Differentiation

McMaster University

Mathematics and Statistics

$$\int_M d\omega = \int_{\partial M} \omega$$

Mathematics 3A03
Real Analysis I

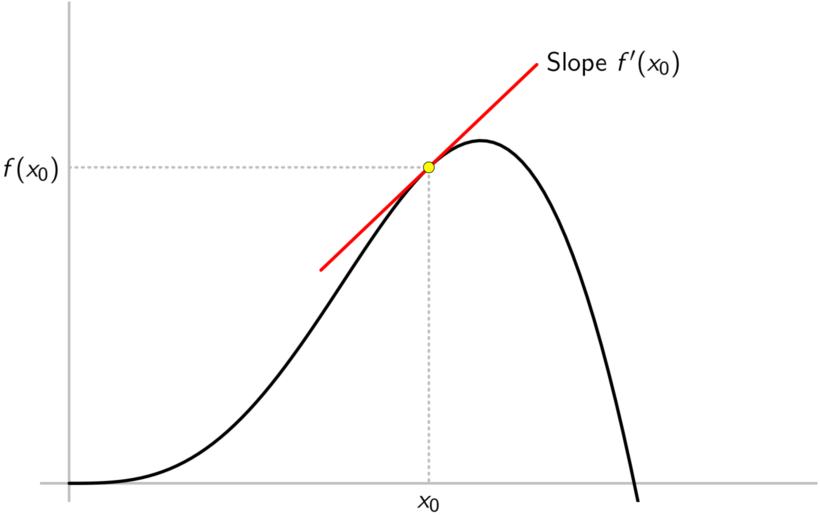
Instructor: David Earn

Lecture 24
Differentiation
Monday 11 March 2019

Announcements

- Assignment 5 will be posted soon.

The Derivative



The Derivative

Definition (Derivative)

Let f be defined on an interval I and let $x_0 \in I$. The **derivative** of f at x_0, denoted by $f'(x_0)$, is defined as

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \,,$$

provided either that this limit exists or is infinite. If $f'(x_0)$ is finite we say that f is **differentiable** at x_0. If f is differentiable at every point of a set $E \subseteq I$, we say that f is differentiable on E. If E is all of I, we simply say that f is a **differentiable function**.

Note: "Differentiable" and "a derivative exists" always mean that the derivative is finite.

The Derivative

Example

$f(x) = x^2$. Find $f'(2)$.

$$f'(2) = \lim_{x\to 2} \frac{x^2 - 4}{x - 2} = \lim_{x\to 2} \frac{(x+2)(x-2)}{x-2} = \lim_{x\to 2} x + 2 = 4$$

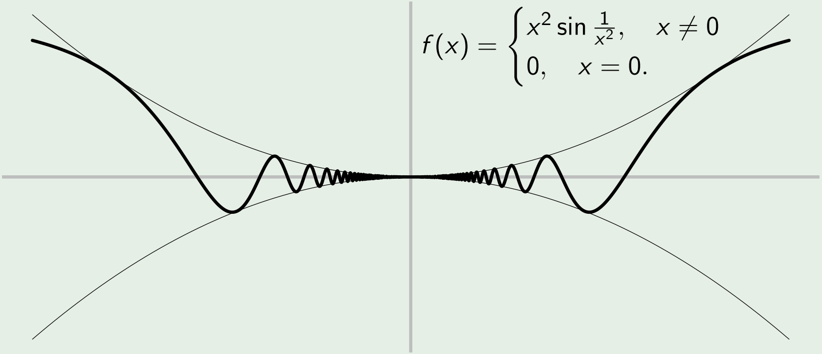
Note:

- In the first two limits, we must have $x \neq 2$.
- But in the third limit, we just plug in $x = 2$.
- Two things are equal, but in one $x \neq 2$ and in the other $x = 2$.
- Good illustration of why it is important to define the meaning of limits rigorously.

