Mathematics and Statistics

$$\int_M d\omega = \int_{\partial M} \omega$$

Mathematics 3A03
Real Analysis I

Instructor: David Earn

Lecture 6
Sequences
Friday 18 January 2019

Announcements

- Solutions to Assignment 1 will be posted later today.
  Study them!
- Assignment 2: check the course web page over the weekend.
- Remember that solutions to assignments and tests from the 2016 and 2017 versions of the course are available on the course web site. Take advantage of these problems and solutions. They provide many useful examples that should help you prepare for tests and the final exam. (However, note that while most of the content of the course is the same this year, there are some differences.)
- No late submission of assignments. No exceptions.
  However, best 5 of 6 assignments will be counted.
  Always due 5 minutes before class on the due date.
- **Note as stated on course info sheet:** *Only a selection of problems on each assignment will be marked; your grade on each assignment will be based only on the problems selected for marking. Problems to be marked will be selected after the due date.*

Announcements for week of 21–25 January 2019

- Office hour on Monday 21 Jan 2019 will be at **3:30pm** (rather than the usual 1:30pm).
- Wednesday's lecture will be given by Niky Hristov.

Sequences

- A **sequence** is a list that goes on forever.
- There is a beginning (a "first term") but no end, *e.g.*,

$$\frac{1}{1},\ \frac{1}{2},\ \frac{1}{3},\ \frac{1}{4},\ \ldots,\ \frac{1}{n},\ \ldots$$

- We use the natural numbers $\mathbb{N}$ to label the terms of a sequence:

$$a_1,\ a_2,\ a_3,\ \ldots,\ a_n,\ \ldots$$

Formal definition of a sequence

Definition (Sequence of Real Numbers)

A **sequence of real numbers** is a function

$$f : \mathbb{N} \to \mathbb{R}\,.$$

A lot of different notation is common for sequences:

$$f(1),\ f(2),\ f(3),\ \dots \qquad \{f(n)\}_{n=1}^{\infty}$$

$$f_1,\ f_2,\ f_3,\ \dots \qquad \{f(n)\}$$

$$\{f(n) : n = 1, 2, 3, \dots\} \qquad \{f_n\}_{n=1}^{\infty}$$

$$\{f(n) : n \in \mathbb{N}\} \qquad \{f_n\}$$

Specifying sequences

There are two main ways to specify a sequence:

1. Direct formula.

Specify $f(n)$ for each $n \in \mathbb{N}$. $\square$

Example (arithmetic progression with common difference d)

Sequence is:

$$c,\ c+d,\ c+2d,\ c+3d,\ \ldots$$

$$\therefore\ f(n) = c + (n-1)d, \qquad n \in \mathbb{N}$$

$$\text{i.e.,}\quad x_n = c + (n-1)d, \qquad n = 1, 2, 3, \ldots$$

Specifying sequences

2. Recursive formula.

Specify first term and function $f(x)$ to **iterate**. □

i.e., Given x_1 and $f(x)$, we have $x_n = f(x_{n-1})$ for all $n > 1$.

$$x_2 = f(x_1), \quad x_3 = f(f(x_1)), \quad x_4 = f(f(f(x_1))), \quad \ldots$$

Example (arithmetic progression with common difference d)

$$x_1 = c, \quad f(x) = x + d$$

$$\therefore \; x_n = x_{n-1} + d, \qquad n = 2, 3, 4, \ldots$$

Note: f is the most typical function name for both the direct and recursive specifications. The correct interpretation of f should be clear from context.

Specifying sequences

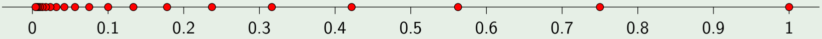
Example (geometric progression with common ratio r)

Sequence is: $c,\ cr,\ cr^2,\ cr^3,\ \ldots$

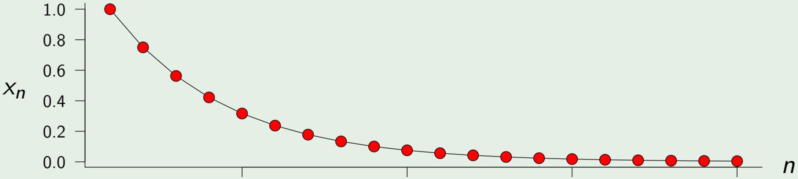
Direct formula: $x_n = f(n) = cr^{n-1}$, $n = 1, 2, 3, \ldots$

Recursive formula: $x_1 = c$, $f(x) = rx$, $x_n = f(x_{n-1})$

Number line representation of $\{x_n\}$ with $c = 1$ and $r = \frac{3}{4}$:



Graph of $f(n)$:



Specifying sequences

Example ($f(n) = 1 + \frac{1}{n^2}$)

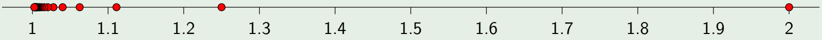
Sequence is: $2, \ \frac{5}{4}, \ \frac{10}{9} \ \frac{17}{16}, \ \ldots$

Direct formula: $x_n = f(n) = 1 + \frac{1}{n^2}, \ n = 1, 2, 3, \ldots$

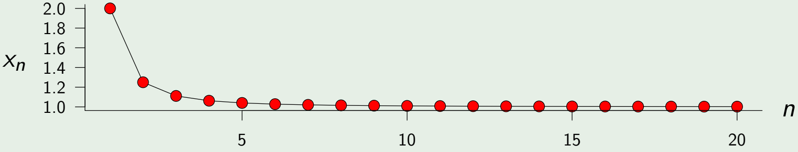
Recursive formula: $x_1 = 2, \quad f(x) = 1 + [1 + (x-1)^{-1/2}]^{-2}$

Get this formula by solving for n in terms of x in
$x = 1 + 1/(n-1)^2 \quad (= x_{n-1})$.
Such an inversion will NOT always be possible.

