McMaster University

Mathematics and Statistics

$$\int_M d\omega = \int_{\partial M} \omega$$

Mathematics 3A03
Real Analysis I

Instructor: David Earn

Lecture 13
Topology of $\mathbb{R}$ I
Monday 4 February 2019

Announcements

- Assignment 3 was posted on Saturday.
  Due Friday 15 Feb 2019 at 1:25pm.
  IMPORTANT CHANGE:
  - For the remainder of the term, assignments must be submitted **electronically**, not as a hardcopy.
  - You should have received a link for Assignment 3 via e-mail from crowdmark. If you have not received such an e-mail, please e-mail earn@math.mcmaster.ca.
  - If you write your solutions by hand, you will need to scan or photograph them to submit them via the online system.
  - If you use LaTeX to create a pdf file, you will need to separate your solutions for each question.
  - Marked assignments will be available online, rather than being returned in tutorial.
- Today: "How big is $\mathbb{R}$?" (see last few slides for Lecture 12) and intro to "Topology of $\mathbb{R}$"

Topology of $\mathbb{R}$

Intervals

Open interval:

$$(a, b) = \{x : a < x < b\}$$

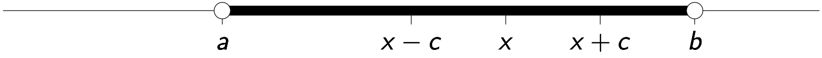
Closed interval:

$$[c, d] = \{x : c \leq x \leq d\}$$

Half-open interval:

$$(e, f] = \{x : e < x \leq f\}$$

Interior point



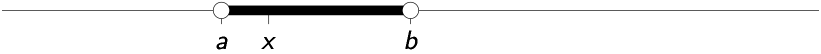
Definition (Interior point)

If $E \subseteq \mathbb{R}$ then x is an **interior point** of E if x lies in an open interval that is contained in E, *i.e.*, $\exists c > 0$ such that $(x-c, x+c) \subset E$.

Interior point examples

Set E	**Interior points?**
$(-1,1)$	Every point
$[0,1]$	Every point *except the endpoints*
$\mathbb{N}$	$\nexists$
$\mathbb{R}$	Every point
$\mathbb{Q}$	$\nexists$
$(-1,1) \cup [0,1]$	Every point *except 1*
$(-1,1) \setminus \{\frac{1}{2}\}$	Every point

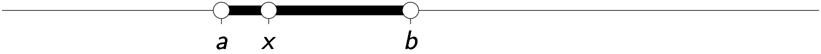
Neighbourhood



Definition (Neighbourhood)

A **neighbourhood** of a point $x \in \mathbb{R}$ is an open interval containing x.

Deleted neighbourhood

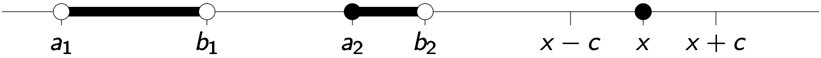


Definition (Deleted neighbourhood)

A **deleted neighbourhood** of a point $x \in \mathbb{R}$ is a set formed by removing x from a neighbourhood of x.

$$(a, b) \setminus \{x\}$$

Isolated point



$$E = (a_1, b_1) \cup [a_2, b_2) \cup \{x\}$$

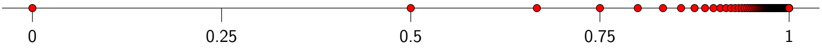
Definition (Isolated point)

If $x \in E \subseteq \mathbb{R}$ then x is an **isolated point** of E if there is a neighbourhood of x for which the only point in E is x itself, *i.e.*, $\exists c > 0$ such that $(x-c, x+c) \cap E = \{x\}$.

