Continuous Functions

McMaster University

Mathematics and Statistics

$$\int_M d\omega = \int_{\partial M} \omega$$

Mathematics 3A03
Real Analysis I

Instructor: David Earn

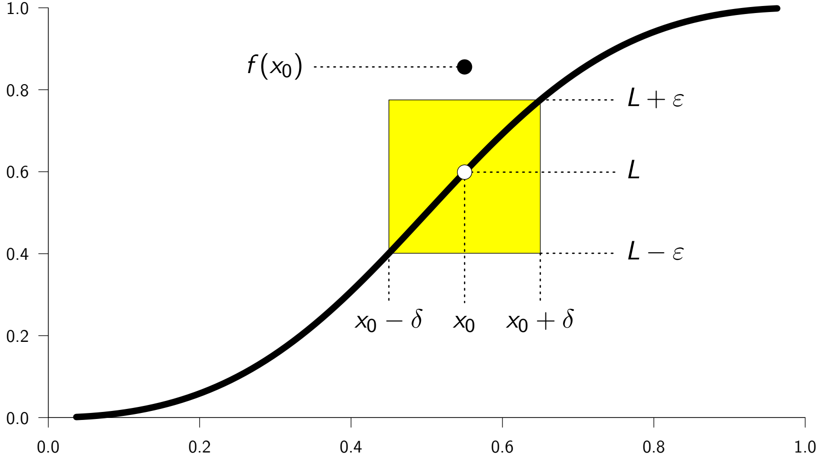
Lecture 18
Continuity
Monday 25 February 2019

Announcements

- A preliminary version of Assignment 4 has been posted on the course web site. More problems will be added soon.
  Due Friday 8 March 2019 at 1:25pm via crowdmark.
  BUT you should do **questions 1 and 2** before Test #1.

- **Math 3A03 Test #1**
  Monday 4 March 2019 at 7:00pm in MDCL 1110
  (room is booked for 90 minutes; you should not feel rushed)

Limits of functions



Limits of functions

Definition (Limit of a function on an interval (a, b))

Let $a < x_0 < b$ and $f : (a, b) \to \mathbb{R}$. Then f is said to **approach the limit L as x approaches x_0**, often written "$f(x) \to L$ as $x \to x_0$" or

$$\lim_{x \to x_0} f(x) = L\,,$$

iff for all $\varepsilon > 0$ there exists $\delta > 0$ such that if $0 < |x - x_0| < \delta$ then $|f(x) - L| < \varepsilon$.

Shorthand version:

$\forall \varepsilon > 0\ \exists \delta > 0\ \ni\ 0 < |x - x_0| < \delta \implies |f(x) - L| < \varepsilon.$

Limits of functions

The function f need not be defined on an entire interval. It is enough for f to be defined on a set with at least one accumulation point.

Definition (Limit of a function with domain $E \subseteq \mathbb{R}$)

Let $E \subseteq \mathbb{R}$ and $f : E \to \mathbb{R}$. Suppose x_0 is a point of accumulation of E. Then f is said to **approach the limit L as x approaches x_0**, *i.e.*,

$$\lim_{x \to x_0} f(x) = L,$$

iff for all $\varepsilon > 0$ there exists $\delta > 0$ such that if $x \in E$, $x \neq x_0$, and $|x - x_0| < \delta$ then $|f(x) - L| < \varepsilon$.

Shorthand version:

$$\forall \varepsilon > 0 \ \exists \delta > 0 \ \ni \ \big(x \in E \ \wedge \ 0 < |x - x_0| < \delta\big) \implies |f(x) - L| < \varepsilon.$$

Limits of functions

Example

Prove directly from the definition of a limit that

$$\lim_{x\to 3}(2x+1) = 7\,.$$

(solution on board)

Proof that $2x+1 \to 7$ as $x \to 3$.

We must show that $\forall\varepsilon > 0\ \exists\delta > 0$ such that $0 < |x-3| < \delta \implies |(2x+1)-7| < \varepsilon$. Given ε, to determine how to choose δ, note that

$$|(2x+1)-7| < \varepsilon \iff |2x-6| < \varepsilon \iff 2\,|x-3| < \varepsilon \iff |x-3| < \frac{\varepsilon}{2}$$

Therefore, given $\varepsilon > 0$, let $\delta = \frac{\varepsilon}{2}$. Then $|x-3| < \delta \implies |(2x+1)-7| = |2x-6| = 2\,|x-3| < 2\frac{\varepsilon}{2} = \varepsilon$, as required. □

Limits of functions

Example

Prove directly from the definition of a limit that

$$\lim_{x\to 2} x^2 = 4\,.$$

(solution on board)

(and on next slide)

Limits of functions

Proof that $x^2 \to 4$ as $x \to 2$.