Number line representation of $\{x_n\}$:



Graph of $f(n)$:



Convergence of sequences

We know from previous experience that:

- $cr^{n-1} \to 0$ as $n \to \infty$ (if $|r| < 1$).
- $1 + \frac{1}{n^2} \to 1$ as $n \to \infty$.

How do we make our intuitive notion of **convergence** *mathematically rigorous*?

Informal definition: "$x_n \to L$ as $n \to \infty$" means "we can make the difference between x_n and L as small as we like by choosing n big enough".

More careful informal definition: "$x_n \to L$ as $n \to \infty$" means "given any *error tolerance*, say ε, we can make the distance between x_n and L smaller than ε by choosing n big enough".

Convergence of sequences

Definition (Limit of a sequence)

A sequence $\{s_n\}$ **converges to** L if, given any $\varepsilon > 0$ there is some integer N such that

$$\text{if } n \geq N \qquad \text{then} \qquad |s_n - L| < \varepsilon\,.$$

In this case, we write $\lim_{n\to\infty} s_n = L$ or $s_n \to L$ as $n \to \infty$ and we say that L is the **limit** of the sequence $\{s_n\}$.

Note: To use this definition to prove that the limit of a sequence is L, we start by imagining that we are given some error tolerance $\varepsilon > 0$. Then we have to find a suitable N, which will depend on ε. This means that *the N that we find will be a function of ε*.

Shorthand:

$$\lim_{n\to\infty} s_n = L \quad \stackrel{\text{def}}{=} \quad \forall\varepsilon > 0 \quad \exists N \in \mathbb{N} \quad \ni \quad n \geq N \implies |s_n - L| < \varepsilon.$$

Convergence of sequences

Convergence terminology:

- A sequence that converges is said to be **convergent**.
- A sequence that is not convergent is said to be **divergent**.

Remark (Sequences in spaces other than $\mathbb{R}$)

The formal definition of a limit of a sequence works in any space where we have a notion of distance if we replace $|s_n - L|$ with $d(s_n, L)$.

Convergence of sequences

Example

Use the formal definition of a limit of a sequence to prove that

$$\frac{n^2+1}{n^2} \to 1 \quad \text{as} \quad n \to \infty\,.$$

(solution on board)

Note: Our strategy here was to solve for n in the inequality $|s_n - L| < \varepsilon$. From this we were able to infer how big N has to be in order to ensure that $|s_n - L| < \varepsilon$ for all $n \geq N$. That much was "rough work". Only after this rough work did we have enough information to be able to write down a rigorous proof.

Mathematics and Statistics

$$\int_M d\omega = \int_{\partial M} \omega$$

Mathematics 3A03
Real Analysis I

Instructor: David Earn

Lecture 7
Sequences II
Monday 21 January 2019

Announcements

- Solutions to Assignment 1 have been posted. **Study them!**
- Assignment 2 has been posted. Due Friday 1 Feb 2019 at 1:25pm.

Convergence of sequences

Example

Use the formal definition of a limit of a sequence to prove that

$$\frac{n^5 - n^3 + 1}{n^8 - n^5 + n + 1} \to 0 \quad \text{as} \quad n \to \infty\,.$$

(solution on board)

Note: In this example, it was not possible to solve for n in the inequality $|s_n - L| < \varepsilon$. Instead, we first needed to bound $|s_n - L|$ by a much simpler expression that is always greater than $|s_n - L|$. If that bound is less than ε then so is $|s_n - L|$.

Uniqueness of limits

Theorem (Uniqueness of limits)

If $\lim_{n\to\infty} s_n = L_1$ *and* $\lim_{n\to\infty} s_n = L_2$ *then* $L_1 = L_2$.

(solution on board)

So, we are justified in referring to "the" limit of a convergent sequence.

Divergence of sequences

Divergence is the logical opposite (negation) of convergence.
We can infer the formal meaning of divergence by taking the *logical negation* of the formal definition of convergence.
Doing so, we find that the sequence $\{s_n\}$ diverges (*i.e.*, does not converge to any $L \in \mathbb{R}$) iff

$$\forall L \in \mathbb{R}, \ \ \exists \varepsilon > 0 \ \text{ such that: } \forall N \in \mathbb{N} \ \ \exists n \geq N \ \ni \ \ |s_n - L| \geq \varepsilon.$$

Notes:

- The n that exists will, in general, depend on L, ε and N.
- This is the meaning of not converging to any limit, but it does not tell us anything about what happens to the sequence $\{s_n\}$ as $n \to \infty$.

Divergence to $\pm\infty$

Definition (Divergence to ∞)

The sequence $\{s_n\}$ of real numbers **diverges to ∞** if, for every real number M there is an integer N such that

$$n \ge N \quad \Longrightarrow \quad s_n \ge M \,,$$

in which case we write $s_n \to \infty$ as $n \to \infty$ or $\lim_{n\to\infty} s_n = \infty$.

Definition (Divergence to $-\infty$)

The sequence $\{s_n\}$ of real numbers **diverges to $-\infty$** if, for every real number M there is an integer N such that

$$n \ge N \quad \Longrightarrow \quad s_n \le M \,.$$

Divergence to ∞

Example

Use the formal definition to prove that

$$\left\{\frac{n^3-1}{n+1}\right\} \quad \text{diverges to } \infty\,.$$

(solution on board)

Approach: Find a lower bound for the sequence that is a simple function of n and show that that can be made bigger than any given M.