Isolated point examples

Set E	**Isolated points?**
$(-1,1)$	$\nexists$
$[0,1]$	$\nexists$
$\mathbb{N}$	Every point
$\mathbb{R}$	$\nexists$
$\mathbb{Q}$	$\nexists$
$(-1,1)\cup[0,1]$	$\nexists$
$(-1,1)\setminus\{\frac{1}{2}\}$	$\nexists$

Accumulation point



$$E = \left\{ 1 - \frac{1}{n} : n \in \mathbb{N} \right\}$$

Definition (Accumulation Point or Limit Point)

If $E \subseteq \mathbb{R}$ then x is an **accumulation point** or **limit point** of E if every neighbourhood of x contains infinitely many points of E,

$$i.e., \qquad \forall c > 0 \qquad (x - c, x + c) \cap (E \setminus \{x\}) \neq \varnothing\,.$$

Notes:

- It is possible but not necessary that $x \in E$.
- The shorthand condition is equivalent to saying that every deleted neighbourhood of x contains at least one point of E.

Accumulation point examples

Set E	**Accumulation points?**
$(-1,1)$	
$[0,1]$	
$\mathbb{N}$	
$\mathbb{R}$	
$\mathbb{Q}$	
$(-1,1)\cup[0,1]$	
$(-1,1)\setminus\{\frac{1}{2}\}$	
$\left\{1-\frac{1}{n} : n\in\mathbb{N}\right\}$	

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$$\int_M d\omega = \int_{\partial M} \omega$$

Mathematics 3A03
Real Analysis I

Instructor: David Earn

Lecture 14
Topology of $\mathbb{R}$ II
Friday 8 February 2019

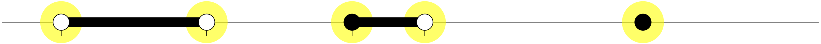
Announcements

- Assignment 3 was posted on Saturday.
  Due Friday 15 Feb 2019 at 1:25pm
  via crowdmark
- **Math 3A03 Test #1**
  Monday 4 March 2019 at 7:00pm in MDCL 1110

Accumulation point examples

Set E	**Accumulation points?**
$(-1,1)$	$[-1,1]$
$[0,1]$	$[0,1]$
$\mathbb{N}$	$\nexists$
$\mathbb{R}$	$\mathbb{R}$
$\mathbb{Q}$	$\mathbb{R}$
$(-1,1)\cup[0,1]$	$[-1,1]$
$(-1,1)\setminus\{\frac{1}{2}\}$	$[-1,1]$
$\left\{1-\frac{1}{n} : n\in\mathbb{N}\right\}$	$\{1\}$

Boundary point



Definition (Boundary Point)

If $E \subseteq \mathbb{R}$ then x is a **boundary point** of E if every neighbourhood of x contains at least one point of E and at least one point not in E, *i.e.*,

$$\forall c > 0 \qquad (x-c, x+c) \cap E \neq \varnothing \\ \wedge \quad (x-c, x+c) \cap (\mathbb{R} \setminus E) \neq \varnothing\,.$$

Note: It is possible but not necessary that $x \in E$.

Definition (Boundary)

If $E \subseteq \mathbb{R}$ then the **boundary** of E, denoted ∂E, is the set of all boundary points of E.

Boundary point examples

Set E	Boundary points?
$(-1,1)$	$\{-1,1\}$
$[0,1]$	$\{0,1\}$
$\mathbb{N}$	$\mathbb{N}$
$\mathbb{R}$	$\nexists$
$\mathbb{Q}$	$\mathbb{R}$
$(-1,1)\cup[0,1]$	$\{-1,1\}$
$(-1,1)\setminus\{\frac{1}{2}\}$	$\{-1,\frac{1}{2},1\}$
$\left\{1-\frac{1}{n} : n\in\mathbb{N}\right\}$	$\left\{1-\frac{1}{n} : n\in\mathbb{N}\right\}\cup\{1\}$

Closed set

Definition (Closed set)

A set $E \subseteq \mathbb{R}$ is **closed** if it contains all of its accumulation points.

