McMaster University

Mathematics and Statistics

$$\int_M d\omega = \int_{\partial M} \omega$$

Mathematics 3A03
Real Analysis I

Instructor: David Earn

Lecture 1
Introduction
Tuesday 3 September 2019

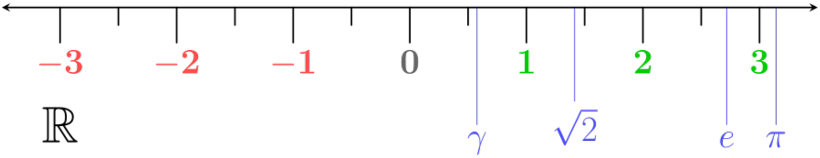
Where to find course information

- The course web site: `http://ms.mcmaster.ca/earn/3A03`
- Click on Course information to download course information as pdf file. *You are expected to read and pay attention to every word of this file.*
- Let's have a look now. . .

What is a "real" number?

What is a "real" number?

- The "Reals" ($\mathbb{R}$) are all the numbers that are needed to fill in the "number line" (so it has no "gaps" or "holes").
- Why aren't the rational numbers ($\mathbb{Q}$) sufficient?



- How do we know that $\sqrt{2}$ is not rational?
- How can we *prove* this?
  <u>*Approach*</u>: "Proof by contradiction."

$\sqrt{2}$ is irrational

Theorem

$\sqrt{2} \notin \mathbb{Q}$.

Proof.

Suppose $\sqrt{2} \in \mathbb{Q}$. Then there exist two positive integers m and n with $\gcd(m, n) = 1$ such that $m/n = \sqrt{2}$.

$$\therefore \left(\frac{m}{n}\right)^2 = (\sqrt{2})^2 \implies \frac{m^2}{n^2} = 2 \implies m^2 = 2n^2.$$

$\therefore m^2$ is even $\implies$ m is even ($\because$ odd numbers have odd squares).

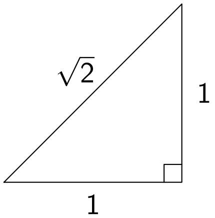
$\therefore m = 2k$ for some $k \in \mathbb{N}$.

$$\therefore 4k^2 = m^2 = 2n^2 \implies 2k^2 = n^2 \implies n \text{ is even}.$$

$\therefore$ 2 is a factor of both m and n. **Contradiction!** $\therefore \sqrt{2} \notin \mathbb{Q}$. $\square$

Does $\sqrt{2}$ exist?

- We have established that $\sqrt{2}$ is not rational.
- But do we really know it exists?
- Can we do without it?
- No. Objects with side length $\sqrt{2}$ exist!



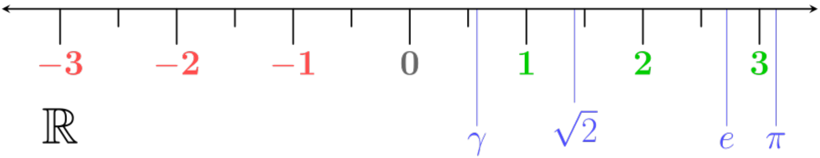
- So irrational numbers are "real".

Poll on rationality

- Please log in (right now) to this web site: https://www.childsmath.ca/childsa/forms/main_login.php
- Click on Math 3A03.
- Click on Take Class Poll.
- After selecting the numbers you think are rational, click the `Submit` button.
- Everybody done?
- Let's Deactivate the poll and View Results

What exactly *are* non-rational real numbers?

- We have solid intuition for what rational numbers are. (Ratios of integers.)
- The number line contains numbers that are not rational.



- Can we *construct* irrational numbers? (Just as we construct rationals as ratios of integers?)
- Do we need to *construct* integers first?
- Maybe we should start with 0, 1, 2, . . .
- But <u>what</u> exactly are we supposed to *construct* numbers <u>from</u>?

Informal introduction to construction of numbers ($\mathbb{N}$)

- Assume we know what a ***set*** is.
- Define $0 \equiv \varnothing = \{\}$ (the empty set)
- Define $1 \equiv \{0\} = \{\varnothing\} = \{\{\}\}$
- Define $2 \equiv \{0, 1\} = \{\{\}, \{\{\}\}\}$
- Define $n + 1 \equiv n \cup \{n\}$ (successor function)
- Define ***natural numbers*** $\mathbb{N} = \{1, 2, 3, \ldots\}$
  - Some books define $\mathbb{N} = \{0, 1, 2, \ldots\}$ and $\mathbb{N}^+ = \{1, 2, 3, \ldots\}$.
  - It is more common to define $\mathbb{N}$ to start with 1.
- Thus, n is defined to be a set containing n elements.