The Derivative

Example

Let f be defined in a neighbourhood I of 0, and suppose $|f(x)| \leq x^2$ for all $x \in I$. Is f necessarily differentiable at 0? *e.g.*,



$$f(x) = \begin{cases} x^2 \sin \frac{1}{x^2}, & x \neq 0 \\ 0, & x = 0. \end{cases}$$

(solution on board)

The Derivative

Definition (One-sided derivatives)

Let f be defined on an interval I and let $x_0 \in I$. The **right-hand derivative** of f at x_0, denoted by $f'_+(x_0)$, is the limit

$$f'_+(x_0) = \lim_{x \to x_0^+} \frac{f(x) - f(x_0)}{x - x_0},$$

provided either that this one-sided limit exists or is infinite. Similarly, the **left-hand derivative** of f at x_0, denoted by $f'_-(x_0)$, is the limit

$$f'_-(x_0) = \lim_{x \to x_0^-} \frac{f(x) - f(x_0)}{x - x_0}.$$

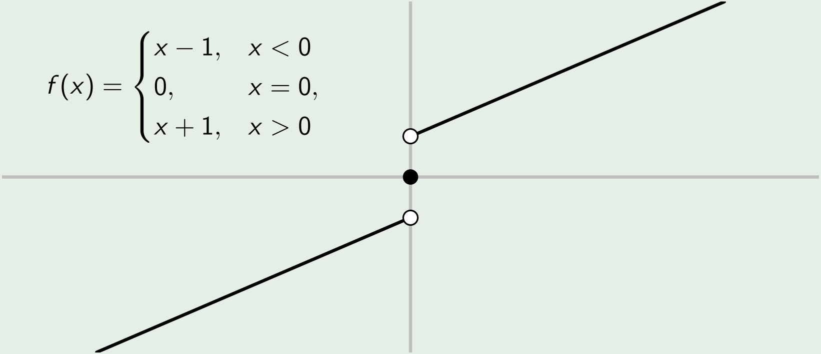
Note:

If $x_0 \in I^\circ$ then f is differentiable at x_0 iff $f'_+(x_0) = f'_-(x_0) \neq \pm\infty$.

The Derivative

Example

$$f(x) = \begin{cases} x-1, & x<0 \\ 0, & x=0, \\ x+1, & x>0 \end{cases}$$



- Same slope from left and right. Why isn't f differentiable???
- $\lim_{x\to 0^-} f'(x) = \lim_{x\to 0^+} f'(x) = \lim_{x\to 0} f'(x) = 1.$
- $f'_-(0) = f'_+(0) = f'(0) = \lim_{x\to 0} \frac{f(x)-f(0)}{x-0} = \infty.$

The Derivative

- Higher derivatives: we write
  - $f'' = (f')'$ if f' is differentiable;
  - $f^{(n+1)} = (f^{(n)})'$ if $f^{(n)}$ is differentiable.

- Other standard notation for derivatives:

$$\frac{df}{dx} = f'(x)$$

$$D = \frac{d}{dx}$$

$$D^n f(x) = \frac{d^n f}{dx} = f^{(n)}(x)$$

The Derivative

Theorem (Differentiable $\implies$ continuous)

If f is defined in a neighbourhood I of x_0 and f is differentiable at x_0 then f is continuous at x_0.

Proof.

Must show $\lim_{x\to x_0} f(x) = f(x_0)$, *i.e.*, $\lim_{x\to x_0} \big(f(x) - f(x_0)\big) = 0$.

$$\begin{aligned}
\lim_{x\to x_0} \big(f(x) - f(x_0)\big) &= \lim_{x\to x_0} \left(\frac{f(x) - f(x_0)}{x - x_0} \times (x - x_0)\right) \\
&= \lim_{x\to x_0} \left(\frac{f(x) - f(x_0)}{x - x_0}\right) \times \lim_{x\to x_0} (x - x_0) \\
&= f'(x_0) \times 0 = 0\,,
\end{aligned}$$

where we have used the theorem on the algebra of limits. □

The Derivative

Theorem (Algebra of derivatives)