Mathematics
and Statistics

$$\int_M d\omega = \int_{\partial M} \omega$$

Mathematics 3A03
Real Analysis I

Instructor: David Earn

Lecture 8
Sequences III
Wednesday 23 January 2019

What we've done so far on sequences

- Definition of convergence.
- Definition of divergence.
- Definition of divergence to $\pm\infty$.
- Examples.

Divergence to ∞

Example (Example from last time)

Use the formal definition to prove that $\left\{ \dfrac{n^3-1}{n+1} \right\}$ diverges to ∞.

Clean proof.

Given $M \in \mathbb{R}^{>0}$, let $N = \lceil M \rceil + 1$. Then $N - 1 = \lceil M \rceil \geq M$. $\therefore \forall n \geq N$, $n - 1 \geq M$. Now observe that

$$\forall n \in \mathbb{N}, \quad n - 1 = \frac{(n-1)(n+1)}{n+1} = \frac{n^2-1}{n+1} \leq \frac{n^3-1}{n+1}\,.$$

$\therefore \forall n \geq N$ we have

$$\frac{n^3-1}{n+1} \geq M\,,$$

as required. $\square$

Sequences of partial sums (a.k.a. Series)

Given a sequence $\{x_n\}$, we define the **sequence of partial sums of $\{x_n\}$** to be $\{s_n\}$, where

$$s_n = \sum_{k=1}^{n} x_k = x_1 + x_2 + \cdots + x_n \,.$$

Note: We can start from any integer, not necessarily $k = 1$.

Boundedness of sequences

A sequence is said to be bounded if its range is a bounded set.

Definition (Bounded sequence)

A sequence $\{s_n\}$ is **bounded** if there is a real number M such that every term in the sequence satisfies $|s_n| \le M$.

Theorem (Every convergent sequence is bounded.)

$$L \in \mathbb{R} \ \wedge \ \lim_{n\to\infty} s_n = L \quad \Longrightarrow \quad \exists M > 0 \ \ni \ |s_n| \le M \ \forall n \in \mathbb{N}.$$

(solution on board)

Note: The converse is FALSE.
Proof? Find a counterexample, *e.g.*, $\{(-1)^n\}$.

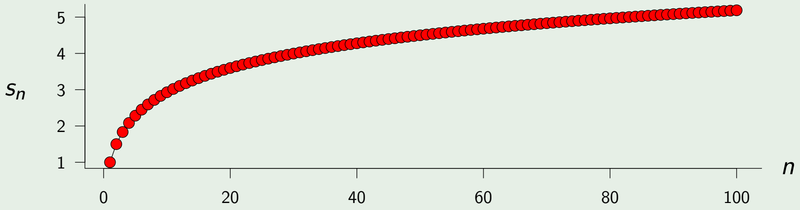
Boundedness of sequences

Corollary (Unbounded sequences diverge)

If $\{s_n\}$ is unbounded then $\{s_n\}$ diverges.

Example (The harmonic series diverges)

Consider the **harmonic series** $s_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$.



Prove that s_n diverges to ∞.

(solution on board)

Harmonic series – idea for proof of divergence

Approach: Group terms and use the corollary above.

$$\underbrace{\underbrace{\underbrace{\underbrace{\left(1+\frac{1}{2}\right)}_{>\ 1\times\frac{1}{2}}}_{s_2\ >\ 1\times\frac{1}{2}} + \underbrace{\left(\frac{1}{3}+\frac{1}{4}\right)}_{>\ 2\times\frac{1}{4}}}_{s_4\ >\ 2\times\frac{1}{2}} + \underbrace{\left(\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}\right)}_{>\ 4\times\frac{1}{8}}}_{s_8\ >\ 3\times\frac{1}{2}} + \cdots$$

$$\implies \qquad s_{2^n} \ > \ n \times \frac{1}{2}$$

Note: These sorts calculations are just "rough work", not a formal proof.
A proof must be a clearly presented coherent argument from beginning to end.

Harmonic series – clean proof of divergence

Proof.

Part (i). Prove (e.g., by induction) that $\quad s_{2^n} > n/2 \quad \forall n \in \mathbb{N}$.

Part (ii). Suppose we are given $M \in \mathbb{R}$.

- If $M \le 0$ then note that $s_n > 0\ \forall n \in \mathbb{N}$.
- If $M > 0$, let $\tilde{N} = 2\lceil M \rceil$ and $N = 2^{\tilde{N}}$. Then, $\forall n \ge N$, we have $s_n \ge s_N = s_{2^{\tilde{N}}} > \tilde{N}/2 = \lceil M \rceil \ge M$, as required.

$\square$

Algebra of limits

Theorem (Algebraic operations on limits)

Suppose $\{s_n\}$ and $\{t_n\}$ are convergent sequences and $C \in \mathbb{R}$.

1. $\displaystyle\lim_{n\to\infty} C\, s_n = C\Big(\lim_{n\to\infty} s_n\Big)$;
2. $\displaystyle\lim_{n\to\infty} (s_n + t_n) = \Big(\lim_{n\to\infty} s_n\Big) + \Big(\lim_{n\to\infty} t_n\Big)$;
3. $\displaystyle\lim_{n\to\infty} (s_n - t_n) = \Big(\lim_{n\to\infty} s_n\Big) - \Big(\lim_{n\to\infty} t_n\Big)$;
4. $\displaystyle\lim_{n\to\infty} (s_n t_n) = \Big(\lim_{n\to\infty} s_n\Big)\Big(\lim_{n\to\infty} t_n\Big)$;
5. *if $t_n \neq 0$ for all n and $\lim_{n\to\infty} t_n \neq 0$ then*
   $$\lim_{n\to\infty}\left(\frac{s_n}{t_n}\right) = \frac{\lim_{n\to\infty} s_n}{\lim_{n\to\infty} t_n} .$$

(solution on board)

Revisit example

Example (previously proved directly from definition)

Use the algebraic properties of limits to prove that

$$\frac{n^5 - n^3 + 1}{n^8 - n^5 + n + 1} \to 0 \quad \text{as} \quad n \to \infty\,.$$

(solution on board)

McMaster University

Mathematics and Statistics

$$\int_M d\omega = \int_{\partial M} \omega$$

Mathematics 3A03
Real Analysis I

Instructor: David Earn

Lecture 9
Sequences IV
Friday 25 January 2019

Announcements

- Assignment 2 is posted.
  Due next Friday, 1 Feb 2019, at 1:25pm.

