2 Differentiation

Differentiation

McMaster University

Mathematics and Statistics

$$\int_M d\omega = \int_{\partial M} \omega$$

Mathematics 3A03
Real Analysis I

Instructor: David Earn

Lecture 2
Differentiation
Wednesday 8 January 2025

Survey to do right now

- Please go to
  https://www.childsmath.ca/childsa/forms/main_login.php
- Click on Math 3A03
- Click on Take Class Poll
- Fill in poll **Survey 2**
- `Submit`.

Announcements

- Results of Survey 1
- Results of Survey 2

Background / reminder

Definition (Cauchy sequence)

A sequence $\{s_n\}$ is said to be a ***Cauchy sequence*** iff for all $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that if $m \geq N$ and $n \geq N$ then $|s_n - s_m| < \varepsilon$.

Poll: another background check

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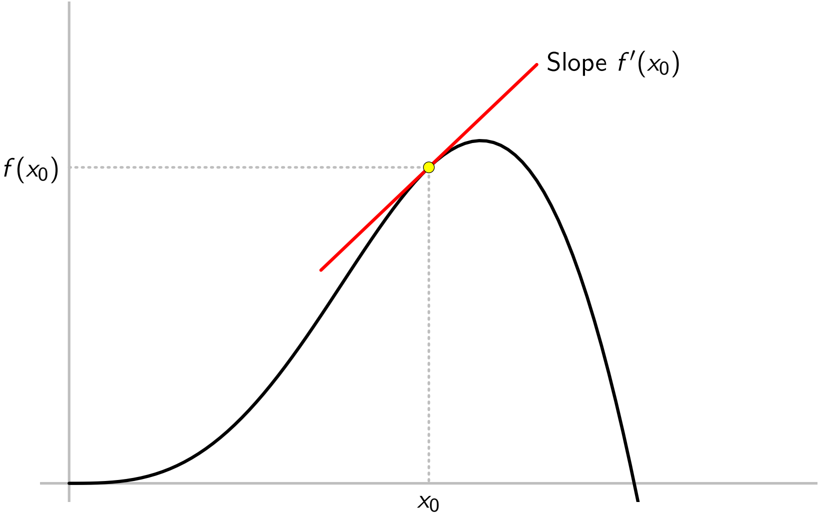
The Derivative



The Derivative

Definition (Derivative)

Let f be defined on an interval I and let $x_0 \in I$. The ***derivative*** of f at x_0, denoted by $f'(x_0)$, is defined as

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0},$$

provided either that this limit exists or is infinite. If $f'(x_0)$ is finite we say that f is ***differentiable*** at x_0. If f is differentiable at every point of a set $E \subseteq I$, we say that f is differentiable on E. If E is all of I, we simply say that f is a ***differentiable function***.

Note: "Differentiable" and "a derivative exists" always mean that the derivative is finite.

The Derivative

Example

$f(x) = x^2$. Find $f'(2)$.

$$f'(2) = \lim_{x\to2} \frac{x^2-4}{x-2} = \lim_{x\to2} \frac{(x+2)(x-2)}{x-2} = \lim_{x\to2} x+2 = 4$$

Note:

- In the first two limits, we must have $x \neq 2$.
- But in the third limit, we just plug in $x = 2$.
- Two things are equal, but in one $x \neq 2$ and in the other $x = 2$.
- Good illustration of why it is important to define the meaning of limits rigorously.

Poll

- Go to
  https://www.childsmath.ca/childsa/forms/main_login.php
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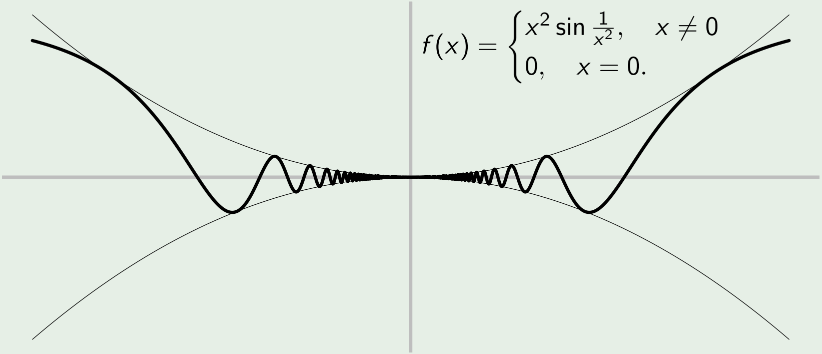
- Fill in poll **Derivatives: Differentiable at 0**
- `Submit`.