Supppose f and g are defined on an interval I and $x_0 \in I$. If f and g are differentiable at x_0 then $f+g$ and fg are differentiable at x_0. If, in addition, $g(x_0) \neq 0$ then f/g is differentiable at x_0. Under these conditions:

1. $(cf)'(x_0) = cf'(x_0)$ *for all* $c \in \mathbb{R}$;
2. $(f+g)'(x_0) = \big(f'+g'\big)(x_0)$;
3. $(fg)'(x_0) = \big(f'g + fg'\big)(x_0)$;
4. $\left(\frac{f}{g}\right)'(x_0) = \left(\frac{gf'-fg'}{g^2}\right)(x_0) \qquad \big(g(x_0) \neq 0\big)$.

(Textbook (TBB) Theorem 7.7, p. 408)

The Derivative

Theorem (Chain rule)

Suppose f is defined in a neighbourhood U of x_0 and g is defined in a neighbourhood V of $f(x_0)$ such that $f(U) \subseteq V$. If f is differentiable at x_0 and g is differentiable at $f(x_0)$ then the composite function $h = g \circ f$ is differentiable at x_0 and

$$h'(x_0) = (g \circ f)'(x_0) = g'\big(f(x_0)\big) f'(x_0)\,.$$

(Textbook (TBB) §7.3.2, p. 411)

TBB provide a very good motivating discussion of this proof, which is quite technical.

The Derivative

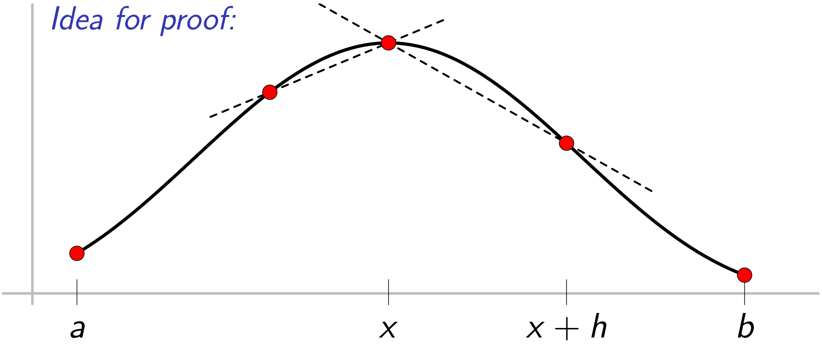
Theorem (Derivative at local extrema)

Let $f : (a, b) \to \mathbb{R}$. If x is a maximum or minimum point of f in (a, b), and f is differentiable at x, then $f'(x) = 0$.

(Textbook (TBB) Theorem 7.18, p. 424)

Note: f need not be differentiable or even continuous at other points.

Idea for proof:



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Mathematics 3A03
Real Analysis I

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Lecture 25
Differentiation II
Wednesday 13 March 2019

Announcements

- Part of Assignment 5 is posted on the course web site (*more to come*). It is due on **Monday 25 March 2019 @ 11:30am via crowdmark**.
- Test 2 is on **Monday 1 April 2019, 7:00pm–8:30pm in MDCL 1110.**
- Assignment 6 will be due on **Monday 8 April 2019 @ 11:30am via crowdmark**.
- Final exam on **Monday 15 April 2019 @ 4:00pm in IWC/2**.

Last time. . .

- Definition of the derivative.
- Proved differentiable $\Longrightarrow$ continuous.
- Discussed algebra of derivatives and chain rule.
- Pictorial argument that derivative is zero at extrema.
- Defined one-sided derivatives
  - Example

The Mean Value Theorem

Theorem (Rolle's theorem)

If f is continuous on $[a,b]$ and differentiable on (a,b), and $f(a)=f(b)$, then there exists $x \in (a,b)$ such that $f'(x)=0$.