Informal introduction to construction of numbers ($\mathbb{N}$)

Historical note:

- We have defined n to be a set containing n elements.
- Logicians first tried to define n as "the set of all sets containing n elements".
- The earlier definition possibly better captures our intuitive notion of what n "really is", but such "sets" are unweildy and create serious challenges for development of mathematical foundations.

Informal introduction to construction of numbers ($\mathbb{N}$)

Order of natural numbers:

- Natural numbers defined as above have the right order:

$$m \leq n \iff m \subseteq n$$

Note: we *define* "$\leq$" on natural numbers via "$\subseteq$" on sets.

Addition and multiplication of natural numbers:

- Still possible to define in terms of sets, but trickier.
- We'll defer this for later, after gaining more experience with rigorous mathematical concepts.
- If you can't wait, see this free e-book:
  "Transition to Higher Mathematics"
  http://openscholarship.wustl.edu/books/10/.

Informal introduction to construction of numbers ($\mathbb{Z}$)

Integers:

- Need additive inverses for all natural numbers.
- Need to define $\cdot, +, -,$ for all pairs of integers.
- Again, possible to define everything via set theory.
- Again, we'll defer this for later.

- For now, we'll assume we "know" what the naturals $\mathbb{N}$ and the integers $\mathbb{Z}$ "are".
- We can then *construct* the rationals $\mathbb{Q}$...

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Mathematics 3A03
Real Analysis I

Instructor: David Earn

Lecture 2
Properties of $\mathbb{R}$
Thursday 5 September 2019

Where to find course information

- The course web site: `http://ms.mcmaster.ca/earn/3A03`
- Click on Course information to download pdf file.
  - **Read it!!**
- Check the course web site regularly!
- **Assignment 1:** You should have received an e-mail from crowdmark. If not, please e-mail earn@math.mcmaster.ca ASAP stating your full name, student number, and when you registered in the course.

What we did last class

- The "Reals" ($\mathbb{R}$) are all the numbers that are needed to fill in the "number line" (so it has no "gaps" or "holes").
- The rationals ($\mathbb{Q}$) have "holes", *e.g.*, $\sqrt{2}$.
- Numbers can be constructed using sets. We will discuss this *informally*. A more formal approach is taken in Math 4L03 (Mathematical Logic) or in this online e-book.
  - The naturals ($\mathbb{N} = \{1, 2, 3, \ldots\}$) can be constructed from $\varnothing$: $0 = \varnothing,\ 1 = \{0\},\ 2 = \{0, 1\}, \ldots,\ n + 1 = n \cup \{n\}$.
  - The integers ($\mathbb{Z}$), and operations on them ($+, -, \cdot$), can also be constructed from sets and set operations (but we deferred that for later).
  - Given $\mathbb{N}$ and $\mathbb{Z}$, we can construct $\mathbb{Q}$...

Bonus participation marks via class polls

- Class polls are administered online at https://www.childsmath.ca/childsa/forms/main_login.php
- Click on Math 3A03, then Take Class Poll, then fill in the poll and $\boxed{\texttt{Submit}}$.
- If you participate in the polls, you can earn bonus marks in your final grade in the course. Your final grade will be increased by 1%, 2% or 3% depending how much you participate. If you participate in
  - 75–89% of class polls $\implies$ 1% bonus;
  - 90–94% of class polls $\implies$ 2% bonus;
  - $\geq$95% of class polls $\implies$ 3% bonus.
- *Note:* Bonus marks are entirely for participation. There are no marks associated with getting the right answer if there is one.

Poll

- Go to https://www.childsmath.ca/childsa/forms/main_login.php
- Click on Math 3A03
- Click on Take Class Poll
- Fill in poll **Lecture 2: Math Background**
- Submit.