The Derivative

Example

Let f be defined in a neighbourhood I of 0, and suppose $|f(x)| \leq x^2$ for all $x \in I$. Is f necessarily differentiable at 0? *e.g.*,



$$f(x) = \begin{cases} x^2 \sin \frac{1}{x^2}, & x \neq 0 \\ 0, & x = 0. \end{cases}$$

The Derivative

Example (Trapping principle)

Suppose $f(x) = \begin{cases} x^2 \sin \frac{1}{x^2}, & x \neq 0 \\ 0, & x = 0. \end{cases}$ Then:

$$\forall x \neq 0 : \quad \left| \frac{f(x) - f(0)}{x - 0} \right| = \left| \frac{f(x)}{x} \right| = \left| \frac{x^2 \sin \frac{1}{x^2}}{x} \right| = \left| x \sin \frac{1}{x^2} \right| \leq |x|$$

Therefore:

$$|f'(0)| = \left| \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} \right| = \lim_{x \to 0} \left| \frac{f(x) - f(0)}{x - 0} \right| \leq \lim_{x \to 0} |x| = 0.$$

$\therefore$ f is differentiable at 0 and $f'(0) = 0$. □

The Derivative

Definition (One-sided derivatives)

Let f be defined on an interval I and let $x_0 \in I$. The ***right-hand derivative*** of f at x_0, denoted by $f'_+(x_0)$, is the limit

$$f'_+(x_0) = \lim_{x \to x_0^+} \frac{f(x) - f(x_0)}{x - x_0},$$

provided either that this one-sided limit exists or is infinite. Similarly, the ***left-hand derivative*** of f at x_0, denoted by $f'_-(x_0)$, is the limit

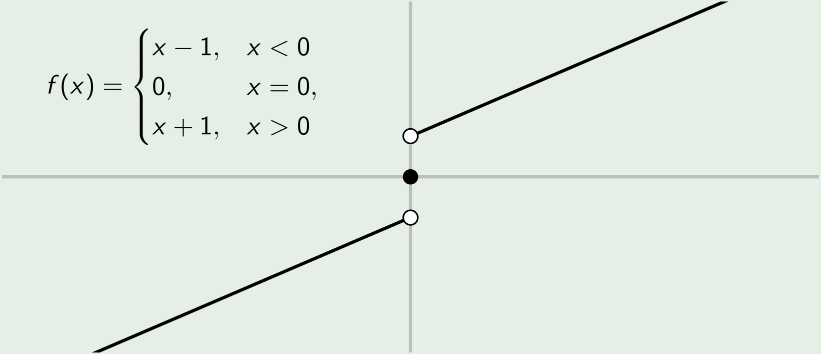
$$f'_-(x_0) = \lim_{x \to x_0^-} \frac{f(x) - f(x_0)}{x - x_0}.$$

Note: If x_0 is not an endpoint of the interval I then f is differentiable at x_0 iff $f'_+(x_0) = f'_-(x_0) \neq \pm\infty$.

The Derivative

Example

$$f(x) = \begin{cases} x-1, & x<0 \\ 0, & x=0, \\ x+1, & x>0 \end{cases}$$



- Same slope from left and right. Why isn't f differentiable???
- $\lim_{x\to 0^-} f'(x) = \lim_{x\to 0^+} f'(x) = \lim_{x\to 0} f'(x) = 1.$
- $f'_-(0) = f'_+(0) = f'(0) = \lim_{x\to 0} \frac{f(x)-f(0)}{x-0} = \infty.$

The Derivative

- Higher derivatives: we write
  - $f'' = (f')'$ if f' is differentiable;
  - $f^{(n+1)} = (f^{(n)})'$ if $f^{(n)}$ is differentiable.