What we've done so far on sequences

- Definition of convergence.
- Definition of divergence.
- Definition of divergence to $\pm\infty$.
- Examples.
- Every convergent sequence is bounded.
- Harmonic series diverges.
- Algebra of limits (more today).

Product Rule for Limits

The 4th item in the algebra of limits theorem was:

Theorem (Product Rule for Limits)

If $s_n \to S$ and $t_n \to T$ as $n \to \infty$ then $s_n t_n \to ST$ as $n \to \infty$.

Proof.

For any $n \in \mathbb{N}$,
$$\begin{aligned} |s_n t_n - ST| &= |s_n t_n - ST + s_n T - s_n T| \\ &= |s_n(t_n - T) + T(s_n - S)| \\ &\le |s_n||t_n - T| + |T||s_n - S| \end{aligned}$$

Now, $\{s_n\}$ converges, so it is bounded by some $M > 0$, *i.e.*, $|s_n| \le M$ $\forall n \in \mathbb{N}$. Therefore, given $\varepsilon > 0$, choose $N \in \mathbb{N}$ such that

$$|t_n - T| < \frac{\varepsilon}{2M} \qquad \text{and} \qquad |s_n - S| < \frac{\varepsilon}{2(1 + |T|)}\,.$$

Then $|s_n t_n - ST| < \varepsilon/2 + \varepsilon/2 = \varepsilon$, as required. □

Quotient Rule for Limits

Quotient Rule was the 5th item in the algebra of limits theorem.

Lemma (Reciprocal Rule for Limits)

If $t_n \neq 0$ $\forall n$ and $t_n \to T \neq 0$ then $1/t_n \to 1/T$.

Proof.

For any $n \in \mathbb{N}$, $\left|\dfrac{1}{t_n} - \dfrac{1}{T}\right| = \left|\dfrac{t_n - T}{t_n T}\right| = |t_n - T| \cdot \dfrac{1}{|t_n|} \cdot \dfrac{1}{|T|}$.

Since $\{t_n\}$ converges, $\exists N_1 \in \mathbb{N}$ such that $\forall n \geq N_1$, $|t_n| > |T|/2$ (details on next slide) and hence $1/|t_n| < 2/|T|$.

Now choose $N \geq N_1$ such that $|t_n - T| < \varepsilon |T|^2/2$. Then

$$\left|\frac{1}{t_n} - \frac{1}{T}\right| = |t_n - T| \cdot \frac{1}{|t_n|} \cdot \frac{1}{|T|} < \frac{\varepsilon |T|^2}{2} \cdot \frac{2}{|T|} \cdot \frac{1}{|T|} = \varepsilon\,,$$

as required. □

Quotient Rule for Limits

Details missing on previous slide: (consider $\varepsilon = \frac{|T|}{2}$)

Since $t_n \to T$, $\exists N_1 \in \mathbb{N}$ such that $\forall n \geq N_1$, $|t_n - T| < \dfrac{|T|}{2}$,

i.e., $-\dfrac{|T|}{2} < t_n - T < \dfrac{|T|}{2}$, i.e., $T - \dfrac{|T|}{2} < t_n < T + \dfrac{|T|}{2}$.

If $T > 0$ this says

$$0 < \frac{T}{2} < t_n < \frac{3T}{2},$$

whereas if $T < 0$ it says

$$\frac{3T}{2} < t_n < \frac{T}{2} < 0.$$

In either case, $\forall n \geq N_1$, we have $0 < \dfrac{|T|}{2} < |t_n|$.

Order properties of limits (§2.8)

Theorem (Limits retain order)

If $\{s_n\}$ and $\{t_n\}$ are convergent sequences then

$$s_n \le t_n \quad \forall n \in \mathbb{N} \qquad \Longrightarrow \qquad \lim_{n\to\infty} s_n \le \lim_{n\to\infty} t_n \,.$$

(solution on board)

Note: If $s_n < t_n$ for all $n \in \mathbb{N}$, can we conclude that

$$\lim_{n\to\infty} s_n < \lim_{n\to\infty} t_n \qquad ?$$

No! No! No! No! No! No!! NO!!!!!!!!!!

Order properties of limits (§2.8)

Theorem (Limits retain bounds)

If $\{s_n\}$ is a convergent sequence then

$$\alpha \le s_n \le \beta \quad \forall n \in \mathbb{N} \qquad \Longrightarrow \qquad \alpha \le \lim_{n\to\infty} s_n \le \beta\,.$$

(solution on board)

Order properties of limits (§2.8)

Theorem (Squeeze Theorem)

If $\{s_n\}$ and $\{t_n\}$ are convergent sequences such that

(i) $s_n \le x_n \le t_n \quad \forall n \in \mathbb{N}$, *($x_n$ is always between them)*

(ii) $\lim\limits_{n\to\infty} s_n = \lim\limits_{n\to\infty} t_n = L$. *(both approach the same limit)*

Then $\{x_n\}$ is convergent and $\lim\limits_{n\to\infty} x_n = L$.

Proof? (What's **WRONG**?).