Informal introduction to construction of numbers ($\mathbb{Q}$)

Rationals:

- *Idea:* Associate $\mathbb{Q}$ with $\mathbb{Z} \times \mathbb{N}$
- Use notation $\dfrac{a}{b} \in \mathbb{Q}$ if $(a, b) \in \mathbb{Z} \times \mathbb{N}$.
- Define equivalence of rational numbers:

$$\frac{a}{b} = \frac{c}{d} \quad \stackrel{\textbf{def}}{=} \quad a \cdot d = b \cdot c$$

- Define order for rational numbers:

$$\frac{a}{b} \leq \frac{c}{d} \quad \stackrel{\textbf{def}}{=} \quad a \cdot d \leq b \cdot c$$

Informal introduction to construction of numbers ($\mathbb{Q}$)

Rationals, continued:

- Define operations on rational numbers:

$$\frac{a}{b} + \frac{c}{d} \quad \stackrel{\text{def}}{=} \quad \frac{ad+bc}{bd}$$

$$\frac{a}{b} \cdot \frac{c}{d} \quad \stackrel{\text{def}}{=} \quad \frac{a \cdot c}{b \cdot d}$$

- Constructed in this way (ultimately from the empty set), $\mathbb{Q}$ satisfies all the standard properties we associate with the rational numbers.

- Formally, $\mathbb{Q}$ is a set of equivalence classes of $\mathbb{Z} \times \mathbb{N}$.
  Extra Challenge Problem: Are "$+$" and "$\cdot$" well-defined on $\mathbb{Q}$?

Properties of the rational numbers ($\mathbb{Q}$)

Addition:

A1 *Closed and commutative under addition.* For any $x, y \in \mathbb{Q}$ there is a number $x + y \in \mathbb{Q}$ and $x + y = y + x$.

A2 *Associative under addition.* For any $x, y, z \in \mathbb{Q}$ the identity

$$(x + y) + z = x + (y + z)$$

is true.

A3 *Existence and uniqueness of additive identity.* There is a unique number $0 \in \mathbb{Q}$ such that, for all $x \in \mathbb{Q}$,

$$x + 0 = 0 + x = x.$$

A4 *Existence of additive inverses.* For any number $x \in \mathbb{Q}$ there is a corresponding number denoted by $-x$ with the property that

$$x + (-x) = 0.$$

Properties of the rational numbers ($\mathbb{Q}$)

Multiplication:

- M1 *Closed and commutative under multiplication.* For any $x, y \in \mathbb{Q}$ there is a number $xy \in \mathbb{Q}$ and $xy = yx$.
- M2 *Associative under multiplication.* For any $x, y, z \in \mathbb{Q}$ the identity $(xy)z = x(yz)$ is true.
- M3 *Existence and uniqueness of multiplicative identity.* There is a unique number $1 \in \mathbb{Q} \setminus \{0\}$ such that, for all $x \in \mathbb{Q}$, $x1 = 1x = x$.
- M4 *Existence of multiplicative inverses.* For any non-zero number $x \in \mathbb{Q}$ there is a corresponding number denoted by x^{-1} with the property that $xx^{-1} = 1$.

Properties of the rational numbers ($\mathbb{Q}$)

Addition and multiplication together:

AM1 *Distributive law.* For any $x, y, z \in \mathbb{Q}$ the identity

$$(x + y)z = xz + yz$$

is true.

The 9 properties (A1–A4, M1–M4, AM1) make the rational numbers $\mathbb{Q}$ a ***field***.

Note: M3 ensures $0 \neq 1$ to exclude the uninteresting case of a field with only one element.

Standard Mathematical Shorthand

Quantifiers

$\forall$	for all
$\exists$	there exists
$\nexists$	there does not exist
$\exists!$	there exists a unique

Logical operands

$\wedge$	logical and
$\vee$	logical or
$\neg$	logical not
$\veebar$	logical exclusive or

Note: $A \veebar B \equiv (A \vee B) \wedge (\neg A \vee \neg B)$

Other shorthand

$\therefore$	therefore	$\because$	because
$\ni$	such that	$\iff$	if and only if
$\equiv$	equivalent	$\Rightarrow\Leftarrow$	contradiction

The field axioms (in mathematical shorthand) for field $\mathbb{F}$

Addition axioms

- **A1** *Closed, commutative.* $\forall x, y \in \mathbb{F}$, $\exists\,(x+y) \in \mathbb{F} \;\wedge\; (x+y) = (y+x)$.
- **A2** *Associative.* $\forall x, y, z \in \mathbb{F}$, $(x+y)+z = x+(y+z)$.
- **A3** *Identity.* $\exists!\; 0 \in \mathbb{F} \ni \forall x \in \mathbb{F}$, $x + 0 = 0 + x = x$.
- **A4** *Inverses.* $\forall x \in \mathbb{F},\; \exists\,(-x) \in \mathbb{F} \ni x + (-x) = 0$.

