McMaster University

Mathematics and Statistics

$$\int_M d\omega = \int_{\partial M} \omega$$

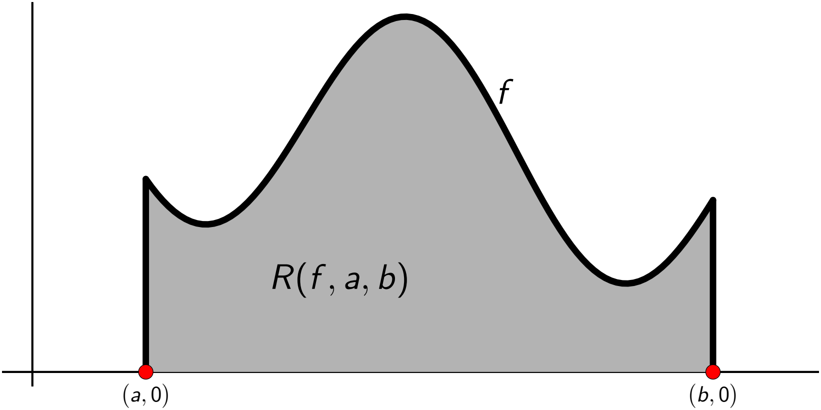
Mathematics 3A03
Real Analysis I

Instructor: David Earn

Lecture 26
Integration
Friday 8 November 2019

Integration

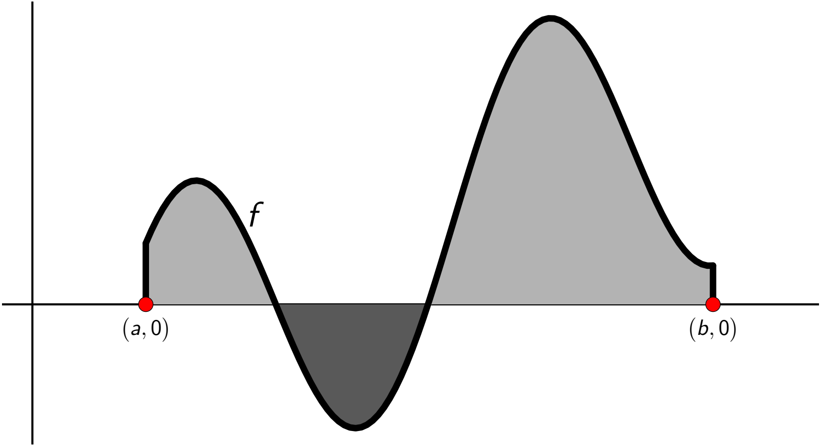
Integration



- "Area of region $R(f, a, b)$" is actually a very subtle concept.
- We will only scratch the surface of it.
- Textbook presentation of integral is different (but equivalent).

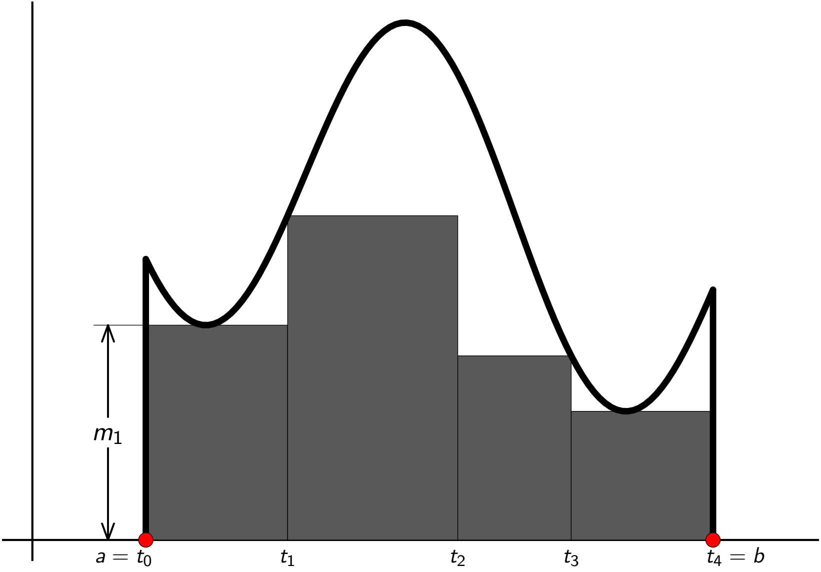
Our treatment is closer to that in M. Spivak "Calculus" (2008).

Integration

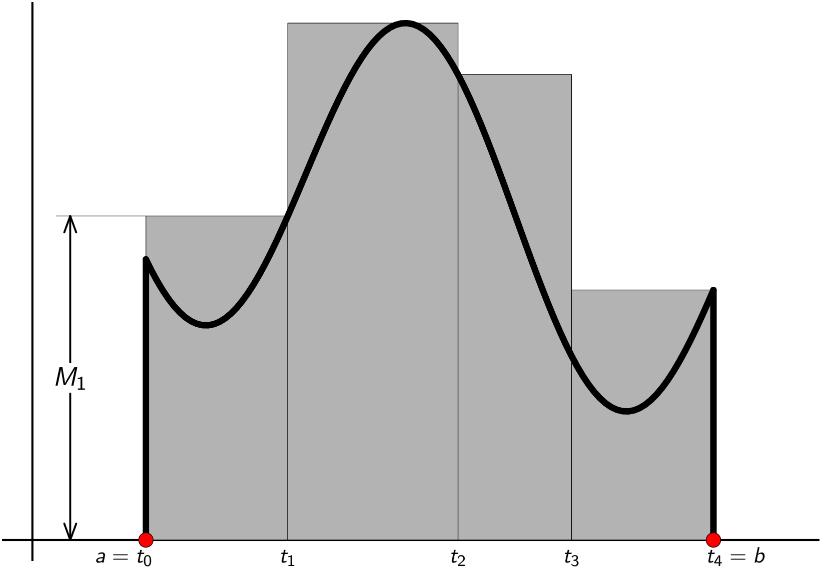


- Contribution to "area of $R(f, a, b)$" is positive or negative depending on whether f is positive or negative.

