18 Continuity

Announcements

- Assignment 3 is posted (and complete).
  Due Tuesday 22 October 2019 at 2:25pm via crowdmark.
- **Math 3A03 Test #1**
  Tuesday 29 October 2019, 5:30–7:00pm, in JHE 264
  (room is booked for 90 minutes; you should not feel rushed)
- Math 3A03 **Final Exam**: Fri 6 Dec 2019, 9:00am–11:30am
  Location: `MDCL 1105`

Continuous Functions

McMaster University

Mathematics and Statistics

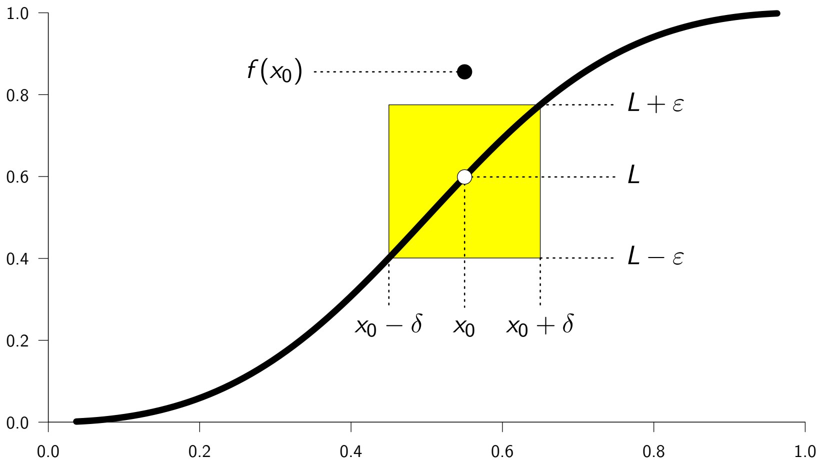
$$\int_M d\omega = \int_{\partial M} \omega$$

Mathematics 3A03
Real Analysis I

Instructor: David Earn

Lecture 18
Continuity
Friday 11 October 2019

Limits of functions



Limits of functions

Definition (Limit of a function on an interval (a, b))

Let $a < x_0 < b$ and $f : (a, b) \to \mathbb{R}$. Then f is said to ***approach the limit L as x approaches x_0***, often written "$f(x) \to L$ as $x \to x_0$" or

$$\lim_{x \to x_0} f(x) = L\,,$$

iff for all $\varepsilon > 0$ there exists $\delta > 0$ such that if $0 < |x - x_0| < \delta$ then $|f(x) - L| < \varepsilon$.

Shorthand version:

$\forall \varepsilon > 0\ \exists \delta > 0 \ni 0 < |x - x_0| < \delta \implies |f(x) - L| < \varepsilon$.

Limits of functions

The function f need not be defined on an entire interval.
It is enough for f to be defined on a set with at least one accumulation point.

Definition (Limit of a function with domain $E \subseteq \mathbb{R}$)

Let $E \subseteq \mathbb{R}$ and $f : E \to \mathbb{R}$. Suppose x_0 is a point of accumulation of E. Then f is said to ***approach the limit L as x approaches x_0***, *i.e.*,

$$\lim_{x \to x_0} f(x) = L\,,$$

iff for all $\varepsilon > 0$ there exists $\delta > 0$ such that if $x \in E$, $x \neq x_0$, and $|x - x_0| < \delta$ then $|f(x) - L| < \varepsilon$.

Shorthand version:

$$\forall \varepsilon > 0\ \exists \delta > 0 \ni \left(x \in E \ \wedge\ 0 < |x - x_0| < \delta\right) \implies |f(x) - L| < \varepsilon.$$

Limits of functions

Example

Prove directly from the definition of a limit that

$$\lim_{x \to 3} (2x + 1) = 7 \,.$$

Proof that $2x + 1 \to 7$ as $x \to 3$.

We must show that $\forall \varepsilon > 0 \; \exists \delta > 0$ such that $0 < |x - 3| < \delta \implies |(2x + 1) - 7| < \varepsilon$. Given ε, to determine how to choose δ, note that

$$|(2x+1) - 7| < \varepsilon \iff |2x - 6| < \varepsilon \iff 2\,|x - 3| < \varepsilon \iff |x - 3| < \frac{\varepsilon}{2}$$

Therefore, given $\varepsilon > 0$, let $\delta = \frac{\varepsilon}{2}$. Then $|x - 3| < \delta \implies |(2x+1) - 7| = |2x - 6| = 2\,|x - 3| < 2\frac{\varepsilon}{2} = \varepsilon$, as required. $\square$

Limits of functions

Example

Prove directly from the definition of a limit that

$$\lim_{x \to 2} x^2 = 4\,.$$

(Solution on next slide)

Limits of functions

Proof that $x^2 \to 4$ as $x \to 2$.

We must show that $\forall\varepsilon > 0\ \exists\delta > 0$ such that $0 < |x-2| < \delta \implies |x^2-4| < \varepsilon$. Given ε, to determine how to choose δ, note that

$$\left|x^2-4\right| < \varepsilon \iff |(x-2)(x+2)| < \varepsilon \iff |x-2|\,|x+2| < \varepsilon.$$

We can make $|x-2|$ as small as we like by choosing δ sufficiently small. Moreover, if x is close to 2 then $x+2$ will be close to 4, so we should be able to ensure that $|x+2| < 5$. To see how, note that

$$\begin{aligned} |x+2| < 5 &\iff -5 < x+2 < 5 \iff -9 < x-2 < 1 \\ &\impliedby -1 < x-2 < 1 \iff |x-2| < 1\,. \end{aligned}$$

Therefore, given $\varepsilon > 0$, let $\delta = \min(1, \frac{\varepsilon}{5})$. Then $|x^2-4| = |(x-2)(x+2)| = |x-2|\,|x+2| < \frac{\varepsilon}{5}5 = \varepsilon$. $\square$

Poll

- Go to https://www.childsmath.ca/childsa/forms/main_login.php
- Click on Math 3A03
- Click on Take Class Poll
- Fill in poll **Lecture 18: ε-δ definition of limit**
- Submit.

Limits of functions

Rather than the ε-δ definition, we can exploit our experience with sequences to define "$f(x) \to L$ as $x \to x_0$".

Definition (Limit of a function via sequences)

Let $E \subseteq \mathbb{R}$ and $f : E \to \mathbb{R}$. Suppose x_0 is a point of accumulation of E. Then

$$\lim_{x \to x_0} f(x) = L$$

iff for every sequence $\{e_n\}$ of points in $E \setminus \{x_0\}$,

$$\lim_{n\to\infty} e_n = x_0 \quad \Longrightarrow \quad \lim_{n\to\infty} f(e_n) = L\,.$$