Multiplication axioms

- **M** *Closed, commutative.* $\forall x, y \in \mathbb{F}$, $\exists\,(xy) \in \mathbb{F} \;\wedge\; (xy) = (yx)$.
- **M** *Associative.* $\forall x, y, z \in \mathbb{F}$, $(xy)z = x(yz)$.
- **M** *Identity.* $\exists!\; 1 \in \mathbb{F}\setminus\{0\} \ni \forall x \in \mathbb{F},\; x1 = 1x = x$.
- **M** *Inverses.* $\forall x \in \mathbb{F}\setminus\{0\}$, $\exists\, x^{-1} \in \mathbb{F} \ni xx^{-1} = 1$.

Distribution axiom

- **AM** *Distribution.* $\forall x, y, z \in \mathbb{F},\; (x+y)z = xz + yz$.

Any collection $\mathbb{F}$ of mathematical objects is called a *field* if it satisfies these 9 algebraic properties.

Poll

- Go to https://www.childsmath.ca/childsa/forms/main_login.php
- Click on Math 3A03
- Click on Take Class Poll
- Fill in poll **Lecture 2: Which are Fields?**
- Submit.

The integers modulo 3 ($\mathbb{Z}_3$)

Imagine a clock that repeats after 3 hours rather than 12 hours.

$\mathbb{Z}_3$ contains the three elements $\{0, 1, 2\}$, with addition and multiplication defined as follows:

$+$	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

$\cdot$	0	1	2
0	0	0	0
1	0	1	2
2	0	2	1

Examples of fields

Set	Field?	Why?
rationals ($\mathbb{Q}$)	**YES**	
integers ($\mathbb{Z}$)	**NO**	no multiplicative inverses
reals ($\mathbb{R}$)	**YES**	
complexes ($\mathbb{C}$)	**YES**	
integers modulo 3 ($\mathbb{Z}_3$)	**YES**	$2^{-1} = 2$

Ordered fields

A field $\mathbb{F}$ is said to be ***ordered*** if the following properties hold:

Order axioms

O1 For any $x, y \in \mathbb{F}$, exactly one of the statements $x = y$, $x < y$ or $y < x$ is true ("***trichotomy***"), *i.e.,*
$\forall x, y \in \mathbb{F},\ \big((x = y) \wedge \neg(x < y) \wedge \neg(y < x)\big) \veebar \big((x \neq y) \wedge [(x < y) \veebar (y < x)]\big)$

O2 For any $x, y, z \in \mathbb{F}$, if $x < y$ is true and $y < z$ is true, then $x < z$ is true, *i.e.,* $\forall x, y, z \in \mathbb{F},\ (x < y) \wedge (y < z) \implies (x < z)$

O3 For any $x, y \in \mathbb{F}$, if $x < y$ is true, then $x + z < y + z$ is also true for any $z \in \mathbb{F}$, *i.e.,* $\forall x, y \in \mathbb{F},\ (x < y) \implies x + z < y + z,\ \forall z \in \mathbb{F}$

O4 For any $x, y, z \in \mathbb{F}$, if $x < y$ is true and $z > 0$ is true, then $xz < yz$ is also true,
i.e., $\forall x, y, z \in \mathbb{F},\ (x < y) \wedge (0 < z) \implies (xz < yz)$

Poll

- Go to https://www.childsmath.ca/childsa/forms/main_login.php
- Click on Math 3A03
- Click on Take Class Poll
- Fill in poll **Lecture 2: Which are ORDERED Fields?**
- $\boxed{\texttt{Submit}}$.

Examples of ordered fields

Field	Ordered?	Why?
rationals ($\mathbb{Q}$)	**YES**	
reals ($\mathbb{R}$)	**YES**	
integers modulo 3 ($\mathbb{Z}_3$)	**NO**	Next slide...
complexes ($\mathbb{C}$)	**NO**	Extra Challenge Problem: *Prove the field $\mathbb{C}$ cannot be ordered.*

The field of integers modulo 3 cannot be ordered

Proposition

$\mathbb{Z}_3$ is not an ordered field.