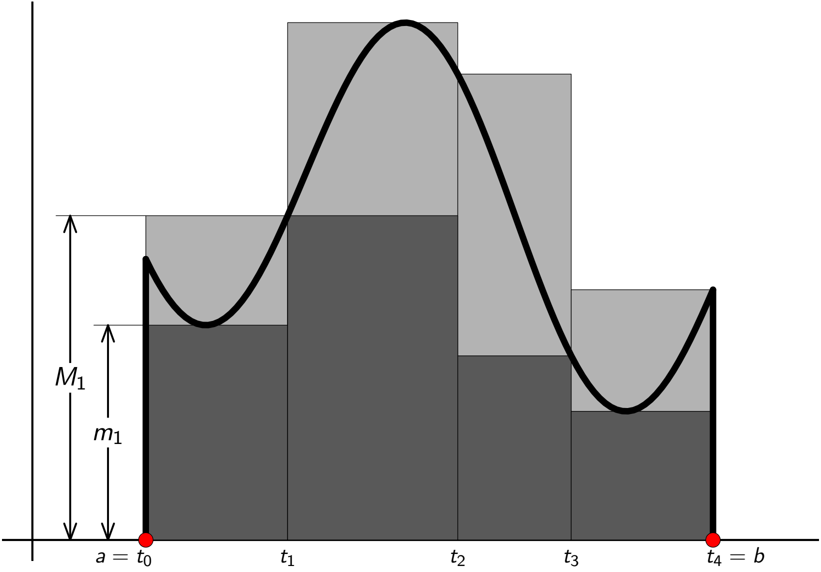
Lower sum



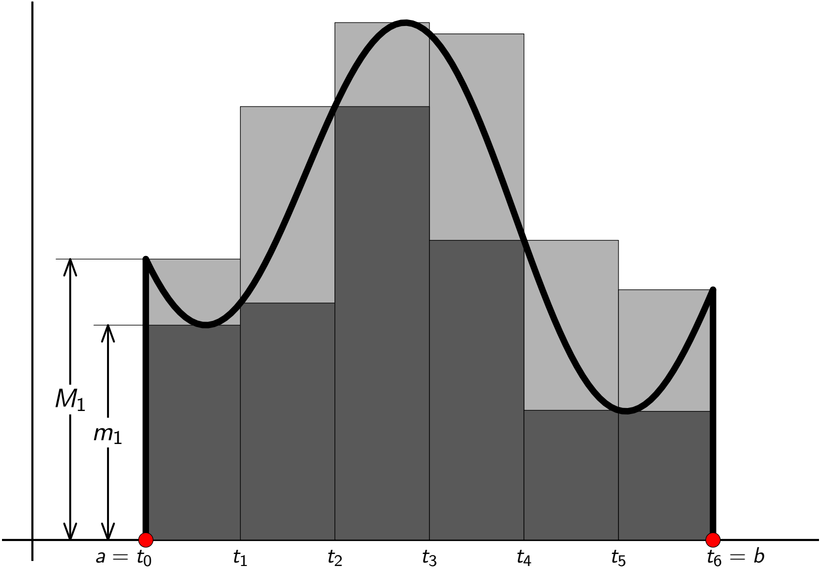
Upper sum



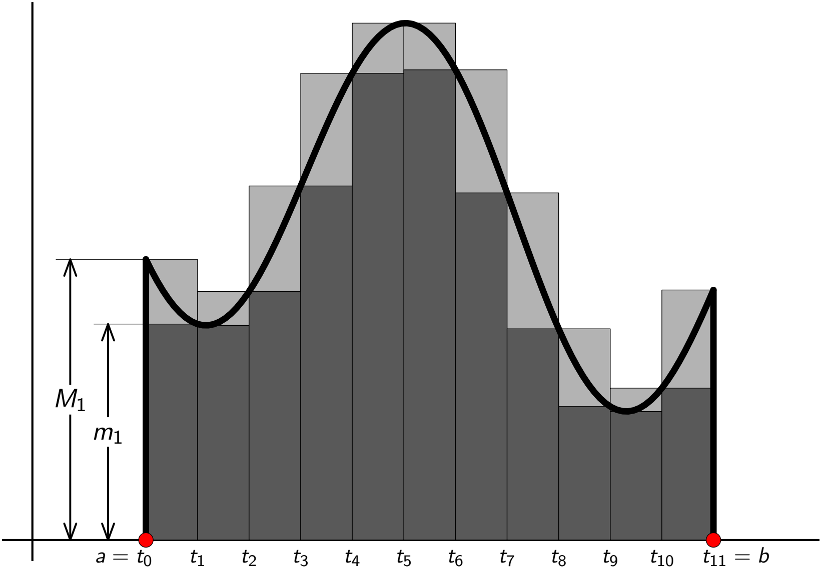
Lower and upper sums



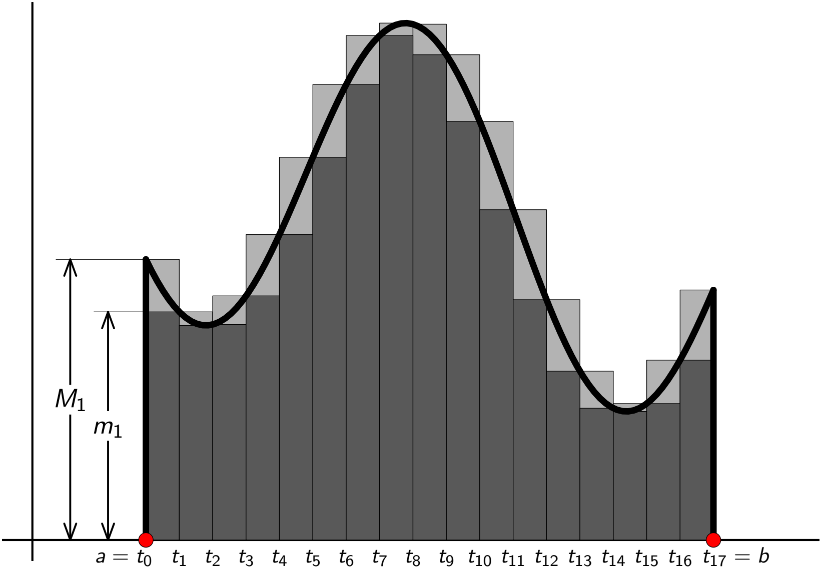
Lower and upper sums



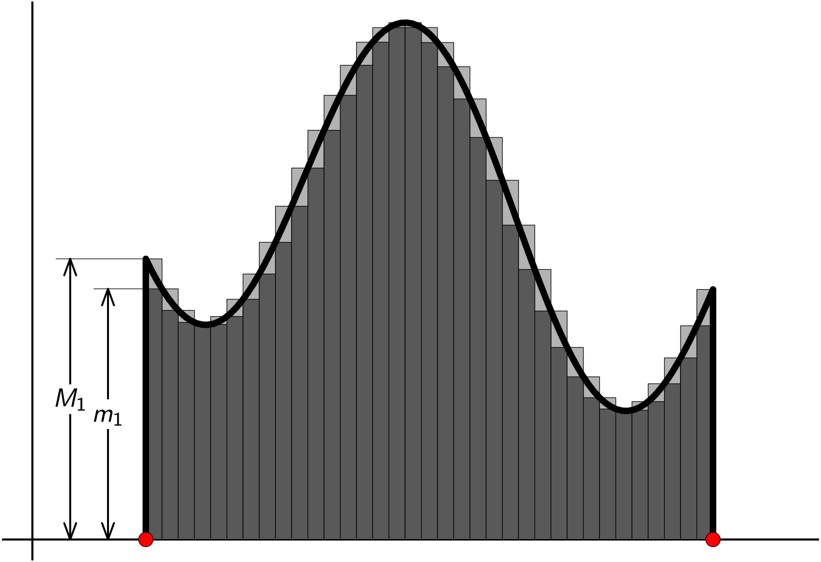
Lower and upper sums



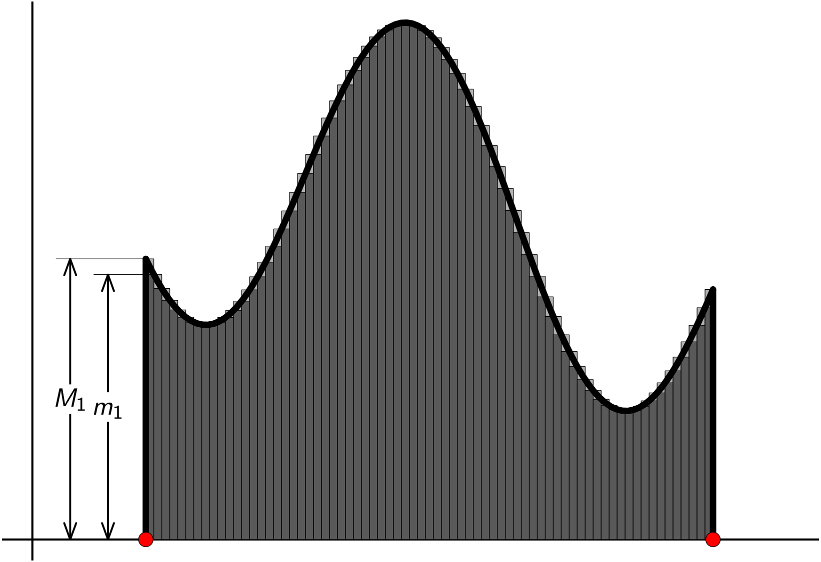
Lower and upper sums



Lower and upper sums



Lower and upper sums



Rigorous development of the integral

Definition (Partition)

Let $a < b$. A ***partition*** of the interval $[a, b]$ is a finite collection of points in $[a, b]$, one of which is a, and one of which is b.

We normally label the points in a partition

$$a = t_0 < t_1 < \cdots < t_{n-1} < t_n = b\,,$$

so the ith subinterval in the partition is

$$[t_{i-1}, t_i]\,.$$

Rigorous development of the integral

Definition (Lower and upper sums)

Suppose f is bounded on $[a, b]$ and $P = \{t_0, \ldots, t_n\}$ is a partition of $[a, b]$. Let

$$m_i = \inf \{ f(x) : x \in [t_{i-1}, t_i] \},$$

$$M_i = \sup \{ f(x) : x \in [t_{i-1}, t_i] \}.$$

The lower sum of f for P, denoted by $L(f, P)$, is defined as

$$L(f, P) = \sum_{i=1}^{n} m_i(t_i - t_{i-1}).$$

The upper sum of f for P, denoted by $U(f, P)$, is defined as

$$U(f, P) = \sum_{i=1}^{n} M_i(t_i - t_{i-1}).$$

Rigorous development of the integral

Relationship between motivating sketch and rigorous definition of lower and upper sums:

- The lower and upper sums correspond to the total areas of rectangles lying below and above the graph of f in our motivating sketch.
- However, these sums have been defined precisely without any appeal to a concept of "area".
- The requirement that f be bounded on $[a, b]$ is essential in order that all the m_i and M_i be well-defined.
- It is also essential that the m_i and M_i be defined as inf's and sup's (rather than maxima and minima) because f was not assumed continuous.

Rigorous development of the integral

Relationship between motivating sketch and rigorous definition of lower and upper sums:

- Since $m_i \le M_i$ for each i, we have