We must show that $\forall \varepsilon > 0\ \exists \delta > 0$ such that $0 < |x-2| < \delta \implies \left|x^2 - 4\right| < \varepsilon$. Given ε, to determine how to choose δ, note that

$$\left|x^2 - 4\right| < \varepsilon \iff |(x-2)(x+2)| < \varepsilon \iff |x-2|\,|x+2| < \varepsilon.$$

We can make $|x-2|$ as small as we like by choosing δ sufficiently small. Moreover, if x is close to 2 then $x+2$ will be close to 4, so we should be able to ensure that $|x+2| < 5$. To see how, note that

$$\begin{aligned} |x+2| < 5 \iff & -5 < x+2 < 5 \iff -9 < x-2 < 1 \\ \Longleftarrow & -1 < x-2 < 1 \iff |x-2| < 1\,. \end{aligned}$$

Therefore, given $\varepsilon > 0$, let $\delta = \min(1, \frac{\varepsilon}{5})$. Then $\left|x^2 - 4\right| = |(x-2)(x+2)| = |x-2|\,|x+2| < \frac{\varepsilon}{5}5 = \varepsilon.$ □

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Mathematics and Statistics

$$\int_M d\omega = \int_{\partial M} \omega$$

Mathematics 3A03
Real Analysis I

Instructor: David Earn

Lecture 19
Continuity II
Wednesday 27 February 2019

Announcements

- A preliminary version of Assignment 4 has been posted on the course web site. More problems will be added soon.
  Due Friday 8 March 2019 at 1:25pm via crowdmark.
  BUT you should do **questions 1 and 2** before Test #1.

- **Math 3A03 Test #1**
  Monday 4 March 2019 at 7:00pm in MDCL 1110
  (room is booked for 90 minutes; you should not feel rushed)
  - Test will cover everything up to the end of the topology section.

- Niky Hristov will hold extra office hours this Friday 1 March 2019, 11:30am–12:30pm and immediately before class on the day of the test, *i.e.*, Monday 4 March 2019, 10:30–11:30am.

- Solutions to $\lim_{x\to 3}(2x+1)=7$ and $\lim_{x\to 2} x^2 = 4$ are now in the slides for the previous lecture.

Limits of functions

Rather than the ε-δ definition, we can exploit our experience with sequences to define "$f(x) \to L$ as $x \to x_0$".

Definition (Limit of a function via sequences)

Let $E \subseteq \mathbb{R}$ and $f : E \to \mathbb{R}$. Suppose x_0 is a point of accumulation of E. Then

$$\lim_{x \to x_0} f(x) = L$$

iff for every sequence $\{e_n\}$ of points in $E \setminus \{x_0\}$,

$$\lim_{n\to\infty} e_n = x_0 \quad \Longrightarrow \quad \lim_{n\to\infty} f(e_n) = L\,.$$

Limits of functions

Lemma (Equivalence of limit definitions)

The ε-δ definition of limits and the sequence definition of limits are equivalent.

(solution on board)

Note: The definition of a limit via sequences is sometimes easier to use than the ε-δ definition.

Proof of Equivalence of ε-δ definition and sequence definition of limit.

Proof (ε-$\delta \implies$ seq).

Suppose the ε-δ definition holds and $\{e_n\}$ is a sequence in $E \setminus \{x_0\}$ that converges to x_0. Given $\varepsilon > 0$, there exists $\delta > 0$ such that if $0 < |x - x_0| < \delta$ then $|f(x) - L| < \varepsilon$. But since $e_n \to x_0$, given $\delta > 0$, there exists $N \in \mathbb{N}$ such that, for all $n \geq N$, $|e_n - x_0| < \delta$. This means that if $n \geq N$ then $x = e_n$ satisfies $0 < |x - x_0| < \delta$, implying that we can put $x = e_n$ in the statement $|f(x) - L| < \varepsilon$. Hence, for all $n \geq N$, $|f(e_n) - L| < \varepsilon$. Thus,

$$e_n \to x_0 \implies f(e_n) \to L\,,$$

as required. □

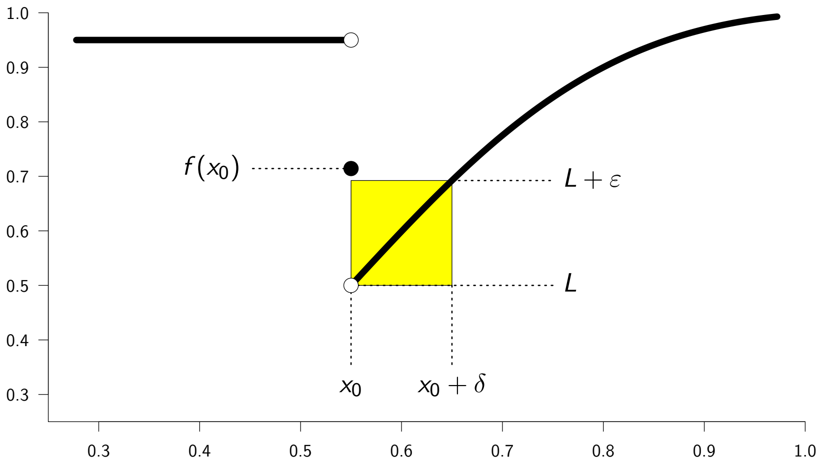
Proof of Equivalence of ε-δ definition and sequence definition of limit.

Proof (seq $\implies$ ε-δ) via contrapositive.

Suppose that as $x \to x_0$, $f(x) \not\to L$ according to the ε-δ definition. We must show that $f(x) \not\to L$ according to the sequence definition.

Since the ε-δ criterion does not hold, $\exists \varepsilon > 0$ such that $\forall \delta > 0$ there is some $x_\delta \in E$ for which $0 < |x_\delta - x_0| < \delta$ and yet $|f(x_\delta) - L| \geq \varepsilon$. This is true, in particular, for $\delta = 1/n$, where n is any natural number. Thus, $\exists \varepsilon > 0$ such that: $\forall n \in \mathbb{N}$, there exists $x_n \in E$ such that $0 < |x_n - x_0| < 1/n$ and yet $|f(x_n) - L| \geq \varepsilon$. This demonstrates that there is a sequence $\{x_n\}$ in $E \setminus \{x_0\}$ for which $x_n \to x_0$ and yet $f(x_n) \not\to L$. Hence, $f(x) \not\to L$ as $x \to x_0$ according to the sequence criterion, as required. $\square$

One-sided limits



One-sided limits

Definition (Right-Hand Limit)

Let $f : E \to \mathbb{R}$ be a function with domain E and suppose that x_0 is a point of accumulation of $E \cap (x_0, \infty)$. Then we write

$$\lim_{x \to x_0^+} f(x) = L$$

if for every $\varepsilon > 0$ there is a $\delta > 0$ so that

$$|f(x) - L| < \varepsilon$$

whenever $x_0 < x < x_0 + \delta$ and $x \in E$.