Definition (Closure of a set)

If $E \subseteq \mathbb{R}$ and E' is the set of accumulation points of E then $\overline{E} = E \cup E'$ is the **closure** of E.

Note: If the set E has no accumulation points, then E is closed because there are no accumulation points to check.

Open set

Definition (Open set)

A set $E \subseteq \mathbb{R}$ is **open** if every point of E is an interior point.

Definition (Interior of a set)

If $E \subseteq \mathbb{R}$ then the **interior** of E, denoted $\text{int}(E)$ or E°, is the set of all interior points of E.

Examples

Set E	Closed?	Open?	$\overline{E}$	E°	∂E
$(-1,1)$	NO	YES	$[-1,1]$	E	$\{-1,1\}$
$[0,1]$	YES	NO	E	$(0,1)$	$\{0,1\}$
$\mathbb{N}$	YES	NO	$\mathbb{N}$	$\varnothing$	$\mathbb{N}$
$\mathbb{R}$	YES	YES	$\mathbb{R}$	$\mathbb{R}$	$\varnothing$
$\varnothing$	YES	YES	$\varnothing$	$\varnothing$	$\varnothing$
$\mathbb{Q}$	NO	NO	$\mathbb{R}$	$\varnothing$	$\mathbb{R}$
$(-1,1)\cup[0,1]$	NO	NO	$[-1,1]$	$(-1,1)$	$\{-1,1\}$
$(-1,1)\setminus\{\frac{1}{2}\}$	NO	YES	$[-1,1]$	E	$\{-1,\frac{1}{2},1\}$
$\left\{1-\frac{1}{n} : n\in\mathbb{N}\right\}$	NO	NO	$E\cup\{1\}$	$\varnothing$	$E\cup\{1\}$

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Mathematics 3A03
Real Analysis I

Instructor: David Earn

Lecture 15
Topology of $\mathbb{R}$ III
Monday 11 February 2019

Announcements

- Assignment 3 is **Due Friday 15 Feb 2019 at 1:25pm via crowdmark**
- **Math 3A03 Test #1**
  Monday 4 March 2019 at 7:00pm in MDCL 1110
  (room is booked for 90 minutes; you should not feel rushed)

Concepts covered recently

- Countable set
- Interval
- Neighbourhood
- Deleted neighbourhood
- Interior point
- Isolated point
- Accumulation point
- Boundary point
- Boundary
- Closed set
- Closure
- Open set
- Interior

Component intervals of open sets

What does the most general open set look like?

Theorem (Component intervals)

If G is an open subset of $\mathbb{R}$ and $G \neq \varnothing$ then there is a unique (possibly finite) sequence of disjoint open intervals $\{(a_n, b_n)\}$ such that

$$G = (a_1, b_1) \cup (a_2, b_2) \cup \cdots \cup (a_n, b_n) \cup \cdots ,$$

$$\text{i.e.,} \quad G = \bigcup_{n=1}^{\infty} (a_n, b_n) .$$

The open intervals (a_n, b_n) are said to be the **component intervals** *of G.*

(Textbook (TBB) Theorem 4.15, p. 231)

Component intervals of open sets

Main ideas of proof of component intervals theorem:

- $x \in G \implies x$ is an interior point of $G \implies$
  - some neighbourhood of x is contained in G, *i.e.*, $\exists c > 0$ such that $(x - c, x + c) \subseteq G$
  - $\exists$ a largest neighbourhood of x that is contained in G: this "**component of** G" is $I_x = (\alpha, \beta)$, where

$$\alpha = \inf\{a : (a, x] \subset G\}, \qquad \beta = \sup\{b : [x, b) \subset G\}$$

  - I_x contains a rational number, *i.e.*, $\exists r \in I_x \cap \mathbb{Q}$
- $\therefore$ We can index all the intervals I_x by rational numbers
- $\therefore$ There are are most countably many intervals that make up G (*i.e.*, G is the union of a sequence of intervals)
- We can choose a disjoint subsequence of these intervals whose union is all of G (see proof in textbook for details).