$$m_i(t_i - t_{i-1}) \le M_i(t_i - t_{i-1}) . \qquad i = 1, \ldots, n.$$

$\therefore$ For <u>any</u> partition P of $[a, b]$ we have

$$L(f, P) \le U(f, P),$$

because

$$L(f, P) = \sum_{i=1}^{n} m_i(t_i - t_{i-1}) ,$$

$$U(f, P) = \sum_{i=1}^{n} M_i(t_i - t_{i-1}) .$$

Poll

- Go to https://www.childsmath.ca/childsa/forms/main_login.php
- Click on Math 3A03
- Click on Take Class Poll
- Fill in poll **Lecture 26: Lower and Upper Sums**
- Submit.

Rigorous development of the integral

Relationship between motivating sketch and rigorous definition of lower and upper sums:

- More generally, if P_1 and P_2 are any two partitions of $[a, b]$, it *ought* to be true that

$$L(f, P_1) \leq U(f, P_2),$$

  because $L(f, P_1)$ should be $\leq$ area of $R(f, a, b)$, and $U(f, P_2)$ should be $\geq$ area of $R(f, a, b)$.

- But "ought to" and "should be" prove nothing, especially since we haven't yet even defined "area of $R(f, a, b)$".

- Before we can *define* "area of $R(f, a, b)$", we need to prove that $L(f, P_1) \leq U(f, P_2)$ for any partitions P_1, P_2 ...

McMaster University

Mathematics and Statistics

$$\int_M d\omega = \int_{\partial M} \omega$$

Mathematics 3A03
Real Analysis I

Instructor: David Earn

Lecture 27
Integration II
Tuesday 12 November 2019

Poll

- Go to https://www.childsmath.ca/childsa/forms/main_login.php
- Click on Math 3A03
- Click on Take Class Poll
- Fill in poll **Lecture 27: Assignment 3 Awaremess**
- $\boxed{\texttt{Submit}}$.

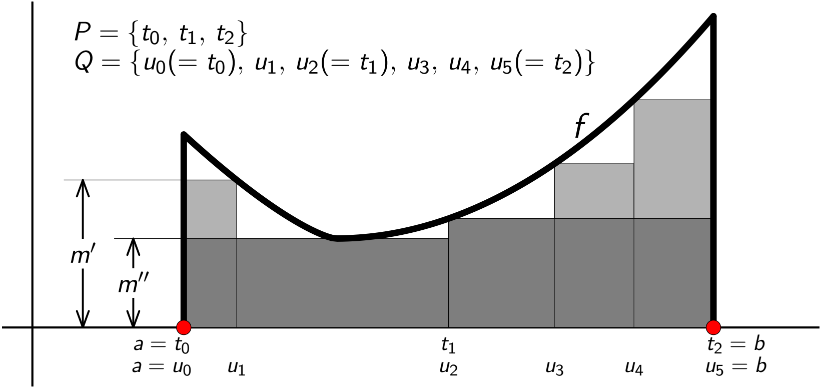
Announcements

- Assignment 4 was due before class today.
- Assignment 5 is due on
  Thursday 21 November 2019 @ 2:25pm via crowdmark.
- Math 3A03 **Test #2**
  Tuesday 26 November 2019, 5:30–7:00pm, in JHE 264
- Assignment 6 will be due on **Tuesday 3 December 2019 @ 2:25pm via crowdmark**.
- Math 3A03 **Final Exam**: Fri 6 Dec 2019, 9:00am–11:30am
  Location: MDCL 1105

Rigorous development of the integral

Lemma

If partition $P \subseteq$ partition Q (i.e., if every point of P is also in Q), then $L(f,P) \le L(f,Q)$ *and* $U(f,P) \ge U(f,Q)$.