$\{s_n\}$ and $\{t_n\}$ are both bounded since they both converge. $\{x_n\}$ is therefore bounded (by the lower bound of $\{s_n\}$ and the upper bound of $\{t_n\}$). $\{x_n\}$ therefore converges, say $x_n \to X$. Hence, by order retension, $L \le X \le L \implies X = L$. □

(solution on board)

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Mathematics and Statistics

$$\int_M d\omega = \int_{\partial M} \omega$$

Mathematics 3A03
Real Analysis I

Instructor: David Earn

Lecture 10
Sequences V
Monday 28 January 2019

Announcements

- Assignment 2.
  Due next Friday, 1 Feb 2019, at 1:25pm.
- Office hour today at **4:30pm** (not the usual 1:30pm)
- Niky Hristov's notes for his lecture (Lecture 8, last Wednesday) are now posted on the Lectures page course web site.

What we've done so far on sequences

- Definition of convergence.
- Definition of divergence.
- Definition of divergence to $\pm\infty$.
- Every convergent sequence is bounded.
- Harmonic series diverges.
- Algebra of limits (sums, products, quotients).
- Order properties of limits; squeeze theorem

Today:

- Proof of Squeeze Theorem
- Absolute value and max/min of limits.
- Monotone convergence.

Order properties of limits (§2.8)

Theorem (Limits of Absolute Values)

If $\{s_n\}$ converges then so does $\{|s_n|\}$, and

$$\lim_{n\to\infty} |s_n| = \left|\lim_{n\to\infty} s_n\right| .$$

(solution on board)

Proof.

Suppose $s_n \to L$. Given $\varepsilon > 0$, choose $N \in \mathbb{N}$ such that $\forall n \geq N, |s_n - L| < \varepsilon$. But for any $x, y \in \mathbb{R}$, we know* from the triangle inequality that $||x| - |y|| \leq |x - y|$. Therefore, $\forall n \geq N$, $||s_n| - |L|| \leq |s_n - L| < \varepsilon$. Thus, $|s_n| \to |L|$, as required. □

* To see this, observe that $|x| = |x - y + y| \leq |x - y| + |y|$, which implies $|x| - |y| \leq |x - y|$. Similarly, $|y| = |y - x + x| \leq |y - x| + |x|$, which implies $|y| - |x| \leq |y - x|$, which can in turn be rewritten $-(|x| - |y|) \leq |x - y|$. Combining these inequalities, we have $||x| - |y|| \leq |x - y|$.

Order properties of limits (§2.8)

Corollary (Max/Min of Limits)

If $\{s_n\}$ *and* $\{t_n\}$ *converge then* $\{\max\{s_n, t_n\}\}$ *and* $\{\min\{s_n, t_n\}\}$ *both converge and*

$$\lim_{n\to\infty} \max\{s_n, t_n\} = \max\left\{\lim_{n\to\infty} s_n,\ \lim_{n\to\infty} t_n\right\},$$

$$\lim_{n\to\infty} \min\{s_n, t_n\} = \min\left\{\lim_{n\to\infty} s_n,\ \lim_{n\to\infty} t_n\right\}.$$

Idea for proof:

$$\forall x, y \in \mathbb{R} \quad \max\{x, y\} = \frac{x+y}{2} + \frac{|x-y|}{2}$$

$$\forall x, y \in \mathbb{R} \quad \min\{x, y\} = \frac{x+y}{2} - \frac{|x-y|}{2}$$

Prove these facts, then use theorems on sums and absolute values of limits.

Monotone convergence (§2.9)

Definition (Monotonic sequence)

The sequence $\{s_n\}$ is **monotonic** iff it satisfies any of the following conditions:

(i) **Increasing**: $s_1 < s_2 < s_3 < \cdots < s_n < s_{n+1} < \cdots$;

(ii) **Decreasing**: $s_1 > s_2 > s_3 > \cdots > s_n > s_{n+1} > \cdots$;

(iii) **Non-decreasing**: $s_1 \le s_2 \le s_3 \le \cdots \le s_n \le s_{n+1} \le \cdots$;

(iv) **Non-increasing**: $s_1 \ge s_2 \ge s_3 \ge \cdots \ge s_n \ge s_{n+1} \ge \cdots$.

Monotone convergence (§2.9)

Theorem (Monotone Convergence Theorem)

A monotonic sequence $\{s_n\}$ is convergent iff it is bounded.
In particular,

(i) $\{s_n\}$ *non-decreasing and unbounded* $\implies s_n \to \infty$;
(ii) $\{s_n\}$ *non-decreasing and bounded* $\implies s_n \to \sup\{s_n\}$;
(iii) $\{s_n\}$ *non-increasing and unbounded* $\implies s_n \to -\infty$;
(iv) $\{s_n\}$ *non-increasing and bounded* $\implies s_n \to \inf\{s_n\}$.

(solution on board)

Subsequences

Definition (Subsequence)

Let $\{s_1, s_2, s_3, \ldots\}$ be a sequence. If $\{n_1, n_2, n_3, \ldots\}$ is an increasing sequence of natural numbers then $\{s_{n_1}, s_{n_2}, s_{n_3}, \ldots\}$ is a **subsequence** of $\{s_1, s_2, s_3, \ldots\}$.

Example (Subsequences)

Consider the sequence $\{s_n\}$ defined by $s_n = n^2$ for all $n \in \mathbb{N}$. What are the first few terms of these subsequences?

- $\{s_n : n \text{ even}\}$ $\quad \{2^2, 4^2, 6^2, \ldots\}$
- $\{s_n : n = 2k+1,\ \exists k \in \mathbb{N}\}$ $\quad \{3^2, 5^2, 7^2, \ldots\}$
- $\{s_{2n+1}\}$ Same as line above
- $\{s_{2^n}\}$ $\quad \{2^2, 4^2, 8^2, \ldots\}$
- $\{s_{n^2}\}$ $\quad \{1^2, 4^2, 9^2, \ldots\}$

Subsequences

Given any sequence $\{s_n\}$, can you always find a subsequence that is monotonic?