Open vs. Closed Sets

Definition (Complement of a set of real numbers)

If $E \subseteq \mathbb{R}$ then the **complement** of E is the set

$$E^{c} = \{x \in \mathbb{R} : x \notin E\} \, .$$

Theorem (Open vs. Closed)

If $E \subseteq \mathbb{R}$ then E is open iff E^{c} is closed.

(Textbook (TBB) Theorem 4.16)

Open vs. Closed Sets

Theorem (Properties of open sets of real numbers)

1. *The sets $\mathbb{R}$ and $\varnothing$ are open.*
2. *Any intersection of a finite number of open sets is open.*
3. *Any union of an arbitrary collection of open sets is open.*
4. *The complement of an open set is closed.*

(Textbook (TBB) Theorem 4.17)

Theorem (Properties of closed sets of real numbers)

1. *The sets $\mathbb{R}$ and $\varnothing$ are closed.*
2. *Any union of a finite number of closed sets is closed.*
3. *Any intersection of an arbitrary collection of closed sets is closed.*
4. *The complement of a closed set is open.*

(Textbook (TBB) Theorem 4.18)

Local vs. Global properties

Definition (Bounded function)

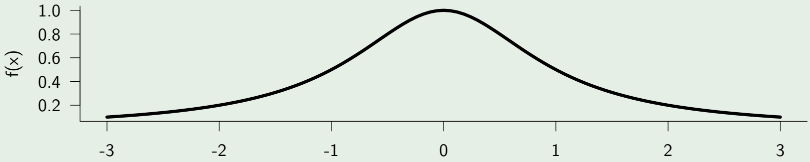
A real-valued function f is **bounded** on the set E if there exists $M > 0$ such that $|f(x)| \leq M$ for all $x \in E$.

(*i.e.*, the function f is bounded on E iff $\{f(x) : x \in E\}$ is a bounded set.)

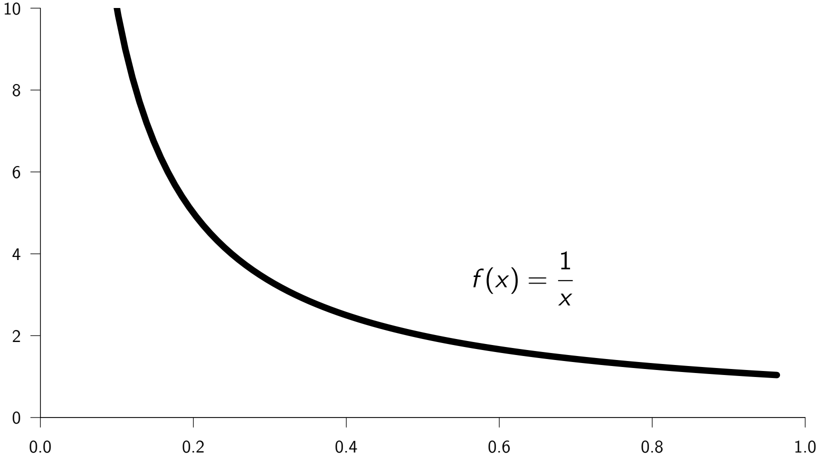
Note: This is a *global* property because there is a single bound M associated with the entire set E.

Example

The function $f(x) = 1/(1 + x^2)$ is bounded on $\mathbb{R}$. *e.g.*, $M = 1$.

