19 Sequences and Series of Functions

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McMaster University

Mathematics and Statistics

$$\int_M d\omega = \int_{\partial M} \omega$$

Mathematics 3A03
Real Analysis I

Instructor: David Earn

Lecture 19
Sequences and Series of Functions
Friday 28 February 2025

Announcements

- New, exciting topic today...

Sequences and Series of Functions

Limits of Functions

We know that it can be useful to represent functions as limits of other functions.

Example

The power series expansion

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

expresses the exponential e^x as a certain limit of the functions

$$1, \qquad 1+\frac{x}{1!}, \qquad 1+\frac{x}{1!}+\frac{x^2}{2!}, \qquad 1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}, \qquad \cdots$$

Our goal is to give meaning to the phrase "*limit of functions*", and discuss how functions behave under limits.

Pointwise Convergence

- There are multiple inequivalent ways to define the limit of a sequence of functions.
- Consequently, there are multiple different notions of what it means for a sequence of functions to converge.
- Some convergence notions are better behaved than others.

We will begin with the simplest notion of convergence.

Definition (Pointwise Convergence)

Suppose $\{f_n\}$ is a sequence of functions defined on a domain $D \subseteq \mathbb{R}$, and let f be another function defined on D. Then $\{f_n\}$ ***converges pointwise on*** D ***to*** f if, for every $x \in D$, the sequence $\{f_n(x)\}$ of real numbers converges to $f(x)$.

What useful properties of functions does *pointwise convergence* preserve?

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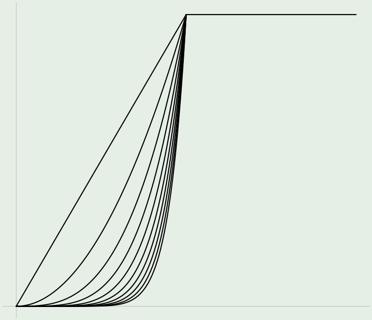
Pointwise Convergence

Example

$$f_n(x) = \begin{cases} x^n & 0 \le x \le 1, \\ 1 & x \ge 1. \end{cases}$$

$$\lim_{n\to\infty} f_n(x) = \begin{cases} 0 & 0 \le x < 1 \\ 1 & x \ge 1 \end{cases}$$



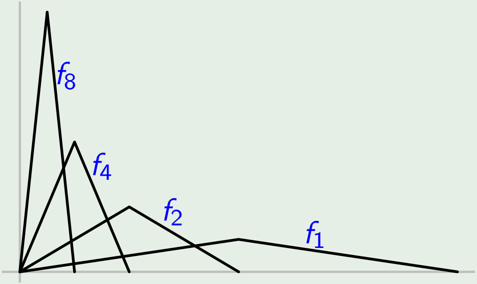
- The limit of this sequence (of continuous functions) is not continuous.
- If we smooth the corner of $f_n(x)$ at $x = 1$, we get a sequence of differentiable functions that converge to a function that is not even continuous.

Pointwise Convergence

Example

Define $f_n(x)$ on $[0,1]$ as follows:

$$f_n(x) = \begin{cases} 2n^2x, & 0 \le x \le \frac{1}{2n} \\ 2n - 2n^2x, & \frac{1}{2n} \le x \le \frac{1}{n} \\ 0, & x \ge \frac{1}{n}. \end{cases}$$



$$\lim_{n\to\infty} f_n(x) = 0 \quad \forall\, x$$

$$\int_0^1 f_n = \frac{1}{2} \quad \forall\, n \in \mathbb{N}$$

$$\int_0^1 \lim_{n\to\infty} f_n = 0$$

Uniform Convergence

A much better behaved notion of convergence is the following.

Definition ($f_n \to f$ uniformly)

Suppose $\{f_n\}$ is a sequence of functions defined on a domain $D \subseteq \mathbb{R}$, and let f be another function defined on D. Then $\{f_n\}$ ***converges uniformly on D to f*** if, for every $\varepsilon > 0$, there is some $N \in \mathbb{N}$ so that, for all $x \in D$,

$$n \geq N \implies |f_n(x) - f(x)| < \varepsilon.$$

Note that $\{f_n\}$ converges uniformly to f if and only if $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ such that

$$n \geq N \implies \sup_{x \in D} |f_n(x) - f(x)| < \varepsilon.$$

uniform convergence $\underset{\not\Longleftarrow}{\Longrightarrow}$ pointwise convergence

Uniform Convergence

The sense in which uniform convergence is better behaved than pointwise convergence is that it does preserve at least some properties of the sequence of functions.

Which properties?

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Uniform Convergence

Theorem (Continuity and Uniform Convergence)

Suppose $\{f_n\}$ is a sequence of functions that converges uniformly on $[a, b]$ to f. If each f_n is continuous on $[a, b]$, then f is continuous on $[a, b]$.

What should our proof strategy be?

Our goal is to show that the limit function f is continuous for all $x \in [a, b]$. So given $x \in [a, b]$, we must show that for any $\varepsilon > 0$ we can find a small enough neighbourhood of x, say $(x - \delta, x + \delta)$ for some small δ, such that $|f(x) - f(y)| < \varepsilon$ if $y \in (x - \delta, x + \delta)$, *i.e.*, if $|x - y| < \delta$.

Somehow we have to manage this using the facts that (i) each f_n is continuous and (ii) $f_n \to f$ uniformly.

The key is that (for any n) if x and y are close then $f_n(x)$ and $f_n(y)$ are close, and, if n is large enough, f_n is (uniformly) close to f throughout $[a, b]$, so continuity is "passed through" to the limit.

Let's make this precise...

Uniform Convergence

Proof: f_n continuous $\forall n$ and $f_n \to f$ uniformly $\implies$ f continuous.

Fix $x \in [a,b]$ and $\varepsilon > 0$. We must show $\exists \delta > 0$ such that if $y \in [a,b]$ and $|x-y| < \delta$ then $|f(x) - f(y)| < \varepsilon$.

Since $f_n \to f$ uniformly, $\exists N \in \mathbb{N} \ni |f_N(y) - f(y)| < \frac{\varepsilon}{3} \ \forall y \in [a,b]$ (in particular, $x \in [a,b]$, so we have $|f_N(x) - f(x)| < \frac{\varepsilon}{3}$).
Fix such an integer N.

Since f_N is continuous, there is some $\delta > 0$ such that if $y \in [a,b]$ satisfies $|x-y| < \delta$, then $|f_N(x) - f_N(y)| < \frac{\varepsilon}{3}$. For such y, we then have

$$\begin{aligned} |f(x) - f(y)| &= |f(x) - f_N(x) + f_N(x) - f_N(y) + f_N(y) - f(y)| \\ &\le |f(x) - f_N(x)| + |f_N(x) - f_N(y)| + |f_N(y) - f(y)| \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon, \end{aligned}$$

as required. □

Uniform Convergence

Theorem (Integrability and Uniform Convergence)

Suppose $\{f_n\}$ is a sequence of functions that converges uniformly on $[a, b]$ to f. If each f_n is integrable on $[a, b]$, then f is integrable and

$$\int_a^b f = \lim_{n \to \infty} \int_a^b f_n .$$