Proof.

Approach: proof by contradiction.

If $\mathbb{Z}_3$ is ordered, then O1 (trichotomy) implies that either $0 < 1$ or $1 < 0$ (and not both).

Suppose $0 < 1$ and $1 \not< 0$. Then O3 $\implies$ $0 + 1 < 1 + 1$,
i.e., $1 < 2$. $\therefore$ O2 (transitivity) $\implies$ $0 < 2$.
Using O3 again, we have $0 + 1 < 2 + 1$, *i.e.*, $1 < 0$. $\Rightarrow\Leftarrow$

Now suppose $1 < 0$. Similarly reach a contradiction (check!).
$\therefore$ $\mathbb{Z}_3$ cannot be ordered. $\square$

Food for thought: Is it possible for any finite field be ordered?

What other properties does $\mathbb{R}$ have?

- $\mathbb{R}$ is an ordered field.
- $\mathbb{R}$ includes numbers that are not in $\mathbb{Q}$, *e.g.*, $\sqrt{2}$.
- What additional properties does $\mathbb{R}$ have?
- Only one more property is required to fully characterize $\mathbb{R}$...
  It is related to *upper and lower bounds*...

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Mathematics and Statistics

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Mathematics 3A03
Real Analysis I

Instructor: David Earn

Lecture 3
Properties of $\mathbb{R}$ II
Friday 6 September 2019

Putnam Competition

- The William Lowell Putnam competition is a university-level mathematics competition held annually for undergraduate students at North American universities. It is organized by the Mathematical Association of America and is taken by over 4,000 participants at more than 500 colleges and universities. More information can be found at

  https://www.math.mcmaster.ca/undergraduate/undergrad-welcome.html

  Follow the Putnam competition link under "Useful Links" at the bottom of the page.
- This year's competition will occur on Saturday Dec. 7. If you are interested in participating or learning more, send email to David Earn, earn@math.mcmaster.ca or Bradd Hart, hartb@mcmaster.ca. In your e-mail ***please state what program and year you are in.***
- There will be an information session **Thursday**, Sept. 12 at 11:30am in HH-312.

Announcements and comments arising from Lecture 2

- My office hours are on Mondays 2:30pm–3:20pm *or by appointment (if you have a conflict on Mondays at 2:30).*
- Tutorials start next week.
- No claim is being made that the field axioms as stated are absolutely minimal (*i.e.*, that there are no redundancies). In fact, we don't need to assume:
  - Identities are unique.
  - Inverses are unique.
  - Commutivity under addition (!).

  Usually a slightly redundant set of axioms is stated to emphasize all the key properties.

More comments arising from Lecture 2

- The property that completes the specification of $\mathbb{R}$ has to somehow fill in all the “holes” in $\mathbb{Q}$.
- It is true that if $x, y \in \mathbb{Q}$ then $\exists r \in \mathbb{R} \setminus \mathbb{Q}$ with $x < r < y$. But this property is not sufficient to characterize $\mathbb{R}$, because it is satisfied by subsets of $\mathbb{R}$.
- To prove that $\mathbb{C}$ is not an ordered field, it is not sufficient to prove that the standard order on $\mathbb{R}$ cannot be extended to $\mathbb{C}$. You must show that it is not possible to define *any* order on $\mathbb{C}$ that makes it an ordered field.

Additional online resources

- Some "Logic Notes" are posted on the Tutorials page of the course web site.
- A sequence of 15 short (3–7 minute) videos covering the very basics of mathematical logic and theorem proving has been posted associated with a course at the University of Toronto:
  - Go to `http://uoft.me/MAT137`, click on the ***Videos*** tab and then on ***Playlist 1***.
  - These videos go at a slower pace than we do, and may be very helpful to you to get your head around the idea of a rigorous mathematical proof.

Bounds

Definition (Upper Bound)

Let $E \subseteq \mathbb{R}$. A number M is said to be an ***upper bound*** for E if $x \leq M$ for all $x \in E$.

A set that has an upper bound is said to be ***bounded above***.