Rigorous development of the integral

Proof of Lemma

As a first step, consider the special case in which the finer partition Q contains only one more point than P:

$$P = \{t_0, \dots, t_n\},$$
$$Q = \{t_0, \dots, t_{k-1}, u, t_k, \dots, t_n\},$$

where

$$a = t_0 < t_1 < \cdots < t_{k-1} < u < t_k < \cdots < t_n = b.$$

Let

$$m' = \inf\{\, f(x) \;:\; x \in [t_{k-1}, u] \,\},$$
$$m'' = \inf\{\, f(x) \;:\; x \in [u, t_k] \,\}.$$

...continued...

Rigorous development of the integral

Proof of Lemma (cont.)

Then $$L(f,P)=\sum_{i=1}^{n} m_i(t_i-t_{i-1})\,,$$

and $$\begin{aligned} L(f,Q)=\sum_{i=1}^{k-1} m_i(t_i-t_{i-1}) &+ m'(u-t_{k-1}) \\ &+ m''(t_k-u)+\sum_{i=k+1}^{n} m_i(t_i-t_{i-1})\,. \end{aligned}$$

$\therefore$ To prove $L(f,P)\le L(f,Q)$, it is enough to show

$$m_k(t_k-t_{k-1})\le m'(u-t_{k-1})+m''(t_k-u)\,.$$

...continued...

Rigorous development of the integral

Proof of Lemma (cont.)

Now note that since

$$\{\, f(x) \,:\, x \in [t_{k-1}, u] \,\} \quad \subseteq \quad \{\, f(x) \,:\, x \in [t_{k-1}, t_k] \,\}\,,$$

the RHS might contain some additional *smaller* numbers, so we must have

$$\begin{aligned} m_k &= \inf \{\, f(x) \,:\, x \in [t_{k-1}, t_k] \,\} \\ &\le \inf \{\, f(x) \,:\, x \in [t_{k-1}, u] \,\} \quad = \quad m'\,. \end{aligned}$$

Thus, $m_k \le m'$, and, similarly, $m_k \le m''$.

$$\begin{aligned} \therefore \quad m_k(t_k - t_{k-1}) &= m_k(t_k - u + u - t_{k-1}) \\ &= m_k(u - t_{k-1}) + m_k(t_k - u) \\ &\le m'(u - t_{k-1}) + m''(t_k - u)\,, \end{aligned}$$

...continued...

Rigorous development of the integral

Proof of Lemma (cont.)

which proves (in this special case where Q contains only one more point than P) that $L(f,P) \le L(f,Q)$.

We can now prove the general case by adding one point at a time.

If Q contains ℓ more points than P, define a sequence of partitions

$$P = P_0 \subset P_1 \subset \cdots \subset P_\ell = Q$$

such that P_{j+1} contains exactly one more point that P_j. Then

$$L(f,P) = L(f,P_0) \le L(f,P_1) \le \cdots \le L(f,P_\ell) = L(f,Q)\,,$$

so $L(f,P) \le L(f,Q)$.

(Proving $U(f,P) \ge U(f,Q)$ is similar: check!) $\square$

Rigorous development of the integral

Theorem (Partition Theorem)

Let P_1 and P_2 be any two partitions of $[a,b]$. If f is bounded on $[a,b]$ then

$$L(f,P_1) \le U(f,P_2)\,.$$

Proof.

This is a straightforward consequence of the partition lemma.

Let $P = P_1 \cup P_2$, *i.e.*, the partition obtained by combining all the points of P_1 and P_2.

Then

$$L(f,P_1) \le L(f,P) \le U(f,P) \le U(f,P_2)\,.$$

$\square$

Rigorous development of the integral

Important inferences that follow from the partition theorem:

- For <u>any</u> partition P', the upper sum $U(f,P')$ is an upper bound for the set of <u>all</u> lower sums $L(f,P)$.

  $\therefore \quad \sup\{L(f,P) : P \text{ a partition of } [a,b]\} \le U(f,P') \qquad \forall P'$

  $\therefore \quad \sup\{L(f,P)\} \le \inf\{U(f,P)\}$

  $\therefore$ For <u>any</u> partition P',

  $$L(f,P') \le \sup\{L(f,P)\} \le \inf\{U(f,P)\} \le U(f,P')$$

- If $\sup\{L(f,P)\} = \inf\{U(f,P)\}$ then we can define "area of $R(f,a,b)$" to be this number.
  - Is it possible that $\sup\{L(f,P)\} < \inf\{U(f,P)\}$?

Rigorous development of the integral

Example

$\exists ?\ f : [a, b] \to \mathbb{R}$ such that $\sup\{L(f,P)\} < \inf\{U(f,P)\}$

Let

$$f(x) = \begin{cases} 1 & x \in \mathbb{Q} \cap [a, b], \\ 0 & x \in \mathbb{Q}^{\mathsf{c}} \cap [a, b]. \end{cases}$$

If $P = \{t_0, \ldots, t_n\}$ then $m_i = 0$ $\quad (\because \quad [t_{i-1}, t_i] \cap \mathbb{Q}^{\mathsf{c}} \neq \varnothing)$,
and $M_i = 1$ $\quad (\because \quad [t_{i-1}, t_i] \cap \mathbb{Q} \neq \varnothing)$.

$\therefore \quad L(f,P) = 0$ and $U(f,P) = b - a$ for <u>any</u> partition P.

$\therefore \quad \sup\{L(f,P)\} = 0 < b - a = \inf\{U(f,P)\}$. $\square$

Can we define "area of $R(f, a, b)$" for such a weird function?

Yes, but not in this course!

Rigorous development of the integral

Definition (Integrable)

A function $f : [a, b] \to \mathbb{R}$ is said to be ***integrable*** on $[a, b]$ if it is bounded on $[a, b]$ and

$$\sup\{L(f, P) : P \text{ a partition of } [a, b]\} = \inf\{U(f, P) : P \text{ a partition of } [a, b]\}.$$

In this case, this common number is called the ***integral*** of f on $[a, b]$ and is denoted

$$\int_a^b f$$

Note: If f is integrable then for any partition P we have

$$L(f, P) \leq \int_a^b f \leq U(f, P),$$

and $\int_a^b f$ is the unique number with this property.

Rigorous development of the integral

- *Notation:*

$$\int_a^b f(x)\,dx \qquad \text{means precisely the same as} \qquad \int_a^b f\,.$$

  - The symbol "dx" has no meaning in isolation just as "$x \to$" has no meaning except in $\lim_{x\to a} f(x)$.

- It is not clear from the definition which functions are integrable.

- The definition of the integral does not itself indicate how to compute the integral of any given integrable function. So far, without a lot more effort we can't say much more than these two things:
  1. If $f(x) \equiv c$ then f is integrable on $[a, b]$ and $\int_a^b f = c \cdot (b - a)$.
  2. The weird example function is <u>not</u> integrable.

Rigorous development of the integral

- A function that is integrable according to our definition is usually said to be ***Riemann integrable***, to distinguish this definition from other definitions of integrability.
- In Math 4A03 you will define "Lebesgue integrable", a more subtle concept that makes it possible to attach meaning to "area of $R(f, a, b)$" for the weird example function (among others), and to precisely characterize functions that are Riemann integrable.

Rigorous development of the integral

Theorem (Equivalent condition for integrability)

A bounded function $f : [a, b] \to \mathbb{R}$ is integrable on $[a, b]$ iff for all $\varepsilon > 0$ there is a partition P of $[a, b]$ such that

$$U(f, P) - L(f, P) < \varepsilon\,.$$

Proof.