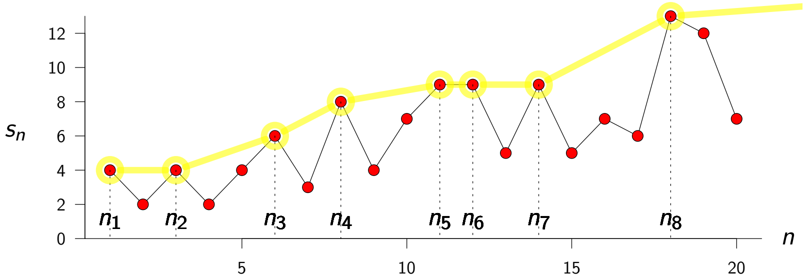
Theorem

Every sequence contains a monotonic subsequence.

Let's draw some pictures to help us visualize how we might construct a proof. . .

Idea for proof that every sequence contains a monotonic subsequence ("point of no return")

Given a sequence $\{s_1, s_2, s_3, \ldots\}$, try to build a subsequence $\{s_{n_1}, s_{n_2}, s_{n_3}, \ldots\}$ that is non-decreasing ($s_{n_1} \le s_{n_2} \le s_{n_3} \le \cdots$) by discarding any terms that are less than the running maximum:



If this works indefinitely then we have a non-decreasing subsequence. But if we can find only finitely many such terms then we're stuck because our subsequence is defined using earlier terms.

McMaster University

Mathematics and Statistics

$$\int_M d\omega = \int_{\partial M} \omega$$

Mathematics 3A03
Real Analysis I

Instructor: David Earn

Lecture 11
Sequences VI
Wednesday 30 January 2019

Announcements

- Niky's Monday office hour has been been switched to Tuesdays at 2:30pm. Note that Math 3D03 students have priority during that hour.
- Please send me an e-mail ASAP if you have a conflict with either of the midterm tests.
- Proof of Limits of Absolute Values theorem is now in the slides for Lecture 10.
- Plan for today:
  - Prove Monotone Convergence Theorem
  - Prove that Every sequence contains a monotonic subsequence.
  - Maybe state and prove Bolzano-Weierstrass Theorem.

Monotone convergence (§2.9)

Theorem (Monotone Convergence Theorem)

A monotonic sequence $\{s_n\}$ is convergent iff it is bounded.
In particular,

(i) $\{s_n\}$ *non-decreasing and unbounded* $\implies s_n \to \infty$;

(ii) $\{s_n\}$ *non-decreasing and bounded* $\implies s_n \to \sup\{s_n\}$;

(iii) $\{s_n\}$ *non-increasing and unbounded* $\implies s_n \to -\infty$;

(iv) $\{s_n\}$ *non-increasing and bounded* $\implies s_n \to \inf\{s_n\}$.

(solution on board)

Proof of Monotone Convergence Theorem

Given a monotonic sequence $\{s_n\}$ we must show that

$$\{s_n\} \text{ converges } \iff \{s_n\} \text{ is bounded}$$

Proof of "$\Longrightarrow$" and part (ii).

$\Longrightarrow$ For <u>any</u> sequence (monotonic or not) convergent implies bounded.

$\Longleftarrow$ [part (ii)] Suppose $\{s_n\}$ is non-decreasing, *i.e.*, $s_n \le s_{n+1}$ for all $n \in \mathbb{N}$. Since $\{s_n\}$ is bounded, it has a <u>least</u> upper bound, say $L = \sup\{s_n\}$. We will now show that $s_n \to L$, *i.e.*, $\forall \varepsilon > 0 \; \exists N \in \mathbb{N} \ni \forall n \ge N$, $|s_n - L| < \varepsilon$.

Before proceeding, note that since $L = \sup\{s_n\}$, it follows that $|s_n - L| < \varepsilon \iff L - s_n < \varepsilon \iff L - \varepsilon < s_n$.

Given $\varepsilon > 0$, choose $N \in \mathbb{N}$ such that $s_N > L - \varepsilon$ (which is possible $\because$ L is the <u>least</u> upper bound of $\{s_n\}$). But $\{s_n\}$ is non-decreasing, so $\forall n \ge N$ we have $s_N \le s_n \implies -s_n \le -s_N \implies L - s_N \le L - s_n < \varepsilon$. $\square$

Proof of Monotone Convergence Theorem

$$\text{Monotonic} \implies \Big[\{s_n\} \text{ converges} \iff \{s_n\} \text{ is bounded} \Big]$$

Proof of parts (i), (iii), (iv).

[part (i)] Suppose $\{s_n\}$ is non-decreasing and *un*bounded. It follows that $\{s_n\}$ diverges, since convergent sequences are bounded. Since $\{s_n\}$ is non-decreasing, it is bounded below (by s_1, for example). Hence $\{s_n\}$ (which is unbounded) must not be bounded above. Consequently, given any $M \in \mathbb{R}$, $\exists N \in \mathbb{N}$ such that $s_N > M$. But $\{s_n\}$ is non-decreasing, so $s_n > M$ for all $n \geq N$, as required.

Proof of [part (iii)] is similar to [part (i)].

Proof of [part (iv)] is similar to [part (ii)]. $\square$

Monotonic subsequences

Given any sequence $\{s_n\}$, can you always find a subsequence that is monotonic?