One-sided limits

One-sided limits can also be expressed in terms of sequence convergence.

Definition (Right-Hand Limit – sequence version)

Let $f : E \to \mathbb{R}$ be a function with domain E and suppose that x_0 is a point of accumulation of $E \cap (x_0, \infty)$. Then we write

$$\lim_{x \to x_0^+} f(x) = L$$

if for every decreasing sequence $\{e_n\}$ of points of E with $e_n > x_0$ and $e_n \to x_0$ as $n \to \infty$,

$$\lim_{n \to \infty} f(e_n) = L\,.$$

Infinite limits

Definition (Right-Hand Infinite Limit)

Let $f : E \to \mathbb{R}$ be a function with domain E and suppose that x_0 is a point of accumulation of $E \cap (x_0, \infty)$. Then we write

$$\lim_{x \to x_0^+} f(x) = \infty$$

if for every $M > 0$ there is a $\delta > 0$ such that $f(x) \geq M$ whenever $x_0 < x < x_0 + \delta$ and $x \in E$.

Properties of limits

There are theorems for limits of functions of a real variable that correspond (and have similar proofs) to the various results we proved for limits of sequences:

- Uniqueness of limits
- Algebra of limits
- Order properties of limits
- Limits of absolute values
- Limits of Max/Min

See Chapter 5 of textbook for details.

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Mathematics and Statistics

$$\int_M d\omega = \int_{\partial M} \omega$$

Mathematics 3A03
Real Analysis I

Instructor: David Earn

Lecture 20
Continuity III
Friday 1 March 2019

Announcements

- All of Assignment 4 has now been posted on the course web site. Due Friday 8 March 2019 at 1:25pm via crowdmark. BUT you should do **questions 1 and 2** before Test #1.

- **Math 3A03 Test #1**
  Monday 4 March 2019 at 7:00pm in MDCL 1110
  (room is booked for 90 minutes; you should not feel rushed)
  - Test will cover everything up to the end of the topology section.

- Niky Hristov will hold an extra office hour immediately before class on Monday, the day of the test, *i.e.*, Monday 4 March 2019, **10:30–11:30am**.

- I will also hold my usual office hour on Monday, **1:30–2:30pm**.

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Tuesday, March 12th 2019
CIBC Hall
5:00PM - 6:30PM

Contact: trepanr@mcmaster.ca

Last time. . .

- Equivalence of ε-δ definition and sequence definition of limit.
- One-sided limit from the right.

Limits of compositions of functions

When is $\lim_{x\to x_0} g\big(f(x)\big) = g\Big(\lim_{x\to x_0} f(x)\Big)$?

Theorem (Limit of composition)

Suppose

$$\lim_{x\to x_0} f(x) = L\,.$$

If g is a function defined in a neighborhood of the point L and

$$\lim_{z\to L} g(z) = g(L)$$

then

$$\lim_{x\to x_0} g\big(f(x)\big) = g\Big(\lim_{x\to x_0} f(x)\Big) = g(L)\,.$$

(Textbook (TBB) §5.2.5)

Limits of compositions of functions – more generally

Note: It is a little more complicated to generalize the statement of this theorem so as to minimize the set on which g must be defined but the proof is no more difficult.

Theorem (Limit of composition)

Let $A, B \subseteq \mathbb{R}$, $f : A \to \mathbb{R}$, $f(A) \subseteq B$, and $g : B \to \mathbb{R}$. Suppose x_0 is an accumulation point of A and

$$\lim_{x \to x_0} f(x) = L\,.$$

Suppose further that g is defined at L. If L is an accumulation point of B and

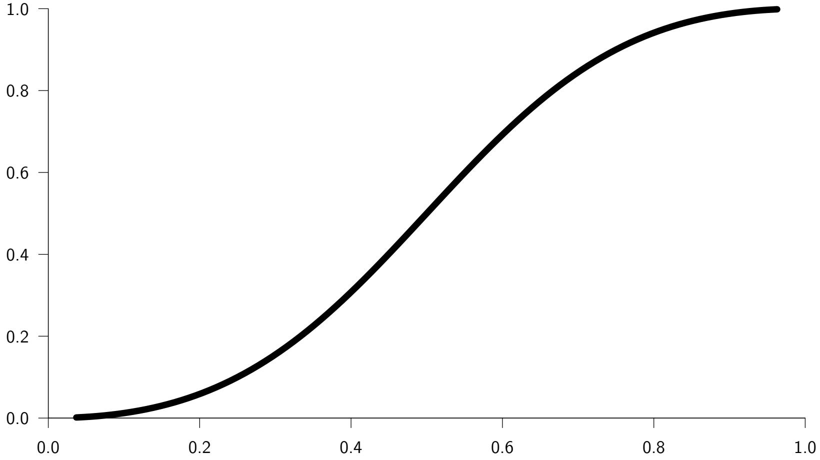
$$\lim_{z \to L} g(z) = g(L)\,,$$

or $\exists \delta > 0$ such that $f(x) = L$ for all $x \in (x_0 - \delta, x_0 + \delta) \cap A$, then

$$\lim_{x \to x_0} g\big(f(x)\big) = g\Big(\lim_{x \to x_0} f(x)\Big) = g(L)\,.$$