2016 Assignment 5. □

Note: This theorem is just a restatement of the definition of integrability. It is often more convenient to work with $\varepsilon > 0$ than with sup's and inf's.

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Mathematics and Statistics

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Mathematics 3A03
Real Analysis I

Instructor: David Earn

Lecture 28
Integration III
Thursday 14 November 2019

Last time. . .

Rigorous development of integral:

- Definition: integrable.
- Example: non-integrable function.
- Theorem: Equivalent "ε-P" definition of integrable.

Poll

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- Fill in poll **Lecture 28: Integrable vs Continuous vs Differentiable**
- Submit.

Integral theorems

Theorem

If f is continuous on $[a, b]$ then f is integrable on $[a, b]$.

Rough work to prepare for proof:

$$U(f, P) - L(f, P) = \sum_{i=1}^{n} (M_i - m_i)(t_i - t_{i-1})$$

Given $\varepsilon > 0$, choose a partition P that is so fine that $M_i - m_i < \varepsilon$ for all i. Then

$$U(f, P) - L(f, P) < \varepsilon \sum_{i=1}^{n} (t_i - t_{i-1}) = \varepsilon(b - a)\,.$$

Not quite what we want. So choose the partition P such that $M_i - m_i < \varepsilon/(b - a)$ for all i. To get that, choose P such that

$$|f(x) - f(y)| < \frac{\varepsilon}{2(b - a)} \qquad \text{if } |x - y| < \max_{1 \le i \le n} (t_i - t_{i-1}),$$

which we can do because f is uniformly continuous on $[a, b]$.

Integral theorems

Proof that continuous $\implies$ integrable

Since f is continuous on the compact set $[a, b]$, it is bounded on $[a, b]$ (which is the first requirement to be integrable on $[a, b]$).

Also, since f is continuous on the compact set $[a, b]$, it is uniformly continuous on $[a, b]$. $\quad \therefore \forall \varepsilon > 0 \; \exists \delta > 0$ such that $\forall x, y \in [a, b]$,

$$|x - y| < \delta \quad \implies \quad |f(x) - f(y)| < \frac{\varepsilon}{2(b-a)} .$$

Now choose a partition of $[a, b]$ such that the length of each subinterval $[t_{i-1}, t_i]$ is less than δ, *i.e.*, $t_i - t_{i-1} < \delta$. Then, for any $x, y \in [t_{i-1}, t_i]$ we have $|x - y| < \delta$ and therefore

. . . continued. . .

Integral theorems

Proof that continuous $\implies$ integrable (cont.)

$$|f(x) - f(y)| < \frac{\varepsilon}{2(b-a)} \qquad \forall x, y \in [t_{i-1}, t_i]\,.$$

$$\therefore \qquad M_i - m_i \le \frac{\varepsilon}{2(b-a)} < \frac{\varepsilon}{b-a} \qquad i = 1, \ldots, n.$$

Since this is true for all i, it follows that

$$\begin{aligned} U(f,P) - L(f,P) &= \sum_{i=1}^{n} (M_i - m_i)(t_i - t_{i-1}) \\ &< \frac{\varepsilon}{b-a} \sum_{i=1}^{n} (t_i - t_{i-1}) = \frac{\varepsilon}{b-a}(b-a) = \varepsilon\,. \end{aligned}$$

□

Properties of the integral

Theorem (Integral segmentation)

Let $a < c < b$. If f is integrable on $[a, b]$, then f is integrable on $[a, c]$ and on $[c, b]$. Conversely, if f is integrable on $[a, c]$ and $[c, b]$ then f is integrable on $[a, b]$. Finally, if f is integrable on $[a, b]$ then

$$\int_a^b f = \int_a^c f + \int_c^b f\,. \qquad (\heartsuit)$$

(a good exercise)

This theorem motivates these definitions:

$$\int_a^a f = 0 \qquad \text{and} \qquad \int_a^b f = -\int_b^a f \quad \text{if } a > b.$$

Then $(\heartsuit)$ holds for any $a, b, c \in \mathbb{R}$.

Properties of the integral

Theorem (Algebra of integrals – a.k.a. $\int_a^b$ is a linear operator)

If f and g are integrable on $[a,b]$ and $c \in \mathbb{R}$ then $f+g$ and cf are integrable on $[a,b]$ and

1. $\displaystyle\int_a^b (f+g) = \int_a^b f + \int_a^b g;$
2. $\displaystyle\int_a^b cf = c\int_a^b f.$

(proofs are relatively easy; good exercises)

Theorem (Integral of a product)

If f and g are integrable on $[a,b]$ then fg is integrable on $[a,b]$.

(proof is much harder; tough exercise)

Properties of the integral

Lemma (Integral bounds)

Suppose f is integrable on $[a,b]$. If $m \le f(x) \le M$ for all $x \in [a,b]$ then

$$m(b-a) \le \int_a^b f \le M(b-a)\,.$$

Proof.

For any partition P, we must have $m \le m_i \; \forall i$ and $M \ge M_i \; \forall i$.

$$\therefore \qquad m(b-a) \le L(f,P) \le U(f,P) \le M(b-a) \qquad \forall P$$

$$\therefore \quad m(b-a) \le \sup\{L(f,P)\} = \int_a^b f = \inf\{U(f,P)\} \le M(b-a)\,.$$

$\square$

Properties of the integral

Theorem (Integrals are continuous)

If f is integrable on $[a,b]$ and F is defined on $[a,b]$ by

$$F(x)=\int_a^x f\,,$$

then F is continuous on $[a,b]$.

Proof

Let's first consider $x_0\in[a,b)$ and show F is continuous from above at x_0, *i.e.*, $\lim_{x\to x_0^+}F(x)=F(x_0)$. If $x\in(x_0,b]$ then

$$(\heartsuit)\qquad\Longrightarrow\qquad F(x)-F(x_0)=\int_a^x f-\int_a^{x_0} f=\int_{x_0}^x f\,.\qquad(*)$$

...continued...

Properties of the integral

Proof (cont.)

Since f is integrable on $[a, b]$, it is bounded on $[a, b]$, so $\exists M > 0$ such that

$$-M \leq f(x) \leq M \qquad \forall x \in [a, b],$$

from which the integral bounds lemma implies

$$-M(x - x_0) \leq \int_{x_0}^{x} f \leq M(x - x_0)\,,$$

$$\therefore \quad (*) \implies -M(x - x_0) \leq F(x) - F(x_0) \leq M(x - x_0)\,.$$

$\therefore$ For any $\varepsilon > 0$ we can ensure $|F(x) - F(x_0)| < \varepsilon$ by requiring $0 \leq x - x_0 < \varepsilon/M$, which proves $\lim_{x \to x_0^+} F(x) = F(x_0)$.

A similar argument starting from $x_0 \in (a, b]$ and $x \in [a, x_0)$ yields $\lim_{x \to x_0^-} F(x) = F(x_0)$. Thus, "integrals are continuous". $\square$

McMaster University

Mathematics and Statistics

$$\int_M d\omega = \int_{\partial M} \omega$$

Mathematics 3A03
Real Analysis I

Instructor: David Earn

Lecture 29
Integration IV
Friday 15 November 2019

Please consider...