Proof.

f continuous on $[a,b] \implies f$ has a max and min value on $[a,b]$. If either a max or min occurs at $x \in (a,b)$ then $f'(x)=0$. If no max or min occurs in (a,b) then they must both occur at the endpoints, a and b. But $f(a)=f(b)$, so f is constant. Hence $f'(x)=0\ \forall x \in (a,b)$. □

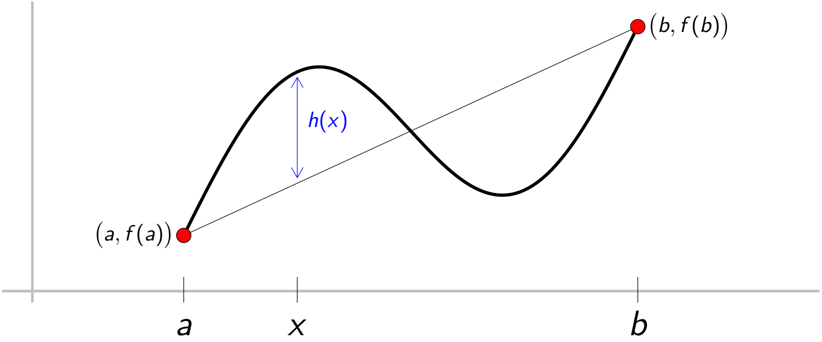
Theorem (Mean value theorem)

If f is continuous on $[a,b]$ and differentiable on (a,b) then there exists $x \in (a,b)$ such that

$$f'(x) = \frac{f(b)-f(a)}{b-a}\,.$$

The Mean Value Theorem

Idea for proof:



Proof.

Apply Rolle's theorem to

$$h(x) = f(x) - \left[f(a) + \left(\frac{f(b) - f(a)}{b - a} \right)(x - a) \right] .$$

□

The Mean Value Theorem

Example

$f'(x) > 0$ on an interval $I \implies f$ strictly increasing on I.

Proof:

Suppose $x_1, x_2 \in I$ and $x_1 < x_2$. We must show $f(x_1) < f(x_2)$.

Since $f'(x)$ exists for all $x \in I$, f is certainly differentiable on the closed subinterval $[x_1, x_2]$.

Hence by the Mean Value Theorem $\exists x_* \in (x_1, x_2)$ such that

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(x_*)\,.$$

But $x_2 - x_1 > 0$ and since $x_* \in I$, we know $f'(x_*) > 0$.

$\therefore\ f(x_2) - f(x_1) > 0,\qquad$ *i.e.,* $\quad f(x_1) < f(x_2)$. □

Intermediate value property for derivatives

Theorem (Darboux's Theorem: IVP for derivatives)

If f is differentiable on an interval I then its derivative f' has the intermediate value property on I.

Notes:

- It is f', not f, that is claimed to have the intermediate value property in Darboux's theorem. This theorem does not follow from the standard intermediate value theorem because the derivative f' is not necessarily continuous.
- *Equivalent (contrapositive) statement of Darboux's theorem:* If a function does not have the intermediate value property on I then it is impossible that it is the derivative of any function on I.
- Darboux's theorem implies that a derivative cannot have jump or removable discontinities. Any discontuity of a derivative must be essential. Recall example of a discontinuous function with IVP.

Intermediate value property for derivatives

Proof of Darboux's Theorem.

Consider $a, b \in I$ with $a < b$.
Suppose first that $f'(a) < 0 < f'(b)$. We will show $\exists x \in (a, b)$ such that $f'(x) = 0$. Since f' exists on $[a, b]$, we must have f continuous on $[a, b]$, so the Extreme Value Theorem implies that f attains its minimum at some point $x \in [a, b]$. This minimum point cannot be an endpoint of $[a, b]$ ($x \neq a$ because $f'(a) < 0$ and $x \neq b$ because $f'(b) > 0$). Therefore, $x \in (a, b)$. But f is differentiable everywhere in (a, b), so, by the theorem on the derivative at local extrema, we must have $f'(x) = 0$.
Now suppose more generally that $f'(a) < K < f'(b)$. Let $g(x) = f(x) - Kx$. Then g is differentiable on I and $g'(x) = f'(x) - K$ for all $x \in I$. In addition, $g'(a) = f'(a) - K < 0$ and $g'(b) = f'(b) - K > 0$, so by the argument above, $\exists x \in (a, b)$ such that $g'(x) = 0$, *i.e.*, $f'(x) - K = 0$, *i.e.*, $f'(x) = K$.
The case $f'(a) > K > f'(b)$ is similar. □