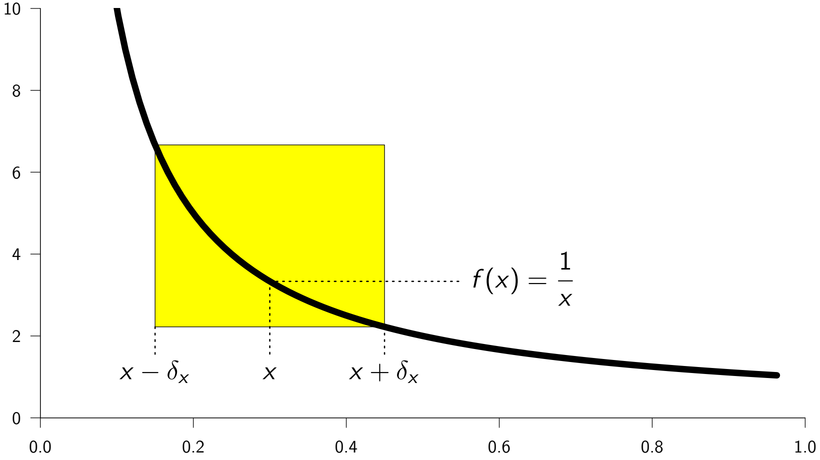


Local vs. Global properties



$f(x) = 1/x$ is <u>not</u> bounded on the interval $E = (0, 1)$.

Local vs. Global properties



$f(x) = 1/x$ is **locally bounded** on the interval $E = (0, 1)$,
i.e., $\forall x \in E$, $\exists \delta_x, M_x > 0 \ni |f(t)| \leq M_x \ \forall t \in (x - \delta_x, x + \delta_x)$.

Local vs. Global properties

Definition (Locally bounded at a point)

A real-valued function f is **locally bounded** at the point x if there is a neighbourhood of x in which f is bounded, *i.e.*, there exists $\delta_x > 0$ and $M_x > 0$ such that $|f(t)| \leq M_x$ for all $t \in (x - \delta_x, x + \delta_x)$.

Definition (Locally bounded on a set)

A real-valued function f is **locally bounded** on the set E if f is locally bounded at each point $x \in E$.

Note: The size of the neighbourhood (δ_x) and the local bound (M_x) depend on the point x.

Local vs. Global properties

Example (Function that is not even locally bounded)

Give an example of a function that is defined on the interval $(0, 1)$ but is not locally bounded on $(0, 1)$.

(solution on board)

Example (Function that is a mess near 0)

Give an example of a function $f(x)$ that is defined everywhere, yet in any neighbourhood of the origin there are infinitely many points at which f is not locally bounded.

(solution on board)

Extra Challenge Problem: Is there a function $f : \mathbb{R} \to \mathbb{R}$ that is not locally bounded anywhere?

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$$\int_M d\omega = \int_{\partial M} \omega$$

Mathematics 3A03
Real Analysis I

Instructor: David Earn

Lecture 16
Topology of $\mathbb{R}$ IV
Wednesday 13 February 2019

Announcements

- Assignment 3 is **Due Friday 15 Feb 2019 at 1:25pm via crowdmark**
- **Math 3A03 Test #1**
  Monday 4 March 2019 at 7:00pm in MDCL 1110
  (room is booked for 90 minutes; you should not feel rushed)

Local vs. Global properties

Example (Function that is not even locally bounded)

Give an example of a function that is defined on the interval $(0, 1)$ but is not locally bounded on $(0, 1)$.

(solution on board)

Example (Function that is a mess near 0)

Give an example of a function $f(x)$ that is defined everywhere, yet in any neighbourhood of the origin there are infinitely many points at which f is not locally bounded.

(solution on board)

Extra Challenge Problem: Is there a function $f : \mathbb{R} \to \mathbb{R}$ that is not locally bounded anywhere?

Local vs. Global properties

- What condition(s) rule out such pathological behaviour?
- When does a property holding locally (near any given point in a set) imply that it holds globally (for the set as a whole)?
- For example: What condition(s) must a set $E \subseteq \mathbb{R}$ satisfy in order that a function f that is locally bounded on E is necessarily bounded on E?
- We will see that the condition we are seeking is that the set E must be "**compact**" . . .