Theorem

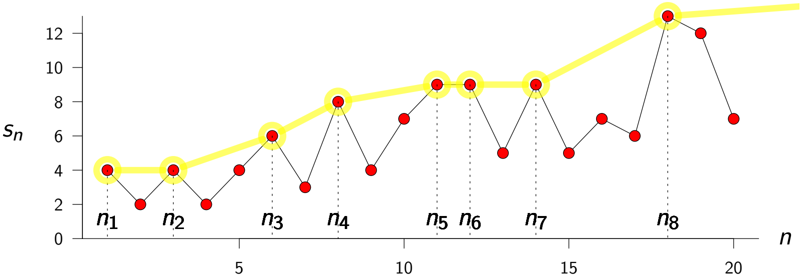
Every sequence contains a monotonic subsequence.

(Textbook (TBB) §2.11, Theorem 2.39, p. 79)

There are no pictures accompanying the proof in the textbook. So let's draw some pictures to help us visualize how we might construct a proof...

Idea for proof that every sequence contains a monotonic subsequence ("point of no return")

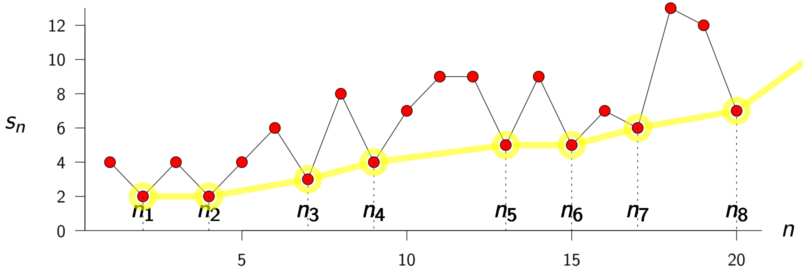
Given a sequence $\{s_1, s_2, s_3, \ldots\}$, try to build a subsequence $\{s_{n_1}, s_{n_2}, s_{n_3}, \ldots\}$ that is <u>non-decreasing</u> ($s_{n_1} \le s_{n_2} \le s_{n_3} \le \cdots$) by discarding any terms that are less than the running maximum:



If this works indefinitely then we have a non-decreasing subsequence. But if we can find only finitely many such terms then we're stuck because our subsequence is defined using <u>earlier</u> terms.

Better idea for proof that every sequence contains a monotonic subsequence ("turn-back point")

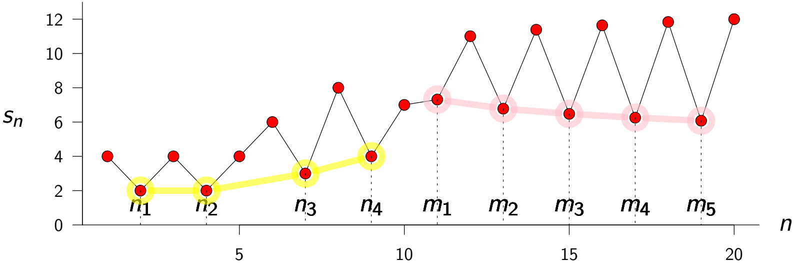
Given a sequence $\{s_1, s_2, s_3, \ldots\}$, try to build a subsequence $\{s_{n_1}, s_{n_2}, s_{n_3}, \ldots\}$ that is non-decreasing ($s_{n_1} \leq s_{n_2} \leq s_{n_3} \leq \cdots$) by identifying terms that are less than or equal to all later terms.



If this works indefinitely then we have a non-decreasing subsequence. What if there are only finitely many such terms? (There might not be any at all!)

Better idea for proof that every sequence contains a monotonic subsequence ("turn-back point")

If there are only finitely many s_{n_i} such that $s_{n_i} \leq s_n \; \forall n > n_i$ ...



... then after the last "turn-back point" (s_{n_4} above) there must be some $m_1 > n_4$ such that s_{m_1} is **not** $\leq$ all later terms, *i.e.*, $\exists m_2 > m_1$ with $s_{m_2} < s_{m_1}$, and similarly for m_2, so there must be a decreasing subsequence $\quad s_{m_1} > s_{m_2} > s_{m_3} > \cdots$

Convergent subsequences

Theorem (Bolzano-Weierstrass theorem)

Every bounded sequence of real numbers contains a convergent subsequence.

Proof.

Suppose $\{x_n\}$ is a bounded sequence. It follows from the previous theorem that $\{x_n\}$ contains a subsequence $\{x_{m_k}\}$ that is monotonic. Since $\{x_n\}$ is bounded, the subsequence $\{x_{m_k}\}$ is bounded as well (by the same bound). Thus, $\{x_{m_k}\}$ is a subsequence of $\{x_n\}$ that is both bounded and monotonic. Hence, it converges by the Monotone Convergence Theorem. □

McMaster University

Mathematics and Statistics

$$\int_M d\omega = \int_{\partial M} \omega$$

Mathematics 3A03
Real Analysis I

Instructor: David Earn

Lecture 12
Sequences VII
Friday 1 February 2019

Announcements

- Assignment 2 was due before class today.
- Assignment 3 will be posted this weekend.
  Due Friday 15 Feb 2019 at 1:25pm.
  IMPORTANT CHANGE:
  - For the remainder of the term, assignments must be submitted **electronically**, not as a hardcopy.
  - By the end of this weekend, you should have received a link for Assignment 3 via e-mail from crowdmark. If you have not received such an e-mail by the end of the weekend, please e-mail earn@math.mcmaster.ca.
  - If you write your solutions by hand, you will need to scan or photograph them to submit them via the online system.
  - If you use LaTeX to create a pdf file, you will need to separate your solutions for each question.
  - Marked assignments will be available online, rather than being returned in tutorial.

Sequences Finale!

Cauchy sequences

Definition (Cauchy sequence)

A sequence $\{s_n\}$ is said to be a **Cauchy sequence** iff for all $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that if $m \geq N$ and $n \geq N$ then $|s_n - s_m| < \varepsilon$.

