

6 Sequences

7 Sequences II

8 Sequences III

9 Sequences IV



Mathematics
and Statistics

$$\int_M d\omega = \int_{\partial M} \omega$$

Mathematics 3A03 Real Analysis I

Instructor: David Earn

Lecture 6

Sequences

Friday 13 September 2019

Poll

- Go to https://www.childsmath.ca/childsa/forms/main_login.php
- Click on [Math 3A03](#)
- Click on [Take Class Poll](#)
- Fill in poll **Lecture 6: Sequence convergence**
- .

Announcements

- [Assignment 1](#) is due via [crowdmark](#) 5 minutes before class on Monday.
- Consider writing the [Putnam competition](#).

Sequences

- A *sequence* is a list that goes on forever.
- There is a beginning (a “first term”) but no end, e.g.,

$$\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots, \frac{1}{n}, \dots$$

- We use the natural numbers \mathbb{N} to label the terms of a sequence:

$$a_1, a_2, a_3, \dots, a_n, \dots$$

Formal definition of a sequence

Definition (Sequence of Real Numbers)

A *sequence of real numbers* is a function

$$f : \mathbb{N} \rightarrow \mathbb{R}.$$

A lot of different notation is common for sequences:

$f(1), f(2), f(3), \dots$	$\{f(n)\}_{n=1}^{\infty}$
f_1, f_2, f_3, \dots	$\{f(n)\}$
$\{f(n) : n = 1, 2, 3, \dots\}$	$\{f_n\}_{n=1}^{\infty}$
$\{f(n) : n \in \mathbb{N}\}$	$\{f_n\}$

Specifying sequences

There are two main ways to specify a sequence:

1. Direct formula.

Specify $f(n)$ for each $n \in \mathbb{N}$.

Example (arithmetic progression with common difference d)

Sequence is:

$$c, c + d, c + 2d, c + 3d, \dots$$

$$\therefore f(n) = c + (n - 1)d, \quad n \in \mathbb{N}$$

$$\text{i.e., } x_n = c + (n - 1)d, \quad n = 1, 2, 3, \dots$$

Specifying sequences

2. Recursive formula.

Specify first term and function $f(x)$ to *iterate*. □

i.e., Given x_1 and $f(x)$, we have $x_n = f(x_{n-1})$ for all $n > 1$.

$$x_2 = f(x_1), \quad x_3 = f(f(x_1)), \quad x_4 = f(f(f(x_1))), \quad \dots$$

Example (arithmetic progression with common difference d)

$$x_1 = c, \quad f(x) = x + d$$

$$\therefore x_n = x_{n-1} + d, \quad n = 2, 3, 4, \dots$$

Note: f is the most typical function name for both the direct and recursive specifications. The correct interpretation of f should be clear from context.

Specifying sequences

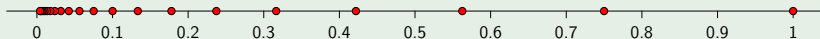
Example (geometric progression with common ratio r)

Sequence is: c, cr, cr^2, cr^3, \dots

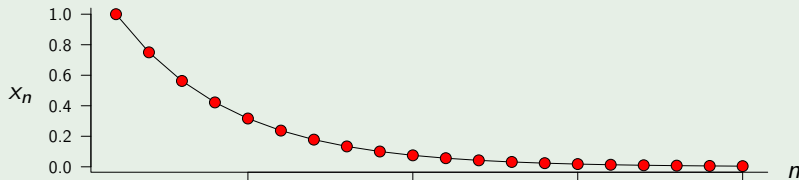
Direct formula: $x_n = f(n) = cr^{n-1}, n = 1, 2, 3, \dots$

Recursive formula: $x_1 = c, f(x) = rx, x_n = f(x_{n-1})$

Number line representation of $\{x_n\}$ with $c = 1$ and $r = \frac{3}{4}$:



Graph of $f(n)$:



Specifying sequences

Example $(f(n) = 1 + \frac{1}{n^2})$

Sequence is: $2, \frac{5}{4}, \frac{10}{9}, \frac{17}{16}, \dots$

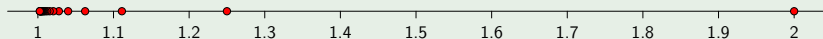
Direct formula: $x_n = f(n) = 1 + \frac{1}{n^2}, n = 1, 2, 3, \dots$

Recursive formula: $x_1 = 2, \quad f(x) = 1 + [1 + (x - 1)^{-1/2}]^{-2}$

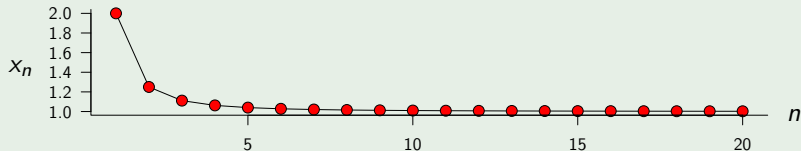
Get this formula by solving for n in terms of x in
 $x = 1 + 1/(n - 1)^2$ ($= x_{n-1}$).