Compactness

Recall the Bolzano-Weierstrass theorem, which we proved when investigating sequences of real numbers:

Theorem (Bolzano-Weierstrass theorem for sequences)

Every bounded sequence in $\mathbb{R}$ contains a convergent subsequence.

For *any set of real numbers*, we define:

Definition (Bolzano-Weierstrass property)

A set $E \subseteq \mathbb{R}$ is said to have the **Bolzano-Weierstrass property** iff any sequence of points chosen from E has a subsequence that converges to a point in E.

Compactness

Theorem (Bolzano-Weierstrass theorem for sets)

A set $E \subseteq \mathbb{R}$ has the Bolzano-Weierstrass property iff E is closed and bounded.

(solution on board) (Textbook (TBB) Theorem 4.21, p. 241)

Notes:

- Why do we need both *closed* and *bounded*? Why didn't we need *closed* in the original version of the Bolzano-Weierstrass theorem (for sequences)?
  - Because we didn't require the limit of the convergent subsequence to be in the set!
- The Bolzano-Weierstrass theorem for sets implies that "If $E \subseteq \mathbb{R}$ is bounded then its closure $\overline{E}$ has the Bolzano-Weierstrass property".
  - The original Bolzano-Weierstrass theorem for sequences is a special case of this statement because any convergent sequence together with its limit is a closed set.

Bijections

The terms **one-to-one** (injective), **onto** (surjective), and **one-to-one correspondence** (bijection) are giving some students trouble.

(Recall, we used bijection in our definition of countable.)

Let's take a step back and recall:

- When we define a **function**, we need three things:
  - the **domain**, *i.e.*, the set to which the function is applied;
  - the **codomain**, *i.e.*, the target set where the values of the function lie;
  - a rule for taking elements of the domain into the codomain.
- If we write $f : A \to B$ then A is the domain and B is the codomain.
- The **range** of a function is the subset of the codomain consisting of all values of the function applied to the domain.

Bijections

Example

Let $f(x) = x^2$, $x \in \mathbb{R}$.

- Is f onto $\mathbb{R}$?
- Is f one-to-one on $\mathbb{R}$? On any interval?
- Is f a bijection?

Example

- Find a bijection between $[0, \infty)$ to $[1, \infty)$.
- Find a different bijection between $[0, \infty)$ to $[1, \infty)$.

Extra Challenge Problem:
Construct a bijection between $[0, 1]$ and $(0, 1)$.

Compactness

Definition (Open Cover)

Let $E \subseteq \mathbb{R}$ and let $\mathcal{U}$ be a family of open intervals. If for every $x \in E$ there exists at least one interval $U \in \mathcal{U}$ such that $x \in U$, *i.e.*,

$$E \subseteq \bigcup \{U : U \in \mathcal{U}\},$$

then $\mathcal{U}$ is called an **open cover** of E.

Example (Open covers of $\mathbb{N}$)

Give examples of open covers of $\mathbb{N}$.

- $\mathcal{U} = \left\{ \left(n - \frac{1}{2}, n + \frac{1}{2} \right) : n = 1, 2, \ldots \right\}$
- $\mathcal{U} = \{(0, \infty)\}$
- $\mathcal{U} = \{(0, \infty), \mathbb{R}, (\pi, 27)\}$

Compactness

Example (Open covers of $\{\frac{1}{n} : n \in \mathbb{N}\}$)

- $\mathcal{U} = \{(0,1),\ (0,2),\ \mathbb{R},\ (\pi, 27)\}$
- $\mathcal{U} = \{(0,2)\}$
- $\mathcal{U} = \left\{\left(\frac{1}{n},\ \frac{1}{n} + \frac{3}{4}\right) : n = 1, 2, \ldots\right\}$

Example (Open covers of $[0,1]$)

- $\mathcal{U} = \{(-2,2)\}$
- $\mathcal{U} = \{(-\frac{1}{2}, \frac{1}{2}),\ (0,2)\}$
- $\mathcal{U} = \left\{\left(\frac{1}{n}, 2\right) : n = 1, 2, \ldots\right\} \cup \left\{\left(-\frac{1}{2}, \frac{1}{2}\right)\right\}$