Theorem (Cauchy criterion)

A sequence of real numbers $\{s_n\}$ is convergent iff it is a Cauchy sequence.

Remark: The proof of the "only if" direction is easy. The proof of the "if" direction contains only one tricky feature: showing that every Cauchy sequence $\{s_n\}$ is bounded.

Cauchy sequences

Proof of Cauchy criterion

"only if": If $\{s_n\}$ converges then, given $\varepsilon > 0$, choose $N \in \mathbb{N}$ such that for all $n \geq N$, $|s_n - L| < \varepsilon/2$. Then, for any $m, n > N$ we have $|s_m - s_n| = |s_m - L + L - s_n| \leq |s_m - L| + |s_n - L| < (\varepsilon/2) + (\varepsilon/2) = \varepsilon$, as required.

"if": If we take $\varepsilon = 1$ in the definition of a Cauchy sequence, we find that there is some N such that $|s_m - s_n| < 1$ for all $m, n > N$. In particular, this means that $|s_m - s_{N+1}| < 1$ for all $m > N$, *i.e.,*

$$s_{N+1} - 1 < s_m < s_{N+1} + 1 \qquad \forall m > N.$$

Thus $\{s_m : m > N\}$ is bounded; moreover, since there are only finitely many other s_i's, the whole sequence $\{s_n\}$ is bounded. Hence, by the Bolzano-Weierstrass theorem, some subsequence of s_n converges; let's write this subsequence as $\{s_{m_k}\}$, and its limit as L.

. . . continued on next slide. . .

Cauchy sequences

Proof of Cauchy criterion (cont'd).

We will show that $\{s_n\}$ converges to L. To prove this, consider any $\varepsilon > 0$. Since the sequence $\{s_n\}$ is Cauchy, there is some $N \in \mathbb{N}$ such that

$$|s_n - s_m| < \frac{\varepsilon}{2} \qquad \text{for all } n, m \geq N.$$

Since the subsequence $\{s_{m_k}\}$ converges to L, there is some N' so that

$$|s_{m_k} - L| < \frac{\varepsilon}{2} \qquad \text{for all } k \geq N'.$$

Now fix an integer k satisfying $k \geq N'$ and $m_k \geq N$. Then $\forall n \geq N$,

$$\begin{aligned} |s_n - L| &\leq |s_n - s_{m_k} + s_{m_k} - L| \\ &\leq |s_n - s_{m_k}| + |s_{m_k} - L| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, \end{aligned}$$

as required. □

Cauchy sequences

Notes:

- The Cauchy criterion is sometimes easier to use in proofs than the original definition of convergence.
- Its significance is more evident in spaces other than $\mathbb{R}$, where Cauchy sequences do not necessarily converge.

Bijectivity

Definition (Bijection)

Let $f : A \to B$ be a function. Then

(i) f is **injective** (or **one-to-one**) if $\forall a_1, a_2 \in A$, $f(a_1) = f(a_2) \implies a_1 = a_2$.

(ii) f is **surjective** (or **onto**) if $\forall b \in B, \exists a \in A$ so that $f(a) = b$.

(iii) Then f is **bijective** if it is both **injective** and **surjective**.

A bijective function is said to be a **bijection**.

Note: A bijection is sometimes called a **one-to-one correspondence**. This termniology can be confusing because it means one-to-one **and** onto.

Countability

Definition (Countable set)

A set S is **countable** if S is finite, or if there is a bijective function $f : \mathbb{N} \to S$.
A set is **uncountable** if it is not countable.

Sequences $\{s_n\}_{n=1}^{\infty}$ are generalizations of the intuitive notion of sets whose elements can be counted.

Example

Suppose S is a subset of $\mathbb{R}$. Then S is countable if and only if S is the range of a sequence.

Countability

Theorem

The natural numbers $\mathbb{N}$ are countable.

(solution on board)

Theorem

The integers $\mathbb{Z}$ are countable.

(solution on board)

Theorem

The rational numbers $\mathbb{Q}$ are countable.

(solution on board)

What about $\mathbb{R}$?...

Countability

Theorem (Cantor)

The real numbers $\mathbb{R}$ are uncountable.

(solution on board)

Notes:

- The main argument in the proof is known as "Cantor's diagonal argument".
- We can infer that not only are some real numbers not rational, but there are "many more" real numbers than rational numbers.
- Cantor's proof depends on there being a binary expansion for any real number number...

Countability

Theorem (Existence and uniqueness of binary expansions)

If $x \in [0,1)$ then there is a sequence $\{a_n\}$ such that $a_n \in \{0,1\}$ $\forall n$ and

$$x = \sum_{n=1}^{\infty} \frac{a_n}{2^n}\,.$$

Specifically,
$$a_n = \left\lfloor \left(x - \sum_{i=1}^{n-1} \frac{a_i}{2^i} \right) 2^n \right\rfloor\,.$$

Moreover, this **binary representation** *is unique unless $x = m/2^k$ for some $k \in \mathbb{N}$ and $m \in \mathbb{N}$, in which case there are exactly two binary representations, the second being given by $\{b_n\}$ where*

$$b_n = \begin{cases} a_n & n < k, \\ 0 & n = k, \\ 1 & n > k. \end{cases}$$

Countability

Theorem (Properties of countable sets)

(i) *Any subset of a countable set is countable.*

(ii) *The union of a sequence of countable sets is countable.*

(iii) *No (non-degenerate*) interval is countable.*

* We normally assume implicitly that the endpoints of intervals are distinct. If the endpoints are the same then the interval is **degenerate**, *e.g.*, $(a, a) = \varnothing$ and $[a, a] = \{a\}$.