

The Derivative

- Higher derivatives: we write
  - $f'' = (f')'$ if f' is differentiable;
  - $f^{(n+1)} = (f^{(n)})'$ if $f^{(n)}$ is differentiable.

- Other standard notation for derivatives:

$$\frac{df}{dx} = f'(x)$$

$$D = \frac{d}{dx}$$

$$D^n f(x) = \frac{d^n f}{dx^n} = f^{(n)}(x)$$

REMINDER: Algebra of limits

Theorem (Algebraic operations on limits of sequences)

Suppose $\{s_n\}$ *and* $\{t_n\}$ *are convergent sequences and* $C \in \mathbb{R}$.

1. $\lim_{n\to\infty} C\, s_n = C\big(\lim_{n\to\infty} s_n\big)$;
2. $\lim_{n\to\infty} (s_n + t_n) = \big(\lim_{n\to\infty} s_n\big) + \big(\lim_{n\to\infty} t_n\big)$;
3. $\lim_{n\to\infty} (s_n - t_n) = \big(\lim_{n\to\infty} s_n\big) - \big(\lim_{n\to\infty} t_n\big)$;
4. $\lim_{n\to\infty} (s_n t_n) = \big(\lim_{n\to\infty} s_n\big)\big(\lim_{n\to\infty} t_n\big)$;
5. *if* $t_n \neq 0$ *for all* n *and* $\lim_{n\to\infty} t_n \neq 0$ *then*
$$\lim_{n\to\infty} \left(\frac{s_n}{t_n}\right) = \frac{\lim_{n\to\infty} s_n}{\lim_{n\to\infty} t_n} .$$

(TBB §2.7, and problem 2.7.4)

REMINDER: Algebra of limits

Theorem (Algebraic operations on limits of functions)

Suppose $f, g : \mathbb{R} \to \mathbb{R}$, $x_0 \in \mathbb{R}$, the limits as $x \to x_0$ of $f(x)$ and $g(x)$ both exist, and $C \in \mathbb{R}$.

1. $\lim_{x \to x_0} C\,f(x) = C\big(\lim_{x \to x_0} f(x)\big)$;
2. $\lim_{x \to x_0} (f(x) + g(x)) = \big(\lim_{x \to x_0} f(x)\big) + \big(\lim_{x \to x_0} g(x)\big)$;
3. $\lim_{x \to x_0} (f(x) - g(x)) = \big(\lim_{x \to x_0} f(x)\big) - \big(\lim_{x \to x_0} g(x)\big)$;
4. $\lim_{x \to x_0} (f(x)g(x)) = \big(\lim_{x \to x_0} f(x)\big)\big(\lim_{x \to x_0} g(x)\big)$;
5. *if* $g(x) \neq 0$ *for* $x \in (x_0 - \delta, x_0 + \delta)$ *for some* $\delta > 0$, *and* $\lim_{x \to x_0} g(x) \neq 0$ *then* $\lim_{x \to x_0} \left(\dfrac{f(x)}{g(x)}\right) = \dfrac{\lim_{x \to x_0} f(x)}{\lim_{x \to x_0} g(x)}$.

The Derivative

Theorem (Differentiable $\Longrightarrow$ continuous)

If f is defined in a neighbourhood I of x_0 and f is differentiable at x_0 then f is continuous at x_0.

Proof.

Must show $\lim_{x \to x_0} f(x) = f(x_0)$, *i.e.*, $\lim_{x \to x_0} (f(x) - f(x_0)) = 0$.

$$\begin{aligned}
\lim_{x \to x_0} (f(x) - f(x_0)) &= \lim_{x \to x_0} \left(\frac{f(x) - f(x_0)}{x - x_0} \times (x - x_0) \right) \\
&= \lim_{x \to x_0} \left(\frac{f(x) - f(x_0)}{x - x_0} \right) \times \lim_{x \to x_0} (x - x_0) \\
&= f'(x_0) \times 0 = 0\,,
\end{aligned}$$

where we have used the theorem on the algebra of limits. □

McMaster University

Mathematics and Statistics

$$\int_M d\omega = \int_{\partial M} \omega$$

Mathematics 3A03
Real Analysis I

Instructor: David Earn

Lecture 3
Differentiation II
Friday 10 January 2025

Announcements

- Lectures are being live streamed and recorded as of today.
- Recordings will be available 24 hours after lectures.
  - Go to https://echo360.ca.
  - Sign in with your `macID@mcmaster.ca` e-mail address.
  - Click on the `Courses` tab.
  - Select "MATH 3A03 - WINTER 2025"
- Course evaluation scheme has changed (next two slides).
  - The course web site and online syllabus have not yet been updated to reflect these changes, but that will happen soon (hopefully over the weekend).

