# 34 What is $\mathbb{R}$?

Mathematics
and Statistics

$$\int_M d\omega = \int_{\partial M} \omega$$

# Mathematics 3A03
# Real Analysis I

Instructor: David Earn

Lecture 34
What is $\mathbb{R}$?
Wednesday 3 April 2019

# Please consider. . .

**5 minute** *Student Respiratory Illness Survey:*

https://surveys.mcmaster.ca/limesurvey/index.php/893454

> *Please complete this anonymous survey to help us monitor the patterns of respiratory illness, over-the-counter drug use, and social contact within the McMaster community. There are no risks to filling out this survey, and your participation is voluntary. You do not need to answer any questions that make you uncomfortable, and all information provided will be kept strictly confidential. Thanks for participating.*
>
> *–Dr. Marek Smieja (Infectious Diseases)*

# What exactly is $\mathbb{R}$ ?

# Construction of the Real Numbers

- Recall that we defined the natural numbers $\mathbb{N}$ as sets: $0 \equiv \varnothing$, $1 \equiv \{0\}$, $2 \equiv \{0,1\}$, *etc.*
- For $m, n \in \mathbb{N}$ we defined $m < n$ to mean $m \subset n$.
- We defined the rational numbers $\mathbb{Q}$ to be ordered pairs of integers (more precisely, $\mathbb{Q}$ is a set of equivalence classes of $\mathbb{Z} \times \mathbb{N}$).
- In the same spirit, we can define real numbers not by determining what they "really are" but instead by settling for a definition that determines their mathematical properties completely.
- So, just as $\mathbb{Z}$ can be built from $\mathbb{N}$, and $\mathbb{Q}$ can be built from $\mathbb{Z}$, we can build $\mathbb{R}$ from $\mathbb{Q}$.
- Richard Dedekind's idea was to construct a real number $\alpha$ as a set of rational numbers, in a way that naturally yields the one property of $\mathbb{R}$ that $\mathbb{Q}$ does not have: least upper bounds. . .

# Construction of the Real Numbers

Dedekind's stroke of genius (on 24 Nov 1858) was to define $\alpha$ as "*the set of rational numbers less than* $\alpha$" in a way that is not circular.

## Definition (Real number)

A ***real number*** is a set $\alpha \subseteq \mathbb{Q}$, with the following four properties:

1. $\forall x \in \alpha$, if $y \in \mathbb{Q}$ and $y < x$, then $y \in \alpha$;
2. $\alpha \neq \varnothing$;
3. $\alpha \neq \mathbb{Q}$;
4. there is no greatest element in $\alpha$,
   *i.e.*, if $x \in \alpha$ then $\exists\, y \in \alpha$ such that $y > x$.

The set of all real numbers is denoted by $\mathbb{R}$.

*Historical note:* Dedekind originally defined a real number $\alpha$ as the pair of sets $(L, R)$ where $L$ is the set of rationals $< \alpha$ and $R$ is the set of rationals $\geq \alpha$. A real number is then described as a ***Dedekind cut***.

# Construction of the Real Numbers

*Example:* $\sqrt{2} = \{q \in \mathbb{Q} : q^2 < 2 \text{ or } q < 0\}$.

With real numbers defined, we can easily define an ordering on $\mathbb{R}$.

**Definition (Order of real numbers)**

If $\alpha, \beta \in \mathbb{R}$ then $\alpha < \beta$ iff $\alpha \subset \beta$. (Similarly for $>$, $\leq$, and $\geq$.)

We now have enough to prove:

**Theorem ($\mathbb{R}$ is complete)**

*If $A \subset \mathbb{R}$, $A \neq \varnothing$, and $A$ is bounded above, then $A$ has a least upper bound.*

We also need to define $+$, $\cdot$, 1 and $\alpha^{-1}$.

Then we can prove that $\mathbb{R}$ *is a complete ordered field* and, moreover, it is the *unique* such field (up to isomorphism).

# Construction of the Real Numbers

## Proof that $\mathbb{R}$ is complete.

Let $\beta = \{x : x \in \alpha \text{ for some } \alpha \in A\} = \bigcup_{\alpha \in A} \alpha$.

Since each $x \in \beta$ is in some set $\alpha \subseteq \mathbb{Q}$, we have $\beta \subseteq \mathbb{Q}$. To verify that $\beta \in \mathbb{R}$, we check the four defining properties:

1. Suppose (i) $x \in \beta$ and (ii) $y < x$. (i) $\implies$ $x \in \alpha$ for some $\alpha \in A$. But $\alpha$ is a real number, so (ii) $\implies$ $y \in \alpha$. Hence $y \in \beta$.
2. Since $A \neq \varnothing$, $\exists \alpha \in A$. Since $\alpha$ is a real number, $\exists x \in \alpha$. This implies $x \in \beta$, so $\beta \neq \varnothing$.
3. Since $A$ is bounded above, there is some real number $\gamma$ such that $\alpha < \gamma$ for every $\alpha \in A$. Since $\gamma$ is a real number, there is some rational number $x \notin \gamma$. But $\alpha < \gamma$ means that $\alpha \subset \gamma$, so it follows that $x \notin \alpha$ for any $\alpha \in A$. This implies $x \notin \beta$, so $\beta \neq \mathbb{Q}$.

*. . . continued. . .*

# Construction of the Real Numbers

**Proof that $\mathbb{R}$ is complete (*continued*).**

4. Suppose $x \in \beta$. Then $x \in \alpha$ for some $\alpha \in A$. Since $\alpha$ does not have a greatest element, $\exists y \in \mathbb{Q}$ with $x < y$ and $y \in \alpha$. But this implies $y \in \beta$; thus $\beta$ does not have a greatest element.

These four points establish that $\beta$ is a real number. It remains to show that $\beta$ is the least upper bound of $A$.

If $\alpha \in A$, then $\alpha \subseteq \beta$, *i.e.*, $\alpha \leq \beta$, so $\beta$ is an upper bound for $A$. On the other hand, if $\gamma$ is an upper bound for $A$, then $\alpha \leq \gamma$ for every $\alpha \in A$; this implies $\alpha \subseteq \gamma$, for every $\alpha \in A$, and hence $\beta \subseteq \gamma$, *i.e.*, $\beta \leq \gamma$. Thus $\beta$ is the least upper bound of $A$. □

# Next time. . .

**An alternative construction of $\mathbb{R}$. . .**

**And much much more. . .**

# Surreal numbers. . .

## . . . Guest lecture by Dr. Jonathan Dushoff

# Winter 2019 Course Evaluations

**Open:** Wednesday March 27, 10:00AM
**Close:** Wednesday April 10, 11:59PM

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