Definition (Lower Bound)

Let $E \subseteq \mathbb{R}$. A number m is said to be a ***lower bound*** for E if $m \leq x$ for all $x \in E$.

A set that has a lower bound is said to be ***bounded below***.

A set that is bounded above and below is said to be ***bounded***.

Maxima and Minima

Definition (Maximum)

Let $E \subseteq \mathbb{R}$. A number M is said to be ***the maximum*** of E if M is an upper bound for E and $M \in E$. If such an M exists we write $M = \max E$.

Definition (Minimum)

Let $E \subseteq \mathbb{R}$. A number m is said to be ***the minimum*** of E if m is a lower bound for E and $m \in E$. If such an m exists we write $m = \min E$.

We refer to "the" maximum and "the" minimum of E because there cannot be more than one of each. *(Proof?)*

Poll

- Go to https://www.childsmath.ca/childsa/forms/main_login.php
- Click on Math 3A03
- Click on Take Class Poll
- Fill in poll **Lecture 3: bounded sets**
- Submit.

Bounds, maxima and minima

Example

Set	bounded below	bounded above	bounded	min	max
$[-1,1]$	**YES**	**YES**	**YES**	-1	1
$[-1,1)$	**YES**	**YES**	**YES**	-1	$\nexists$
$[-1,\infty)$	**YES**	**NO**	**NO**	-1	$\nexists$
$[-1,-\frac{1}{4})\cup(\frac{1}{2},1]$	**YES**	**YES**	**YES**	-1	1
$\mathbb{N}$	**YES**	**NO**	**NO**	1	$\nexists$
$\mathbb{R}$	**NO**	**NO**	**NO**	$\nexists$	$\nexists$
$\varnothing$	**YES**	**YES**	**YES**	$\nexists$	$\nexists$

Least upper bounds

Definition (Least Upper Bound/Supremum)

A number M is said to be the ***least upper bound*** or ***supremum*** of a set E if

(i) M is an upper bound of E, and
(ii) if $\widetilde{M}$ is an upper bound of E then $M \le \widetilde{M}$.

If M is the least upper bound of E then we write $M = \sup E$.

Note: We can refer to "the" least upper bound of E because there cannot be more than one. (Proof?)

What sets have least upper bounds?

Least upper bounds

Example

Set	bounded above	sup
$[-1, 1]$	**YES**	1
$[-1, 1)$	**YES**	1
$\varnothing$	**YES**	$\nexists$
$\{x \in \mathbb{R} : x^2 < 2\}$	**YES**	$\sqrt{2}$
$\{x \in \mathbb{Q} : x^2 < 2\}$	**YES**	$\notin \mathbb{Q}$

Least upper bounds

The property that any set that is bounded above has a least upper bound is what distinguishes the real numbers $\mathbb{R}$ from the rational numbers $\mathbb{Q}$.

Does this realization allow us to finish constructing $\mathbb{R}$?

YES, but we will delay the construction until later in the course.

For now, we will simply annoint the least upper bound property as an axiom:

Completeness Axiom

If $E \subseteq \mathbb{R}$, $E \neq \varnothing$, and E is bounded above, then E has a least upper bound (*i.e.*, $\sup E$ exists and $\sup E \in \mathbb{R}$).

$\mathbb{R}$ is a complete ordered field

- Any field $\mathbb{F}$ that satisfies the order axioms and the completeness axiom is said to be a ***complete ordered field***.
- $\mathbb{R}$ is a complete ordered field.
- Are there any other complete ordered fields?
- Extra Challenge Problem:
  Prove that $\mathbb{R}$ is the <u>only</u> complete ordered field.

Greatest lower bounds

Definition (Greatest Lower Bound/Infimum)

A number m is said to be the ***greatest lower bound*** or ***infimum*** of a set E if

(i) m is a lower bound of E, and

(ii) if $\widetilde{m}$ is a lower bound of E then $\widetilde{m} \leq m$.

If m is the greatest lower bound of E then we write $m = \inf E$.

Greatest lower bounds

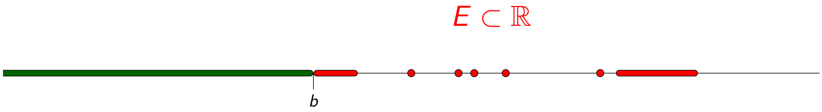
- The existence of least upper bounds was taken as an axiom.
- The existence of greatest lower bounds then follows.