Course evaluation will be revised as follows:

- We will not have quizzes
- 5% for participating in at least 80% of in-class polls
- 15% for participating in assignments, based on multiple choice (MC) questions:

$$\text{assignment mark} = \frac{\text{number MC questions answered}}{\text{total number MC questions assigned}}$$

- 30% for midterm test on Thurs 27 Feb 2025
- 50% for final exam in April
- *Note:* If your final exam mark is better than your midterm test mark then the final exam mark will replace the midterm test mark.
- *Important:* Do NOT skip the midterm. Even if you don't feel well prepared, write it for practice so you are better prepared for writing the final exam.
- If you must miss the midterm (*e.g.*, illness or accepting a Nobel prize), your final exam mark will replace it.

Tentative plan for assignments

- There will be regular assignments.
- Each question will have a multiple choice component (probably on childsmath). Only participation counts for marks; you will get the same credit for correct and incorrect answers, or for selecting "I haven't had time to think about this yet".
- Optionally, full solutions/proofs can be written up and submitted on crowdmark. Feedback will be given, but no marks. The purpose is to help you prepare better for the test and exam.
- If you're not sure if your proof is complete, or you got stuck and don't know how to complete it, make that clear in the document that you submit on crowdmark, so the TA can focus on the help you need.
- Always try your best to solve problems on your own first. But if you used stackexchange or ChatGPT or whatever for help, provide a URL to your source if possible, so it is easier for the TA to provide the help you need.
- Make the best possible use of the TA's time: say what you think you do or don't understand.

Last time. . .

- Definition of the derivative.
    - Example: Trapping Principle
- Defined one-sided derivatives
    - Example
- Proved differentiable $\Longrightarrow$ continuous.

More on the derivative

Theorem (Algebra of derivatives)

Suppose f and g are defined on an interval I and $x_0 \in I$. If f and g are differentiable at x_0 then $f + g$ and fg are differentiable at x_0. If, in addition, $g(x_0) \neq 0$ then f/g is differentiable at x_0. Under these conditions:

1. $(cf)'(x_0) = cf'(x_0)$ *for all* $c \in \mathbb{R}$;
2. $(f+g)'(x_0) = (f'+g')(x_0)$;
3. $(fg)'(x_0) = (f'g + fg')(x_0)$;
4. $\left(\frac{f}{g}\right)'(x_0) = \left(\frac{gf' - fg'}{g^2}\right)(x_0) \qquad (g(x_0) \neq 0)$.

(TBB Theorem 7.7, p. 408)

The Derivative

Theorem (Chain rule)

Suppose f is defined in a neighbourhood U of x_0 and g is defined in a neighbourhood V of $f(x_0)$ such that $f(U) \subseteq V$. If f is differentiable at x_0 and g is differentiable at $f(x_0)$ then the composite function $h = g \circ f$ is differentiable at x_0 and

$$h'(x_0) = (g \circ f)'(x_0) = g'(f(x_0))f'(x_0)\,.$$

Informally, if $y = f(x)$ and $z = g(y)$ then $\frac{\mathrm{d}z}{\mathrm{d}x} = \frac{\mathrm{d}z}{\mathrm{d}y}\frac{\mathrm{d}y}{\mathrm{d}x}$.

(TBB §7.3.2, p. 411)

Why the chain rule is plausible

The derivative of $g \circ f$ at x_0 is the limit as $x \to x_0$ of the difference quotient

$$\frac{g(f(x)) - g(f(x_0))}{x - x_0} = \frac{g(f(x)) - g(f(x_0))}{f(x) - f(x_0)} \cdot \frac{f(x) - f(x_0)}{x - x_0} \qquad (\spadesuit)$$

Recall: $f'(x_0)$ exists $\implies$ f continuous at x_0
$\implies f(x) \to f(x_0)$ as $x \to x_0$.

Can we take the limit as $x \to x_0$ and conclude that $(g \circ f)'(x_0) = g'(f(x_0)) \cdot f'(x_0)$?