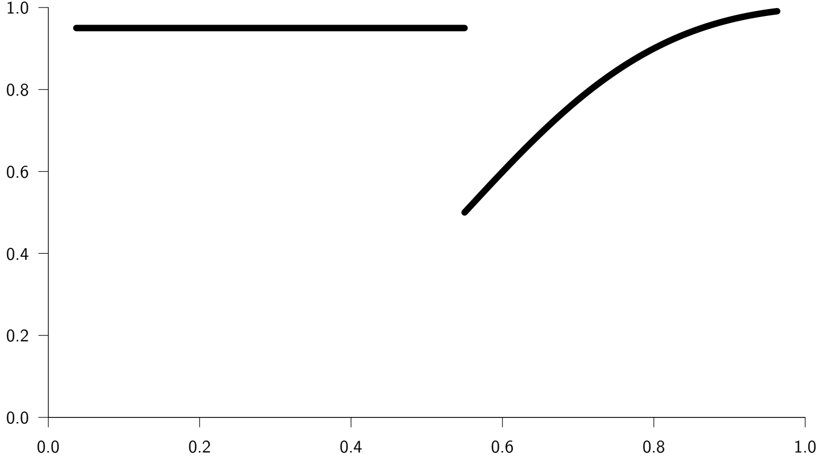
Continuity

Intuitively, a function f is **continuous** if you can draw its graph without lifting your pencil from the paper...



Continuity

and **discontinuous** otherwise. . .



Continuity

In order to develop a rigorous foundation for the theory of functions, we need to be more precise about what we mean by "continuous".

The main challenge is to define "continuity" in a way that works consistently on sets other than intervals (and generalizes to spaces that are more abstract than $\mathbb{R}$).

We will define:

- continuity at a single point;
- continuity on an open interval;
- continuity on a closed interval;
- continuity on more general sets.

Pointwise continuity

Definition (Continuous at an interior point of the domain of f)

If the function f is defined in a neighbourhood of the point x_0 then we say f is **continuous at x_0** iff

$$\lim_{x \to x_0} f(x) = f(x_0)\,.$$

This definition works more generally provided x_0 is a point of accumulation of the domain of f (notation: $\mathrm{dom}(f)$).

We will also consider a function to be continuous at any isolated point in its domain.

Pointwise continuity

Definition (Continuous at any $x_0 \in \text{dom}(f)$ – limit version)

If $x_0 \in \text{dom}(f)$ then f is **continuous at x_0** iff x_0 is either an isolated point of $\text{dom}(f)$ or x_0 is an accumulation point of $\text{dom}(f)$ and $\lim_{x\to x_0} f(x) = f(x_0)$.

Definition (Continuous at any $x_0 \in \text{dom}(f)$ – sequence version)

If $x_0 \in \text{dom}(f)$ then f is **continuous at x_0** iff for any sequence $\{x_n\}$ in $\text{dom}(f)$, if $x_n \to x_0$ then $f(x_n) \to f(x_0)$.

Definition (Continuous at any $x_0 \in \text{dom}(f)$ – ε-δ version)

If $x_0 \in \text{dom}(f)$ then f is **continuous at x_0** iff for any $\varepsilon > 0$ there exists $\delta > 0$ such that if $x \in \text{dom}(f)$ and $|x - x_0| < \delta$ then $|f(x) - f(x_0)| < \varepsilon$.

Pointwise continuity

Example

Suppose $f : A \to \mathbb{R}$. In which cases is f continuous on A?

- $A = (0,1) \cup \{2\}, \quad f(x) = x;$
- $A = \{0\} \cup \{\frac{1}{n} : n \in \mathbb{N}\} \cup \{2\}, \quad f(x) = x;$
- $A = \{\frac{1}{n} : n \in \mathbb{N}\} \cup \{2\}, \quad f(x) =$ whatever you like.

Example

Is it possible for a function f to be discontinuous at every point of $\mathbb{R}$ and yet for its restriction to the rational numbers ($f\big|_{\mathbb{Q}}$) to be continuous at every point in $\mathbb{Q}$?

Extra Challenge Problem:

Prove or disprove: There is a function $f : \mathbb{R} \to \mathbb{R}$ that is continuous at every irrational number and discontinuous at every rational number.

Continuity on an interval

Definition (Continuous on an open interval)

The function f is said to be **continuous on** (a, b) iff

$$\lim_{x \to x_0} f(x) = f(x_0) \qquad \text{for all } x_0 \in (a, b)\,.$$

Definition (Continuous on a closed interval)

The function f is said to be **continuous on** $[a, b]$ iff it is continuous on the open interval (a, b), and

$$\lim_{x \to a^+} f(x) = f(a) \qquad \text{and} \qquad \lim_{x \to b^-} f(x) = f(b)\,.$$

Continuity on an arbitrary set $E \subseteq \mathbb{R}$

Definition (Continuous on a set E)

The function f is said to be **continuous on E** iff f is continuous at each point $x \in E$.

Example

- Every polynomial is continuous on $\mathbb{R}$.
- Every rational function is continuous on its domain (*i.e.*, avoiding points where the denominator is zero).

These facts are painful to prove directly from the definition.
But they follow easily if from the theorem on the algebra of limits.