- Other standard notation for derivatives:

$$\frac{df}{dx} = f'(x)$$

$$D = \frac{d}{dx}$$

$$D^n f(x) = \frac{d^n f}{dx^n} = f^{(n)}(x)$$

REMINDER: Algebra of limits

Theorem (Algebraic operations on limits of sequences)

Suppose $\{s_n\}$ *and* $\{t_n\}$ *are convergent sequences and* $C \in \mathbb{R}$.

1. $\lim_{n\to\infty} C\,s_n = C\left(\lim_{n\to\infty} s_n\right)$;
2. $\lim_{n\to\infty} (s_n + t_n) = \left(\lim_{n\to\infty} s_n\right) + \left(\lim_{n\to\infty} t_n\right)$;
3. $\lim_{n\to\infty} (s_n - t_n) = \left(\lim_{n\to\infty} s_n\right) - \left(\lim_{n\to\infty} t_n\right)$;
4. $\lim_{n\to\infty} (s_n t_n) = \left(\lim_{n\to\infty} s_n\right)\left(\lim_{n\to\infty} t_n\right)$;
5. *if* $t_n \neq 0$ *for all* n *and* $\lim_{n\to\infty} t_n \neq 0$ *then*
   $$\lim_{n\to\infty} \left(\frac{s_n}{t_n}\right) = \frac{\lim_{n\to\infty} s_n}{\lim_{n\to\infty} t_n} .$$

(TBB §2.7, and problem 2.7.4)

REMINDER: Algebra of limits

Theorem (Algebraic operations on limits of functions)

Suppose $f, g : \mathbb{R} \to \mathbb{R}$, $x_0 \in \mathbb{R}$, *the limits as* $x \to x_0$ *of* $f(x)$ *and* $g(x)$ *both exist, and* $C \in \mathbb{R}$.

1. $\lim_{x \to x_0} C\, f(x) = C\big(\lim_{x \to x_0} f(x)\big)$;
2. $\lim_{x \to x_0} (f(x) + g(x)) = \big(\lim_{x \to x_0} f(x)\big) + \big(\lim_{x \to x_0} g(x)\big)$;
3. $\lim_{x \to x_0} (f(x) - g(x)) = \big(\lim_{x \to x_0} f(x)\big) - \big(\lim_{x \to x_0} g(x)\big)$;
4. $\lim_{x \to x_0} (f(x) g(x)) = \big(\lim_{x \to x_0} f(x)\big)\big(\lim_{x \to x_0} g(x)\big)$;
5. *if* $g(x) \neq 0$ *for* $x \in (x_0 - \delta, x_0 + \delta)$ *for some* $\delta > 0$, *and* $\lim_{x \to x_0} g(x) \neq 0$ *then* $\lim_{x \to x_0} \left(\dfrac{f(x)}{g(x)}\right) = \dfrac{\lim_{x \to x_0} f(x)}{\lim_{x \to x_0} g(x)}$.

The Derivative

Theorem (Differentiable $\implies$ continuous)

If f is defined in a neighbourhood I of x_0 and f is differentiable at x_0 then f is continuous at x_0.

Proof.

Must show $\lim_{x \to x_0} f(x) = f(x_0)$, *i.e.*, $\lim_{x \to x_0} (f(x) - f(x_0)) = 0$.

$$\begin{aligned}
\lim_{x \to x_0} (f(x) - f(x_0)) &= \lim_{x \to x_0} \left(\frac{f(x) - f(x_0)}{x - x_0} \times (x - x_0) \right) \\
&= \lim_{x \to x_0} \left(\frac{f(x) - f(x_0)}{x - x_0} \right) \times \lim_{x \to x_0} (x - x_0) \\
&= f'(x_0) \times 0 = 0 \,,
\end{aligned}$$

where we have used the theorem on the algebra of limits. □