5 minute *Student Respiratory Illness Survey:*

https://surveys.mcmaster.ca/limesurvey/index.php/893454

> *Please complete this anonymous survey to help us monitor the patterns of respiratory illness, over-the-counter drug use, and social contact within the McMaster community. There are no risks to filling out this survey, and your participation is voluntary. You do not need to answer any questions that make you uncomfortable, and all information provided will be kept strictly confidential. Thanks for participating.*
>
> *–Dr. Marek Smieja (Infectious Diseases)*

Announcements

- Assignment 5 is due on **Thursday 21 November 2019 @ 2:25pm via crowdmark**.
- Math 3A03 **Test #2**
  Tuesday 26 November 2019, 5:30–7:00pm, in JHE 264
- Assignment 6 will be due on **Tuesday 3 December 2019 @ 2:25pm via crowdmark**.
- Math 3A03 **Final Exam**: Fri 6 Dec 2019, 9:00am–11:30am
  Location: MDCL 1105

Last time...

Rigorous development of integral:

- continuous $\implies$ integrable.
- Integral segmentation.
- Algebra of integrals.
- Integral bounds lemma.
- Integrals are continuous.

Fundamental Theorem of Calculus

Theorem (First Fundamental Theorem of Calculus)

Let f be integrable on $[a, b]$, and define F on $[a, b]$ by

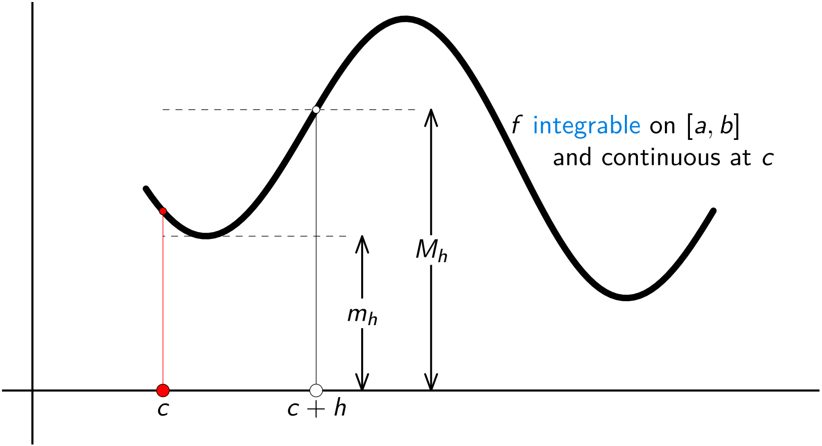
$$F(x) = \int_a^x f \,.$$

If f is continuous at $c \in [a, b]$, then F is differentiable at c, and

$$F'(c) = f(c) \,.$$

Note: If $c = a$ or b, then $F'(c)$ is understood to mean the right- or left-hand derivative of F.

Fundamental Theorem of Calculus



$$F(c+h) - F(c) \simeq f(c+h) \cdot h$$

and $\lim\limits_{h \to 0} f(c+h) = f(c)$

$$\implies \quad \lim_{h \to 0} \frac{F(c+h) - F(c)}{h} = f(c)$$

Fundamental Theorem of Calculus

Proof of First Fundamental Theorem of Calculus

Suppose $c \in [a, b)$, and $0 < h \leq b - c$. Then the integral segmentation theorem implies

$$F(c+h) - F(c) = \int_c^{c+h} f \,.$$

Motivated by the sketch, define

$$m_h = \inf \{ \, f(x) \, : \, x \in [c, c+h] \, \} \,,$$
$$M_h = \sup \{ \, f(x) \, : \, x \in [c, c+h] \, \} \,.$$

Then the integral bounds lemma implies

$$m_h \cdot h \leq \int_c^{c+h} f \leq M_h \cdot h \,,$$

...continued...

Fundamental Theorem of Calculus

Proof of First Fundamental Theorem of Calculus (cont.)

and hence

$$m_h \le \frac{F(c+h)-F(c)}{h} \le M_h\,.$$

This inequality is true for any integrable function. However, because f is continuous at c, we have

$$\lim_{h\to 0^+} m_h = \lim_{h\to 0^+} M_h = f(c)\,,$$

so the squeeze theorem implies

$$F'_+(c) = \lim_{h\to 0^+} \frac{F(c+h)-F(c)}{h} = f(c)\,.$$

A similar argument for $c \in (a,b]$ and $c-a \le h < 0$ yields $F'_-(c) = f(c)$. □

Fundamental Theorem of Calculus

Corollary

If f is continuous on $[a, b]$ and $f = g'$ for some function g, then

$$\int_a^b f = g(b) - g(a)\,.$$

Proof.

Let $F(x) = \int_a^x f$. Then $\forall x \in [a, b]$, $F'(x) = f(x)$ (by FFTC).

$$\implies F' = f = g'.$$

$\therefore\ \exists c \in \mathbb{R}$ such that $F = g + c$ (2016 Assignment 5).

$\therefore\ F(a) = g(a) + c.$ But $F(a) = \int_a^a f = 0$, so $c = -g(a)$.

$\therefore\ F(x) = g(x) - g(a)$.

This is true, in particular, for $x = b$, so $\int_a^b f = g(b) - g(a)$. $\square$

Poll

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Fundamental Theorem of Calculus

Theorem (Second Fundamental Theorem of Calculus)

If f is integrable on $[a, b]$ and $f = g'$ for some function g, then

$$\int_a^b f = g(b) - g(a) \, .$$

Notes:

- This looks like the corollary to the first fundamental theorem, except that f is only assumed integrable, not continuous.
- Recall from Darboux's theorem that if $f = g'$ for some g then f has the intermediate value property, but f need not be continuous.
- g' exists on $[a, b] \implies$ Mean Value Theorem applies to g.
- The proof of the second fundamental theorem is completely different from the corollary to the first, because we cannot use the first fundamental theorem (which assumed f is continuous).

Fundamental Theorem of Calculus

Proof of Second Fundamental Theorem of Calculus

Let $P = \{t_0, \dots, t_n\}$ be any partition of $[a, b]$. By the Mean Value Theorem, for each $i = 1, \dots, n$, $\exists\, x_i \in [t_{i-1}, t_i]$ such that

$$g(t_i) - g(t_{i-1}) = g'(x_i)(t_i - t_{i-1}) = f(x_i)(t_i - t_{i-1})\,.$$

Define m_i and M_i as usual. Then $m_i \le f(x_i) \le M_i\ \forall i$, so

$$m_i(t_i - t_{i-1}) \le f(x_i)(t_i - t_{i-1}) \le M_i(t_i - t_{i-1})\,,$$

$$\textit{i.e.,}\quad m_i(t_i - t_{i-1}) \le g(t_i) - g(t_{i-1}) \le M_i(t_i - t_{i-1})\,.$$

$$\therefore\quad \sum_{i=1}^{n} m_i(t_i - t_{i-1}) \le \sum_{i=1}^{n} \Big(g(t_i) - g(t_{i-1})\Big) \le \sum_{i=1}^{n} M_i(t_i - t_{i-1})$$

$$\textit{i.e.,}\qquad L(f, P) \le g(b) - g(a) \le U(f, P)$$

for <u>any</u> partition P. $\quad\therefore\quad g(b) - g(a) = \int_a^b f\,.$ $\square$

What useful things can we do with integrals?