Continuity of compositions of functions

Theorem (Continuity of $f \circ g$ at a point)

If g is continuous at x_0 and f is continuous at $g(x_0)$ then $f \circ g$ is continuous at x_0.

Consequently, if g is continuous at x_0 and f is continuous at $g(x_0)$ then

$$\lim_{x \to x_0} f\big(g(x)\big) = f\Big(\lim_{x \to x_0} g(x)\Big)\,.$$

Theorem (Continuity of $f \circ g$ on a set)

If g is continuous on $A \subseteq \mathbb{R}$ and f is continuous on $g(A)$ then $f \circ g$ is continuous on A.

Continuity of compositions of functions

Example

Use the theorem on continuity of $f \circ g$, and the theorem on the algebra of limits, to prove that

1. the polynomial $x^8 + x^3 + 2$ is continuous on $\mathbb{R}$;
2. the rational function $\dfrac{x^2+2}{x^2-2}$ is continuous on $\mathbb{R} \setminus \{-\sqrt{2}, \sqrt{2}\}$.
3. the function $\sqrt{\dfrac{x^2+2}{x^2-2}}$ is continuous on its domain.

McMaster University

Mathematics and Statistics

$$\int_M d\omega = \int_{\partial M} \omega$$

Mathematics 3A03
Real Analysis I

Instructor: David Earn

Lecture 21
Continuity IV
Monday 4 March 2019

Announcements

- All of Assignment 4 has been posted on crowdmark.
  Due Friday 8 March 2019 at 1:25pm via crowdmark.
  BUT make sure to have done **questions 1 and 2** before tonight's test.
- **Math 3A03 Test #1**
  TONIGHT 4 March 2019 at 7:00pm in MDCL 1110
  (room is booked for 90 minutes; you should not feel rushed)
  - Test covers everything up to the end of the topology section.
- I will hold my usual office hour today, **1:30–2:30pm**.
- Let's look at tonight's test.

Last time. . .

- Limits of compositions.
- Continuity at a point and on a set.
- Continuity of compositions.

Uniform continuity

In the ε-δ definition of continuity, the δ that must exist depends on ε **AND** on the point x_0, *i.e.*, $\delta = \delta(f, \varepsilon, x_0)$.

Definition (Uniformly continuous)

If $f : A \to \mathbb{R}$ then f is said to be **uniformly continuous on A** iff for every $\varepsilon > 0$ there exists $\delta > 0$ such that if $x, y \in A$ and $|x - y| < \delta$ then $|f(x) - f(y)| < \varepsilon$.

Note: This is a stronger form of continuity: Given any $\varepsilon > 0$, there is a single $\delta > 0$ that works for the entire set A.
(δ still depends on f and ε.)

Uniform continuity

Example

Prove that $f(x) = 2x + 1$ is uniformly continuous on $\mathbb{R}$.

(solution on board)

Proof.

We must show that $\forall \varepsilon > 0$, $\exists \delta > 0$ such that if $x, y \in \mathbb{R}$ and $|x - y| < \delta$ then $|(2x+1) - (2y+1)| < \varepsilon$. But note that

$$|(2x+1) - (2y+1)| = |2x - 2y| = 2\,|x-y|\ ,$$

so if we choose $\delta = \varepsilon/2$ then we have

$$|(2x+1) - (2y+1)| = 2\,|x-y| < 2 \cdot \frac{\varepsilon}{2} = \varepsilon\,,$$

as required. □

Uniform continuity

Example

Prove that $f(x) = \sqrt{x}$ is uniformly continuous on $[\frac{1}{8}, 1]$.

(solution on board)

Proof.

We must show that $\forall \varepsilon > 0$, $\exists \delta > 0$ such that if $x, y \in [\frac{1}{8}, 1]$ and $|x - y| < \delta$ then $\left|\sqrt{x} - \sqrt{y}\right| < \varepsilon$. But note that

$$\left|\sqrt{x} - \sqrt{y}\right| = \left|\left(\sqrt{x} - \sqrt{y}\right)\frac{\sqrt{x} + \sqrt{y}}{\sqrt{x} + \sqrt{y}}\right|$$

$$= \left|\frac{x - y}{\sqrt{x} + \sqrt{y}}\right| \le \left|\frac{x - y}{\sqrt{\frac{1}{8}} + \sqrt{\frac{1}{8}}}\right| = \left|\frac{x - y}{\frac{1}{\sqrt{2}}}\right| = \sqrt{2}\,|x - y| \,,$$

so taking $\delta = \varepsilon/\sqrt{2}$, we have $\left|\sqrt{x} - \sqrt{y}\right| < \sqrt{2} \cdot \frac{\varepsilon}{\sqrt{2}} = \varepsilon$. □

Uniform continuity

Example

Is $f(x) = \sqrt{x}$ uniformly continuous on $[0, 1]$?

Note: The proof on the previous slide fails if the lower limit is 0, but that doesn't establish that the function is not uniformly continuous. We need to show that $\exists \varepsilon > 0$ such that $\forall \delta > 0$, $\exists x, y \in [0, 1]$ such that $|x - y| < \delta$ and yet $\left|\sqrt{x} - \sqrt{y}\right| \geq \varepsilon$.

McMaster University

Mathematics and Statistics

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Mathematics 3A03
Real Analysis I

Instructor: David Earn

Lecture 22
Continuity V
Wednesday 6 March 2019

Uniform continuity

Theorem (Unif. cont. on a bounded interval $\implies$ bounded)

If f is uniformly continuous on a bounded interval I then f is bounded on I.