Theorem

If $E \subseteq \mathbb{R}$, $E \neq \varnothing$, and E is bounded below, then E has a greatest lower bound (i.e., $\inf E$ exists and $\inf E \in \mathbb{R}$).

Proof?

Idea of proof:



$L = \{\ell \in \mathbb{R} : \ell \text{ is a lower bound of } E\}$

Greatest lower bounds

Theorem

If $E \subseteq \mathbb{R}$, $E \neq \varnothing$, and E is bounded below, then E has a greatest lower bound (i.e., $\inf E$ exists and $\inf E \in \mathbb{R}$).

Proof. *Recall* graphical idea of proof.

Let $L = \{\ell \in \mathbb{R} : \ell$ is a lower bound of $E\}$. Then:

- $L \neq \varnothing$ ($\because$ E is bounded below).
- L is bounded above ($\because x \in E \implies x$ an upper bound for L).
- $\therefore$ L has a least upper bound, say $b = \sup L$.

Now show $b = \inf E$. First show $b \in L$ (*i.e.*, $x \in E \implies b \leq x$). Suppose $x \in E$ and $b \not\leq x$; then by O1 (trichotomy), we must have $b > x$. Now $b = \sup L$ and $x < b$, so x is not an upper bound of L, *i.e.*, there is some $\ell \in L$ such that $x < \ell$. But then ℓ is not a lower bound of E. $\Rightarrow\Leftarrow$ $\therefore$ $b \in L$ and b is also $\max L$, *i.e.*, $b = \inf E$. $\square$

Comment on least upper bounds and greatest lower bounds

- The proof above shows that:

$$\inf E = \sup\{x \in \mathbb{R} : x \text{ is a lower bound of } E\}$$

- Similarly:

$$\sup E = \inf\{x \in \mathbb{R} : x \text{ is a upper bound of } E\}$$

Some notational abuse concerning sup and inf

By convention, for convenience, we (and your textbook) sometimes write:

$$\begin{aligned} \inf \mathbb{R} &= -\infty \\ \sup \mathbb{R} &= \infty \\ \inf \varnothing &= \infty \\ \sup \varnothing &= -\infty \end{aligned}$$

This is an ***abuse of notation***, since $\varnothing$ and $\mathbb{R}$ do not have least upper or greatest lower bounds in $\mathbb{R}$. ∞ *is not a real number.*

If you are asked "*What is the least upper bound of* $\mathbb{R}$*?*" how should you answer?
Correct answer: "$\mathbb{R}$ is not bounded above so it does not have a least upper bound."

Consequences of the real number axioms (§§1.7–1.9)

Theorem (Archimedean property)

The set of natural numbers $\mathbb{N}$ has no upper bound.

Proof.

Suppose $\mathbb{N}$ is bounded above. Then it has a least upper bound, say $B = \sup \mathbb{N}$. Thus, for all $n \in \mathbb{N}$, $n \leq B$. But if $n \in \mathbb{N}$ then $n + 1 \in \mathbb{N}$, hence $n + 1 \leq B$ for all $n \in \mathbb{N}$, *i.e.*, $n \leq B - 1$ for all $n \in \mathbb{N}$. Thus, $B - 1$ is an upper bound for $\mathbb{N}$, contradicting B being the least upper bound. $\square$

Consequences of the real number axioms (§§1.7–1.9)

Theorem (Equivalences of the Archimedean property)

1. *The set of natural numbers $\mathbb{N}$ has no upper bound.*
2. *Given any $x \in \mathbb{R}$, there exists $n \in \mathbb{N}$ such that $n > x$.*

   i.e., No matter how large a real number x is, there is always a natural number n that is larger.

3. *Given any $x > 0$ and $y > 0$, there exists $n \in \mathbb{N}$ such that $nx > y$.*

   i.e., Given any positive number y, no matter how large, and any positive number x, no matter how small, one can add x to itself sufficiently many times so that the result exceeds y (i.e., $nx > y$ for some $n \in \mathbb{N}$).

4. *Given any $x > 0$, there exists $n \in \mathbb{N}$ such that $\frac{1}{n} < x$.*

   i.e., Given any positive number x, no matter how small, one can always find a fraction $1/n$ that is smaller than x.