- Compute areas of complicated shapes: find anti-derivatives and use the second fundamental theorem of calculus.
- Define trigonometric functions (rigorously).
- Define logarithm and exponential functions (rigorously).

Mathematics
and Statistics

$$\int_M d\omega = \int_{\partial M} \omega$$

Mathematics 3A03
Real Analysis I

Instructor: David Earn

Lecture 35
Integration V
Friday 29 November 2019

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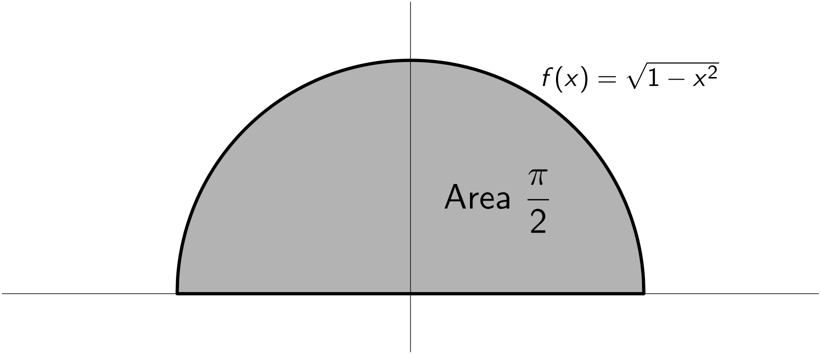
> *Please complete this anonymous survey to help us monitor the patterns of respiratory illness, over-the-counter drug use, and social contact within the McMaster community. There are no risks to filling out this survey, and your participation is voluntary. You do not need to answer any questions that make you uncomfortable, and all information provided will be kept strictly confidential. Thanks for participating.*
>
> *–Dr. Marek Smieja (Infectious Diseases)*

You can monitor what percentage of students has completed the course evaluation at https://evals.mcmaster.ca/stats/index.php?fac=SCIENCE.

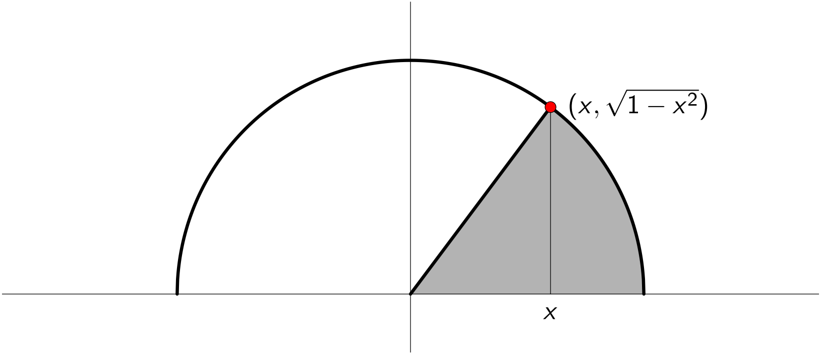
What is π ?



Definition

$$\pi \;\equiv\; 2\int_{-1}^{1} \sqrt{1-x^2}\; dx\,.$$

What are cos and sin ?

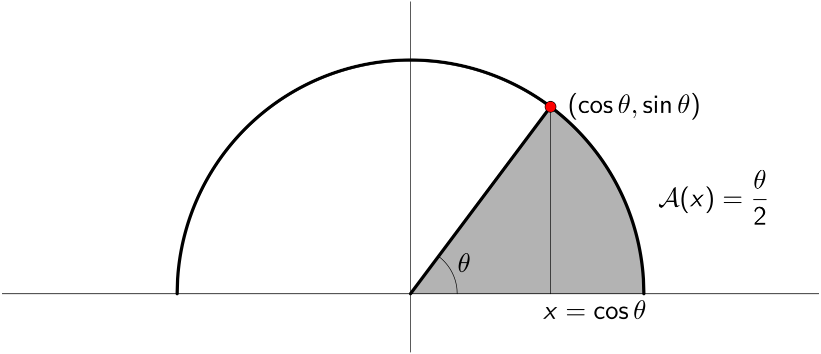


Definition (Sectoral area)

If $x \in [-1,1]$ then $\quad \mathcal{A}(x) = \dfrac{x\sqrt{1-x^2}}{2} + \displaystyle\int_x^1 \sqrt{1-t^2}\,dt\,.$

Note: $\mathcal{A}(-1) = \pi/2, \quad \mathcal{A}(1) = 0.$

What are cos and sin ?



Length of circular arc swept out by angle θ: θ

Area of sectoral region swept out by angle θ: $\theta/2$

So, if $\theta \in [0, \pi]$ then we <u>define</u> $\cos\theta$ to be the unique number in $[-1, 1]$ such that $\mathcal{A}(\cos\theta) = \theta/2$, and we <u>define</u> $\sin\theta$ to be $\sqrt{1 - (\cos\theta)^2}$.

We must prove: given $x \in [0, \pi]$ $\exists!\, y \in [-1, 1]$ such that $\mathcal{A}(y) = x/2$.

What are cos and sin ?

Proof that $\forall\, x \in [0,\pi]\ \exists!\, y \in [-1,1]$ *such that* $\mathcal{A}(y) = x/2$*:*

Existence: $\mathcal{A}(1) = 0$, $\mathcal{A}(-1) = \pi/2$, and $\mathcal{A}$ is continuous. Hence by the intermediate value theorem $\exists\, y \in [-1,1]$ such that $\mathcal{A}(y) = x/2$.

Uniqueness: $\mathcal{A}$ is differentiable on $(-1,1)$ and $\mathcal{A}'(x) < 0$ on $(-1,1)$.
$\therefore$ On $(-1,1)$, $\mathcal{A}$ is decreasing, and hence one-to-one.

Definition (cos and sin)

If $x \in [0,\pi]$ then $\cos x$ is the unique number in $[-1,1]$ such that $\mathcal{A}(\cos x) = x/2$, and $\sin x = \sqrt{1-(\cos x)^2}$.

These definitions are easily extended to all of $\mathbb{R}$:

- For $x \in [\pi, 2\pi]$, define $\cos x = \cos(2\pi - x)$ and $\sin x = -\sin(2\pi - x)$.
- Then, for $x \in \mathbb{R} \setminus [0,2\pi]$ define $\cos x = \cos(x \mod 2\pi)$ and $\sin x = \sin(x \mod 2\pi)$.