(solution on board)

Clean proof.

Suppose f is uniformly continuous on the interval I with endpoints a, b (where $a < b$). Then, given $\varepsilon > 0$ we can find $\delta > 0$ such that if $x, y \in I$ and $|x - y| < \delta$ then $|f(x) - f(y)| < \varepsilon$.

Moreover, given any $\delta > 0$ and any $c > 0$, we can find $n \in \mathbb{N}$ such that $0 < \frac{c}{n} < \delta$.

Choose $n \in \mathbb{N}$ such that if $x, y \in I$ and $|x - y| < 2\left(\frac{b-a}{n}\right)$ then $|f(x) - f(y)| < 1$.

Continued...

Uniform continuity

Clean proof (continued).

Divide I into n subintervals with endpoints

$$x_i = a + i\left(\frac{b-a}{n}\right), \qquad i = 0, 1, \ldots, n.$$

For $0 \le i \le n-1$, define $I_i = [x_i, x_{i+1}] \cap I$ (we intersect with I in case $a \notin I$ or $b \notin I$), and note that $\forall x, y \in I_i$ we have $|x - y| \le \frac{b-a}{n} < 2\left(\frac{b-a}{n}\right)$ and hence $|f(x) - f(y)| < 1 \;\; \forall x, y \in I_i$.

Let $\overline{x}_i = (x_i + x_{i+1})/2$ (the midpoint of interval I_i). Then, in particular, we have $|f(x) - f(\overline{x}_i)| < 1 \;\; \forall x \in I_i$, *i.e.*,

$$f(\overline{x}_i) - 1 < f(x) < f(\overline{x}_i) + 1 \qquad \forall x \in I_i\,.$$

Thus, f is bounded on I_i and therefore has a LUB and GLB on I_i.

Continued...

Uniform continuity

Clean proof (continued).

Therefore, for $i = 0, 1, \ldots, n-1$, define

$$m_i = \inf\{f(x) : x \in I_i\}\,,$$
$$M_i = \sup\{f(x) : x \in I_i\}\,,$$

and let

$$m = \min\{m_i : i = 0, 1, \ldots, n-1\}\,,$$
$$M = \max\{M_i : i = 0, 1, \ldots, n-1\}\,.$$

Then

$$m \le f(x) \le M \qquad \forall x \in I = \bigcup_{i=1}^{n-1} I_i\,,$$

i.e., f is bounded on the entire interval I. □

Uniform continuity

Theorem (Cont. on a closed interval $\implies$ unif. cont.)

If $f : [a, b] \to \mathbb{R}$ is continuous then f is uniformly continuous.

(Textbook (TBB) Theorem 5.48, p. 323)

Corollary (Continuous on a closed interval $\implies$ bounded)

If $f : [a, b] \to \mathbb{R}$ is continuous then f is bounded.

Proof.

Combine the above two theorems. $\square$

Uniform continuity

Although stated in terms of a closed interval $[a, b]$, we have proved something more general.

Theorem

A continuous function on a compact set is uniformly continuous.

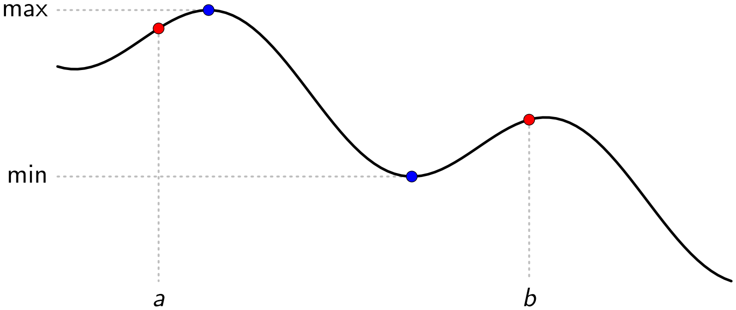
The converse is also true:

Theorem

If every continuous function on a set E is uniformly continuous then E is compact.

Recall that compactness is associated with global properties (as opposed to local properties). Uniform continuity is a global property in that a single δ is sufficient for an entire set.

Extreme Value Theorem



Theorem (Extreme value theorem)

A continuous function on a closed interval $[a, b]$ has a maximum and minimum value on $[a, b]$.

Extreme Value Theorem

More generally:

Theorem

A continuous function on a compact set has a maximum and minimum value.

Extreme Value Theorem

Theorem

A continuous function on a compact set has a maximum and minimum value.

Proof (by contradiction).

Since f is continuous on the compact set $[a, b]$, it is bounded on $[a, b]$. This means that the **range** of f, *i.e.*, the set

$$f([a, b]) \stackrel{\textbf{def}}{=} \{f(x) : x \in [a, b]\}$$

is bounded. This set is not $\varnothing$, so it has a LUB α. Since $\alpha \geq f(x)$ for $x \in [a, b]$, it suffices to show that $\alpha = f(y)$ for some $y \in [a, b]$.

Suppose instead that $\alpha \neq f(y)$ for any $y \in [a, b]$, *i.e.*, $\alpha > f(y)$ for all $y \in [a, b]$. Then the function g defined by ...

Extreme Value Theorem

Proof of Extreme Value Theorem (continued).

$$g(x) = \frac{1}{\alpha - f(x)}\,, \qquad x \in [a,b],$$

is positive and continuous on $[a,b]$, since the denominator of the RHS is always positive. On the other hand, α is the LUB of $f([a,b])$; this means that

$$\forall \varepsilon > 0 \quad \exists x \in [a,b] \quad \ni \quad \alpha - f(x) < \varepsilon\,.$$

Since $\alpha - f(x) > 0$, this, in turn, means that

$$\forall \varepsilon > 0 \quad \exists x \in [a,b] \quad \ni \quad g(x) > \frac{1}{\varepsilon}\,.$$

But *this* means that g is not bounded on $[a,b]$, ...