Intermediate value property for derivatives

Example ($f'(x) \neq 0\ \forall x \in I \implies f \nearrow$ or $\searrow$ on I)

If f is differentiable on an interval I and $f'(x) \neq 0$ for all $x \in I$ then f is either increasing or decreasing on the entire interval I.

Proof:
Suppose $\exists a, b \in I$ such that $f'(a) < 0$ and $f'(b) > 0$.

Then, from Darboux's theorem, $\exists c \in I$ such that $f'(c) = 0$. $\Rightarrow\Leftarrow$

$\therefore$ Either "$\exists a \in I \ni f'(a) < 0$" is FALSE
or "$\exists b \in I \ni f'(b) > 0$" is FALSE.

$\therefore$ Since we know $f'(x) \neq 0\ \forall x \in I$, it must be that
either $f'(x) > 0\ \forall x \in I$ or $f'(x) < 0\ \forall x \in I$,
i.e., either f is increasing on I or decreasing on I. □

The Derivative of an Inverse

Example (Sufficient condition for *non*-differentiable inverse)

Suppose f is continuous and one-to-one on an interval I. If $x \in I$ and $f'(x) = 0$ then f^{-1} is not differentiable at $y = f(x)$.

Proof: By definition, the inverse function satisfies

$$f(f^{-1}(y)) = y\,.$$

Suppose that f is differentiable at y. Then, by the Chain Rule,

$$f'(f^{-1}(y)) \cdot (f^{-1})'(y) = 1\,.$$

But $f^{-1}(y) = x$, and $f'(x) = 0$, so

$$0 \cdot (f^{-1})'(y) = 1\,,$$

which is impossible! $\Rightarrow\Leftarrow$. $\square$

The Derivative of an Inverse

Theorem (Inverse function theorem)

If f is differentiable on an interval I and $f'(x) \neq 0$ $\forall x \in I$, then

1. *f is one-to-one on I;*
2. *f^{-1} is differentiable on $J = f(I)$;*
3. $$(f^{-1})'\big(f(x)\big) = \frac{1}{f'(x)} \quad \text{for all } x \in I,$$
   i.e., $$(f^{-1})'(y) = \frac{1}{f'\big((f^{-1}(y)\big)} \quad \text{for all } y \in J.$$

(Textbook (TBB) Theorem 7.32, p. 445)

The Derivative of an Inverse

Key insights for proof of inverse function theorem:

- Darboux's theorem $\implies$ $f \nearrow$ or $\searrow$ on I $\implies$ f is $1:1$ on I
- If $\quad y = f(x)$ and $y_0 = f(x_0)$

  then $\quad x = f^{-1}(y)$ and $x_0 = f^{-1}(y_0)$,

$$\text{so} \quad \frac{f^{-1}(y) - f^{-1}(y_0)}{y - y_0} = \frac{x - x_0}{f(x) - f(x_0)} = \frac{1}{\frac{f(x)-f(x_0)}{x-x_0}}\,.$$

- Since f continuous at x_0, we know $x \to x_0 \implies y \to y_0$.
- But we need $y \to y_0 \implies x \to x_0$, *i.e.*, f^{-1} continuous at y_0.
- In fact, f continuous and either $\nearrow$ or $\searrow$ on $I \implies f^{-1}$ continuous on $J = f(I)$. (more generally, *cf.* Invariance of Domain thm)