Game values and (sur)real numbers

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1 Introduction

Game theory and theory of games

- **Game theory** is the theory of games with *imperfect information*
  - Nash equilibria and so on
- **Theory of games** (or combinatorial game theory) is the theory of games with *perfect* information
  - . . . accidentally led to some of the most beautiful theories of analysis

Resources

- *On Numbers and Games*, Conway
- *Surreal Numbers*, Knuth
- *Winning Ways*, Berlekamp, Conway, Guy

Review

- We define the real numbers by:
  - Building the integers as nested sets
  - Building the rationals as equivalence classes of ordered pairs of integers
  - Building the reals as cuts of the rationals
- With deterministic games, we build all this at once
  - . . . and much more!

2 Games

Hackenbush

- Draw a picture
  - bLue lines can be removed by Left
  - Red lines can be removed by Right
  - greeN lines can be removed by aNyone
- On your turn, you remove one line
  - Lines no longer connected to ground are removed

Domineering

- On your turn, you place a domino on some sort of grid
  - Left places verticaL dominoes
  - Right places hoRizontal dominoes

Defining games

- Intuition: if you're playing a game, you have a set of moves
  - A move changes the game to a different game
  - If you don't have a move you lose!
- A **game** is
  - a set of options for the Left player, and a set of options for the Right player
    - $X = (X^L \mid X^R)$
  - Options are *previously defined* games

Adding games

- Intuition: to play the sum of two games, you move in one of them when it's your turn
- $A + B = (A + b^L, a^L + B \mid A + b^R, a^R + B)$

Negatives

- The **negative** of a game reverses the roles of Left and Right
  - $A = (A^L \mid A^R)$
  - $-A \equiv (-A^R \mid -A^L)$
- Again, relying on beautiful induction

3 Ordering games

- Intuition: adding game A to an existing game can't hurt Left *unless* Right has a good move
  - IOW, unless Right can move in A to a game that doesn't hurt Right
- $A \geq 0$ *unless*
  - Some option $a^R \leq 0$ **Def:** $-a^R \geq 0$
- This is a perfectly complete definition (induction again!) that tells you the outcome of any game

Game analysis

- $A \geq 0$ $A \leq 0 \implies A = 0$: second-player win
- $A \geq 0$ $A \not\leq 0 \implies A > 0$: Left wins
- $A \not\geq 0$ $A \leq 0 \implies A < 0$: Right wins
- $A \not\geq 0$ $A \not\leq 0 \implies A||B$: first-player win

Partial ordering

- Intuition: A is better (for Left) than B if A + the negative of B is good for left
- $A \geq B \iff A - B \geq 0$
- **Def:** $A - B = A + (-B)$

4 Values

- Two games have the same value if they have the same effect when added to any other game
- Which is the same as saying if they're equal under the partial ordering above
- Thus, a game value is an *equivalence class* of games
  - Like you learned about with the rationals

5 Numbers

- The values I've defined are a very cool group.
- But not very numerical:
  - $* + * = 0$

What is a (surreal) number?

- Intuition: a game is number-like if you never want to move. There's a certain advantage for a given player, and they "spend" it by moving
  - We build number-like games recursively.
- A number-game is: a set of options for the Left player, and a set of options for the Right player
  - $x = (x^L \mid x^R)$, s.t. no $x^L \geq$ any x^R
  - Options are *previously defined* number-games
- A number is a value associated with a class of number-games

Numbers

- We create the natural numbers as $n + 1 = (n|)$
  - Negative integers are then defined by negation rule
- We can create any finite *binary* expansion
  - $(2k + 1)/2^{n+1} = (k/2^n \mid (k + 1)/2^n)$
  - e.g., $7/16 = (3/8 \mid 1/2)$

The limit

- What happens if we take the limit of all numbers we can make in a finite number of steps?
- We can get all the reals ...
  - e.g., $1/3 = (0, 1/4, 5/16, \ldots \mid 1, 1/2, 3/8, \ldots)$
- plus some very weird stuff
  - $\omega = (0, 1, 2, \ldots \mid)$
  - $1/\omega = (0|1, 1/2, 1/4, \ldots)$

0.999...

- Is 0.999... really equal to 1?
- Depends on your definitions
- What is 0.1111... (base 2) as a game?

Multiplication

- Intuition: no real game intuition
- Motivation: $(x - x^S)(y - y^S)$ has a known sign
- ... and construct division
  - Insane simultaneous induction on simpler quotients, and on the main quotient
- The surreal numbers are a *field*

A wild and woolly set of infinities

- You can take as many limits as you want
- A collection of infinite numbers (the ordinals, plus you can add divide, multiply and root them)
- A similar collection of infinitely small numbers (infinitesimals)
  - The real numbers are just a subfield

Surreal arithmetic

- $\omega - 1$,
- $\omega/2$, $\sqrt{(}\omega)$
- Even crazier stuff: $\sqrt[3]{\omega - 1} - \pi/\omega$

6 Beyond numbers

Micro-infinitesimals

- If we allow values that aren't numbers, we have infinitesimals that are smaller than the smallest infinitesimal numbers

Temperature

- Cold games are games where moving makes the position worse for your side
  - Number games are games that are (recursively) cold
  - Red-blue hackenbush
- Neutral games are games where the positions are the same for left and Right
  - The theory of Nim values
  - Green hackenbush
- Hot games are games where there can be a positive value to moving
  - Example: domineering

Conclusion

- We can define a bewildering array of games with a simple, recursive definition
- By defining addition, we can organize these into values, which form a group under sensible game addition
- By recursively requiring making a move to have a cost, we identify a subset that we call the surreal numbers
  - these contain the reals, the infinite ordinals and a consistent set of infinitesimals
  - These surreal numbers form a field
- There are also interesting game values that are *not* numbers
- Game values are the best thing

Beyond the conclusions

- The option framework is sort of a generalization of
  - the Cantor framework for the ordinals
    - (building up, never a right option)
  - the Dedekind framework for the reals
    - (filling in, always a right option)

Simplicity theorem (numbers)

- The value of $(x^L \mid x^R)$ is the simplest, non-prohibited value
- Prohibited: if if is larger than some x^R or less than some x^L
- Simplest: earliest created; it has no options that are not prohibited
  - …or else those would be simpler, non-prohibited values

More simplicity

- If no non-prohibited value already exists, then the value is
  - $(x^L + x^R)/2$, if both exist
  - $x^L + 1$, if only x^L exists
  - …