Trigonometric theorems

Given the rigorous definition of cos and sin, we can prove:

1. cos and sin are differentiable on $\mathbb{R}$. Moreover, $\cos' = -\sin$ and $\sin' = \cos$.
2. sec, tan, csc and cot can all be defined in the usual way and have all the usual properties.
3. The inverse function theorem allows us to define and compute the derivatives of all the inverse trigonometric functions.
4. If f is twice differentiable on $\mathbb{R}$, $\quad f'' + f = 0, \quad f(0) = a$ and $f'(0) = b, \quad$ then $\quad f = a\cos + b\sin$.
5. For all $x, y \in \mathbb{R}$,

$$\sin(x+y) = \sin x \cos y + \cos x \sin y\,,$$
$$\cos(x+y) = \cos x \cos y - \sin x \sin y\,.$$

Something deep that you know enough to prove

Extra Challenge Problem:

Prove that π is irrational.

What are log and exp ?

Consider the function

$$f(x) = 10^x\,.$$

What exactly is this function?

In our mathematically naïve previous life, we just assumed that $f(x)$ is well-defined $\forall\, x \in \mathbb{R}$, and that f has a well-defined inverse function,

$$f^{-1}(x) = \log_{10}(x)\,.$$

But how are 10^x and $\log_{10}(x)$ defined for irrational x ?

Let's review what we know...

What are log and exp ?

$$n \in \mathbb{N} \quad \Longrightarrow \quad 10^n = \underbrace{10 \cdots 10}_{n \text{ times}}$$

$$n, m \in \mathbb{N} \quad \Longrightarrow \quad 10^n \cdot 10^m = 10^{n+m}$$

When we extend 10^x to $x \in \mathbb{Q}$, we want this product rule to be preserved:

$$10^0 \cdot 10^n = 10^{0+n} = 10^n \quad \Longrightarrow \quad 10^0 = 1$$

$$10^{-n} \cdot 10^n = 10^0 = 1 \quad \Longrightarrow \quad 10^{-n} = \frac{1}{10^n}$$

$$\underbrace{10^{1/n} \cdots 10^{1/n}}_{n \text{ times}} = 10^{\overbrace{1/n \cdots 1/n}} \!\!\!\!\!\!\!\! \underbrace{}_{n \text{ times}} = 10^1 = 10 \quad \Longrightarrow \quad 10^{1/n} = \sqrt[n]{10}$$

What are log and exp ?

Finally, to define 10^q for all $q \in \mathbb{Q}$, note that we must have

$$\left(10^{\frac{1}{n}}\right)^m = \underbrace{10^{\frac{1}{n}} \cdots 10^{\frac{1}{n}}}_{m \text{ times}} = 10^{\overbrace{\frac{1}{n}+\cdots+\frac{1}{n}}^{}}\!\!\!\!\!\!\!\!\!\!\!\!\underbrace{\phantom{\frac{1}{n}+\cdots+\frac{1}{n}}}_{m \text{ times}} = 10^{\frac{m}{n}} \quad \Longrightarrow \quad 10^{\frac{m}{n}} \overset{\text{def}}{=} \left(\sqrt[n]{10}\right)^m$$

Now we're stuck. *How do we extend this scheme to irrational x?*

We need a more sophisticated idea.

Let's try to find a function on all of $\mathbb{R}$ that satisfies

$$f(x+y) = f(x) \cdot f(y)\,, \qquad \forall x, y \in \mathbb{R},$$
$$\text{and} \qquad f(1) = 10\,.$$

It then follows that $\quad f(0) = 1 \quad$ and, $\forall\, x \in \mathbb{Q}, \quad f(x) = [f(1)]^x$.

What additional properties can we impose on $f(x)$ that will lead us to a sensible definition of $f(x)$ for all $x \in \mathbb{R}$?

What are log and exp ?

One approach is to insist that f is *differentiable*.

Then we can compute

$$
\begin{aligned}
f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{f(x)\cdot f(h)-f(x)}{h} \\
&= f(x)\cdot\lim_{h\to 0}\frac{f(h)-1}{h} \quad = \quad f(x)\cdot f'(0) \quad \equiv \quad \alpha\, f(x)
\end{aligned}
$$

So $f'(x) = \alpha\, f(x)$, *i.e.*, we have f' in terms of unknowns f and α.
So what?!?

Let's look at the inverse function, f^{-1} (think "$\log_{10}$"):

$$
(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))} = \frac{1}{\alpha\, f(f^{-1}(x))} = \frac{1}{\alpha\, x}
$$

Holy \$#@%! We have a simple formula for the derivative of f^{-1}!

What are log and exp ?

Since we want $\log_{10} 1 = 0$, we should <u>define</u> $\log_{10} x$ as $(1/\alpha)\int_1^x t^{-1}\,dt$. *Great idea, but we don't know what α is.*

So, let's ignore α ...

(and hope that what we end up with is log to some "natural" base).

Definition (Logarithm function)

If $x > 0$ then

$$\log x = \int_1^x \frac{1}{t}\,dt\,.$$

This function is strictly increasing ($\log'(x) > 0$ for all $x > 0$) so we can now define:

Definition (Exponential function)

$$\exp = \log^{-1}\,.$$

What are log and exp ?

With these rigorous defintions of log and exp, we can prove the following as theorems:

1. If $x, y > 0$ then $\log(xy) = \log x + \log y$.
2. If $x, y > 0$ then $\log(x/y) = \log x - \log y$.
3. If $n \in \mathbb{N}$ and $x > 0$ then $\log(x^n) = n \log x$.
4. For all $x \in \mathbb{R}$, $\exp'(x) = \exp(x)$.
5. For all $x, y \in \mathbb{R}$, $\exp(x + y) = \exp(x) \cdot \exp(y)$.
6. For all $x \in \mathbb{Q}$, $\exp(x) = [\exp(1)]^x$.

The last theorem above motivates:

Definition

$$\begin{aligned} e &= \exp(1)\,, \\ e^x &= \exp(x) \qquad \text{for all } x \in \mathbb{R}. \end{aligned}$$

What are log and exp ?

We can now give a rigorous definition of 10^x for any $x \in \mathbb{R}$.
In fact, we can do this for any $a > 0$.

Definition (a^x)

If $a > 0$ and x is <u>any real number</u> then

$$a^x = e^{x \log a} .$$

We then have the following <u>theorems</u> for any $a > 0$:

1. $(a^x)^y = a^{xy}$ for all $x, y \in \mathbb{R}$;
2. $a^0 = 1; \quad a^1 = a$;
3. $a^{x+y} = a^x \cdot a^y$ for all $x, y \in \mathbb{R}$;
4. $a^{-x} = 1/a^x$ for all $x \in \mathbb{R}$;
5. if $a > 1$ then a^x is increasing on $\mathbb{R}$;
6. if $0 < a < 1$ then a^x is decreasing on $\mathbb{R}$.

Using the integral to define useful functions rigorously

- Just as we defined 10^x via the definition of $\log x = \int_1^x \frac{1}{t}\,dt$, we could have defined the trigonometric functions starting from

$$\arcsin x = \int_0^x \frac{1}{\sqrt{1-t^2}}\,dt\,, \qquad -1 < x < 1,$$

rather than the more complicated definition of cos via $\mathcal{A}(x)$. Many common functions are defined as integrals of rational functions of square roots.

- Any compositions of trig functions, log, exp, rational functions and radicals, are called ***elementary functions***.

- Most functions that turn up a lot in applications can be defined rigorously via integrals of elementary functions. Such functions are collectively called ***special functions***.