- What if $f(x) = 0$ for all x?
- What if f is a constant function?
- What if $f(x) = f(x_0)$ for some $x \neq x_0$?
- Can we use $(\spadesuit)$ to prove the chain rule?

Poll

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- Click on Math 3A03
- Click on Take Class Poll
- Fill in poll **Derivatives: Chain Rule**
- Submit.

REMINDER: limits of functions

Theorem (Equivalence of ε-δ and sequence definitions of limits)

Let $a < x_0 < b$, $I = (a, b)$, and $f : I \setminus \{x_0\} \to \mathbb{R}$. Then the following two definitions of

$$\lim_{x \to x_0} f(x) = L$$

are equivalent:

1. *for all $\varepsilon > 0$ there exists $\delta > 0$ such that if $0 < |x - x_0| < \delta$ then $|f(x) - L| < \varepsilon$.*
2. *for every sequence $\{x_n\}$ of points in $I \setminus \{x_0\}$,*

$$\lim_{n \to \infty} x_n = x_0 \quad \Longrightarrow \quad \lim_{n \to \infty} f(x_n) = L\,.$$

Note: The deleted neighbourhood ($I \setminus \{x_0\}$) can be replaced by any set on which f is defined and x_0 is an accumulation point.

Proof of the chain rule.

1. Suppose there is an open interval I, with $x_0 \in I$, and $f(x) \neq f(x_0)$ for all $x \in I \setminus \{x_0\}$. Then we can take the limit $x \to x_0$ in (♠) and we get the chain rule.

2. Next suppose that no open interval like the one hypothesized above exists. Then, in any open interval containing x_0, there must be at least one point $x \neq x_0$ for which $f(x) = f(x_0)$. Therefore, we can construct a sequence of open intervals I_n, with lengths decreasing to 0, such that each I_n contains x_0 and a point $x_n \neq x_0$ with $f(x_n) = f(x_0)$. Therefore, since $f'(x_0)$ exists, and we recall the previous slide, we can compute $f'(x_0)$ via

$$f'(x_0) = \lim_{n\to\infty} \frac{f(x_n) - f(x_0)}{x_n - x_0} = \lim_{n\to\infty} \frac{0}{x_n - x_0} = 0.$$

We can also show that $(g \circ f)'(x_0) = 0$, using the sequence definition on the previous slide. *Try to fill in this last detail*, or look it up (TBB §7.3.2, p. 411).

Note: TBB's proof leaves out the proof that $f'(x_0) = 0$ in case 2 above. □

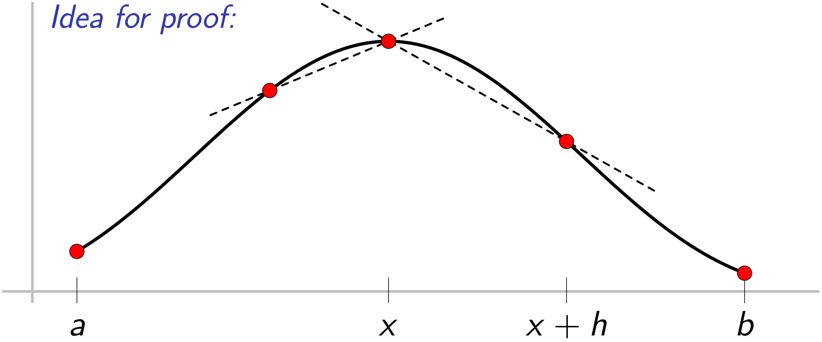
More on the derivative

Theorem (Derivative at local extrema)

Let $f : (a,b) \to \mathbb{R}$*. If* x *is a maximum or minimum point of* f *in* (a,b)*, and* f *is differentiable at* x*, then* $f'(x) = 0$.

Note: f need not be differentiable or even continuous at other points.

Idea for proof:



More on the derivative

Proof that the derivative vanishes at local extrema.

If f has a local maximum at $x \in (a, b)$, then for sufficiently small $h > 0$ we must have

$$\frac{f(x+h) - f(x)}{h} \leq 0 \leq \frac{f(x) - f(x-h)}{h}$$

Since f is differentiable at x, it is left and right differentiable at x, so we can evaluate the limits as $h \to 0$ to obtain

$$f'_+(x) \leq 0 \leq f'_-(x)\,.$$

But since f is differentiable at x, the left and right derivatives must be equal, hence $f'(x) = 0$. □