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Mathematics 3A03
Real Analysis I

Instructor: David Earn

Lecture 17
Topology of $\mathbb{R}$ V
Friday 15 February 2019

Announcements

- Assignment 3 was **Due TODAY at 1:25pm via crowdmark**
  Solutions will be posted over the weekend.
- Assignment 4 will be posted over the weekend.
  Due Friday 8 March 2019 at 1:25pm via crowdmark
  BUT you should do it before Test #1.
- **Math 3A03 Test #1**
  Monday 4 March 2019 at 7:00pm in MDCL 1110
  (room is booked for 90 minutes; you should not feel rushed)

Compactness

Definition (Heine-Borel Property)

A set $E \subseteq \mathbb{R}$ is said to have the **Heine-Borel property** if every open cover of E can be reduced to a finite subcover. That is, if $\mathcal{U}$ is an open cover of E, then there exists a finite subfamily $\{U_1, U_2, \ldots, U_n\} \subseteq \mathcal{U}$, such that $E \subseteq U_1 \cup U_2 \cup \cdots \cup U_n$.

When does any open cover of a set E have a finite subcover?

Theorem (Heine-Borel Theorem)

A set $E \subseteq \mathbb{R}$ has the Heine-Borel property iff E is both closed and bounded.

(Textbook (TBB) pp. 249–250)

Compactness

Definition (Compact Set)

A set $E \subseteq \mathbb{R}$ is said to be **compact** if it has any of the following equivalent properties:

1. E is closed and bounded.
2. E has the Bolzano-Weierstrass property.
3. E has the Heine-Borel property.

Note: In spaces other than $\mathbb{R}$, these three properties are not necessarily equivalent. Usually the Heine-Borel property is taken as the definition of compactness.

Compactness

Example

Prove that the interval $(0, 1]$ is not compact by showing that it is not closed or not bounded.

(solution on board)

Example

Prove that the interval $(0, 1]$ is not compact by showing that it does not have the Bolzano-Weierstrass property.

(solution on board)

Example

Prove that the interval $(0, 1]$ is not compact by showing that it does not have the Heine-Borel property.

(solution on board)

Compactness

Example (Classic non-trivial compactness argument)

Let E be a compact subset of $\mathbb{R}$. Prove that if $f : E \to \mathbb{R}$ is locally bounded on E then f is bounded on E.

(solution on board)

Bolzano-Weierstrass approach: Textbook (TBB) p. 242
Heine-Borel approach: Textbook (TBB) p. 251

Example (Converse of above example)

Let $E \subseteq \mathbb{R}$. If every function $f : E \to \mathbb{R}$ that is locally bounded on E is bounded on E, then E is compact.

(solution on board)

Note: Contrapositive of converse is: If $E \subseteq \mathbb{R}$ is not compact then $\exists f : E \to \mathbb{R} \ni f$ is locally bounded on E but not bounded on E.

Complements and Closures problem

Example

How many distinct sets can be obtained from $E = [0, 1]$ by applying the complement and closure operations?

Consider this sequence of sets: $E_1 = [0, 1]$,
$E_2 = E_1^c = (-\infty, 0) \cup (1, \infty)$, $E_3 = \overline{E_2} = (-\infty, 0] \cup [1, \infty)$,
$E_4 = E_3^c = (0, 1)$, $E_5 = \overline{E_4} = E_1$.

Does this prove the answer is 4?

Extra Challenge Problem

If $E \subseteq \mathbb{R}$, how many distinct sets can be obtained by taking complements or closures of E and its successors? Put another way, if $\{E_n\}$ is a sequence of sets produced by taking the complement or closure of the previous set, how many distinct sets can such a sequence contain? If the answer is finite, find a set E that generates the maximum number in this way.