Extreme Value Theorem

Proof of Extreme Value Theorem (continued).

contradicting the theorem that a continuous function on a compact set is bounded. $\Rightarrow\Leftarrow$

Therefore, $\alpha = f(y)$ for some $y \in [a, b]$,
i.e., f has a maximum on $[a, b]$.

A similar argument shows that f has a minimum on $[a, b]$. $\square$

McMaster University

Mathematics and Statistics

$$\int_M d\omega = \int_{\partial M} \omega$$

Mathematics 3A03
Real Analysis I

Instructor: David Earn

Lecture 23
Continuity VI
Friday 8 March 2019

Uniform continuity is stronger than continuity

Theorem (Uniform continuity $\implies$ continuity)

Suppose $f : E \to \mathbb{R}$ is uniformly continuous. Then f is continuous.

Proof.

f uniformly continuous means $\forall \varepsilon > 0 \; \exists \delta > 0$ such that if $x, y \in E$ and $|x - y| < \delta$ then $|f(x) - f(y)| < \varepsilon$. If we fix any point $y \in E$ then this is the definition of continuity at y, *i.e.*, f is continuous at each $y \in E$. □

Note: Converse is false!

Example (Continuous $\not\Longrightarrow$ uniformly continuous)

$f(x) = 1/x$ on is continuous on $(0, 1)$ but not uniformly continuous on $(0, 1)$.

Key theorems about uniform continuity

1. Uniformly continuous on a bounded interval $\implies$ bounded
   - Proved last time.

2. Uniformly continuous on a compact set $\implies$ bounded
   - Generalization of 1 in case of closed interval $[a, b]$.

3. Continuous on a compact set $\implies$ uniformly continuous
   - Mentioned last time for a closed interval $[a, b]$ and a general compact set.

4. Continuous on a compact set $\implies$ bounded
   - Combine 3 with 2.

Note: Continuity is a *local* property, whereas uniform continuity is a *global* property.

Key theorems relating continuity and compactness

1. Continuous on a compact set $\Longrightarrow$ uniformly continuous.
   - Also stated on previous slide.
2. Continuous image of a compact set is compact.
   - Not discussed in class but a great exercise and important result.
3. Extreme Value Theorem
   - Proved last time.

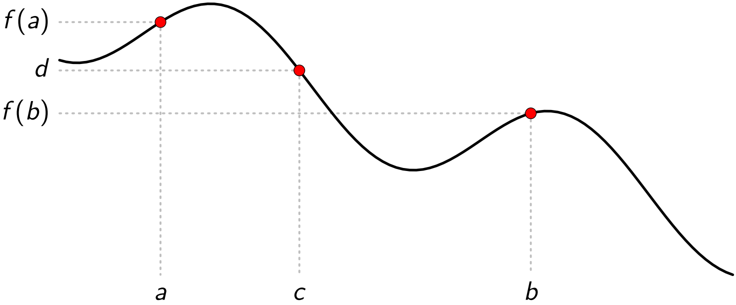
Announcements

- Assignment 4 was due at 1:25pm today.
- Solutions to Test 1 were posted on Wednesday.

Today:

- Intermediate Value Theorem
  - Another intuitively obvious theorem that is hard to prove!
  - Key theorem wrt continuity.
  - *Not* related to compactness

Intermediate Value Theorem



Definition (Intermediate Value Property (IVP))

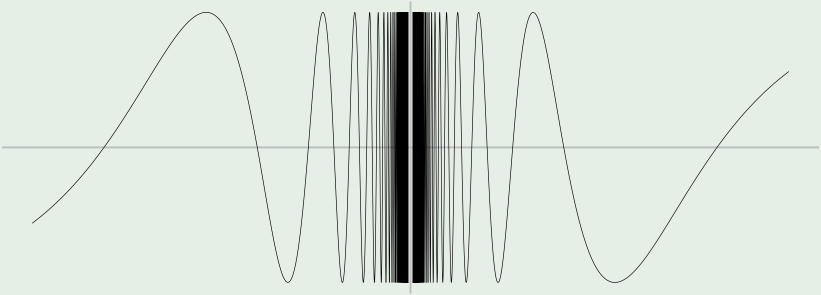
A function f defined on an interval I is said to have the **intermediate value property (IVP)** on I iff for each $a, b \in I$ with $f(a) \neq f(b)$, and for each d between $f(a)$ and $f(b)$, there exists c between a and b for which $f(c) = d$.

Intermediate Value Theorem

Question: If a function has the IVP on an interval I, must it be continuous on I?

Example

$$f(x) = \begin{cases} \sin\frac{1}{x} & x \neq 0, \\ 0 & x = 0. \end{cases}$$



Intermediate Value Theorem

Theorem (Intermediate Value Theorem (IVT))

If f is continuous on an interval I then f has the intermediate value property (IVP) on I.

(solution after proving the neighbourhood sign lemma)

Note: The interval I in the statement of the IVT does not have to be closed and it does not have to be bounded.
Unlike the extreme value theorem, the IVT is not a theorem about functions defined on compact sets.

Intermediate Value Theorem

Lemma (Neighbourhood sign)

Suppose I is an interval and $f : I \to \mathbb{R}$ is continuous at $a \in I$. If $f(a) > 0$ then f is positive in a neighbourhood of a. Similarly, if $f(a) < 0$, then f is negative in a neighbourhood of a.

Proof.

Consider the case $f(a) > 0$. Since f is continuous at a, given $\varepsilon > 0$ $\exists\delta > 0$ such that if $|x - a| < \delta$ then $|f(x) - f(a)| < \varepsilon$. Since $f(a) > 0$ we can take $\varepsilon = f(a)$. Thus, $\exists\delta > 0$ such that if $|x - a| < \delta$ then $|f(x) - f(a)| < f(a)$, *i.e.*,

$$|x - a| < \delta \implies -f(a) < f(x) - f(a) < f(a) \implies 0 < f(x) < 2f(a)\,.$$

In particular, $f(x) > 0$ in a neighbourhood* of radius δ about a.

The case $f(a) < 0$ is similar: take $\varepsilon = -f(a)$. □

*The neighbourhood is $(a - \delta, a + \delta)$, unless a is an endpoint of the set on which f is defined, in which case the neighbourhood is either $[a, a + \delta)$ or $(a - \delta, a]$.

Intermediate Value Theorem

The Intermediate Value Theorem follows directly from the following lemma, which is what we'll prove:

Lemma (Existence of roots)

If f is continuous on $[a, b]$ and $f(a) < 0 < f(b)$ then there exists $x \in [a, b]$ such that $f(x) = 0$.

How does Intermediate Value Property follow?

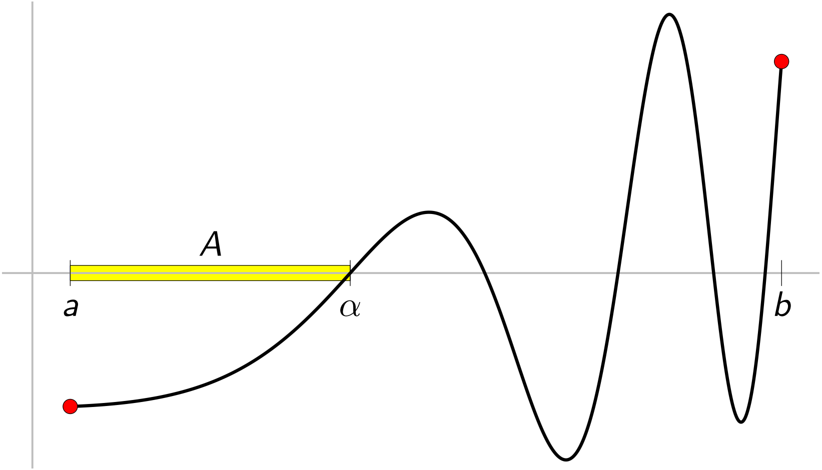
If $f(a) < M < f(b)$ for some $M \in \mathbb{R}$, then apply the lemma to $g(x) = f(x) - M$.

If $f(a) > M > f(b)$ for some $M \in \mathbb{R}$, then apply the lemma to $g(x) = M - f(x)$.

What if the interval I on which f is continuous is not a closed interval?

Intermediate Value Theorem

Idea for proof of root existence lemma:



Intermediate Value Theorem

Sketch of proof of root existence lemma:

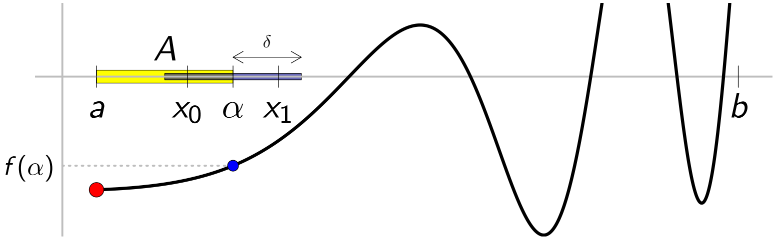
1. (i) $A = \{x : a \le x \le b, \text{and } f \text{ is negative on the interval } [a, x]\}$;
   (ii) $\alpha = \sup(A)$ exists;
   (iii) neighbourhood sign lemma $\implies a < \alpha < b$.

2. Prove by contradiction that $f(\alpha) < 0$ is impossible.
   To guide this argument, it helps to draw a picture that is consistent with the assumption that $f(\alpha) < 0$. This picture is not really correct because it represents an assumption that we will prove to be false.

3. Prove by contradiction that $f(\alpha) > 0$ is impossible.

Intermediate Value Theorem

Picture to guide proof by contradiction that it is impossible that $f(\alpha) < 0$:



- Given $f(\alpha) < 0$, the neighbourhood sign lemma implies $\exists \delta > 0$ such that $f(x) < 0$ on $(\alpha - \delta, \alpha + \delta)$.
- For any $x_0 \in (\alpha - \delta, \alpha)$, since $x_0 < \alpha$, we must have $x_0 \in A$, *i.e.*, $f(x) < 0$ on $[a, x_0]$. Otherwise, α would not be the least upper bound of A.
- Now pick any $x_1 \in (\alpha, \alpha + \delta)$. We know $x_1 \notin A$ because $\alpha < x_1$. But $f(x) < 0$ on $[x_0, x_1]$ since $[x_0, x_1] \subset (\alpha - \delta, \alpha + \delta)$ and $f(x) < 0$ on $[a, x_0]$ because $x_0 \in A$. Hence $f(x) < 0$ on $[a, x_1]$, *i.e.*, $x_1 \in A$. $\Rightarrow\Leftarrow$