Such an inversion will NOT always be possible.

Number line representation of $\{x_n\}$:



Graph of $f(n)$:



Convergence of sequences

We know from previous experience that:

- $cr^{n-1} \rightarrow 0$ as $n \rightarrow \infty$ (if $|r| < 1$).

- $1 + \frac{1}{n^2} \rightarrow 1$ as $n \rightarrow \infty$.

How do we make our intuitive notion of *convergence* mathematically rigorous?

Informal definition: “ $x_n \rightarrow L$ as $n \rightarrow \infty$ ” means “we can make the difference between x_n and L as small as we like by choosing n big enough”.

More careful informal definition: “ $x_n \rightarrow L$ as $n \rightarrow \infty$ ” means “given any *error tolerance*, say ε , we can make the *distance* between x_n and L smaller than ε by choosing n big enough”.

Convergence of sequences

Definition (Limit of a sequence)

A sequence $\{s_n\}$ **converges to** L if, given any $\varepsilon > 0$ there is some integer N such that

$$\text{if } n \geq N \quad \text{then} \quad |s_n - L| < \varepsilon.$$

In this case, we write $\lim_{n \rightarrow \infty} s_n = L$ or $s_n \rightarrow L$ as $n \rightarrow \infty$ and we say that L is the **limit** of the sequence $\{s_n\}$.

Note: To use this definition to prove that the limit of a sequence is L , we start by imagining that we are given some error tolerance $\varepsilon > 0$. Then we have to find a suitable N , which will depend on ε . This means that *the N that we find will be a function of ε .*

Shorthand:

$$\lim_{n \rightarrow \infty} s_n = L \stackrel{\text{def}}{=} \forall \varepsilon > 0 \quad \exists N \in \mathbb{N} \quad \forall n \geq N \implies |s_n - L| < \varepsilon.$$

Convergence of sequences

Convergence terminology:

- A sequence that converges is said to be *convergent*.
- A sequence that is not convergent is said to be *divergent*.

Remark (Sequences in spaces other than \mathbb{R})

The *formal definition of a limit of a sequence* works in any space where we have a *notion of distance* if we replace $|s_n - L|$ with $d(s_n, L)$.

Convergence of sequences

Example

Use the [formal definition of a limit of a sequence](#) to prove that

$$\frac{n^2 + 1}{n^2} \rightarrow 1 \quad \text{as } n \rightarrow \infty.$$

(solution on board)

Note: Our strategy here was to solve for n in the inequality $|s_n - L| < \varepsilon$. From this we were able to infer how big N has to be in order to ensure that $|s_n - L| < \varepsilon$ for all $n \geq N$. That much was “rough work”. Only after this rough work did we have enough information to be able to write down a rigorous proof.

Convergence of sequences

Example

Use the [formal definition of a limit of a sequence](#) to prove that

$$\frac{n^5 - n^3 + 1}{n^8 - n^5 + n + 1} \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

(solution on board)

Note: In this example, it was not possible to solve for n in the inequality $|s_n - L| < \varepsilon$. Instead, we first needed to bound $|s_n - L|$ by a much simpler expression that is always greater than $|s_n - L|$. If that bound is less than ε then so is $|s_n - L|$.



Mathematics
and Statistics

$$\int_M d\omega = \int_{\partial M} \omega$$

Mathematics 3A03 Real Analysis I

Instructor: David Earn

Lecture 7
Sequences II
Tuesday 17 September 2019

Poll

- Go to https://www.childsmath.ca/childsa/forms/main_login.php
- Click on [Math 3A03](#)
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- .

Announcements

- If you are interested in becoming a volunteer notetaker to support students with disabilities, please go to <https://sas.mcmaster.ca/volunteer-notetaking/>.
- **Solutions to Assignment 1** will be posted soon. **Study them!**
- **Assignment 2** will be posted soon. Due in two weeks.
- No late submission of assignments. No exceptions. However, best 5 of 6 assignments will be counted. *Always due 5 minutes before class on the due date.*
- **Note as stated on course info sheet:** *Only a selection of problems on each assignment will be marked; your grade on each assignment will be based only on the problems selected for marking. Problems to be marked will be selected after the due date.*

Announcements continued. . .

- Remember that solutions to assignments and tests from previous years are available on the [course web site](#). Take advantage of these problems and solutions. They provide many useful examples that should help you prepare for tests and the final exam. (However, note that while most of the content of the course is the same this year, there are some differences.)

Uniqueness of limits

Theorem (Uniqueness of limits)

If $\lim_{n \rightarrow \infty} s_n = L_1$ and $\lim_{n \rightarrow \infty} s_n = L_2$ then $L_1 = L_2$.

(solution on board)

So, we are justified in referring to “the” limit of a convergent sequence.

Divergence of sequences

Divergence is the logical opposite (negation) of convergence. We can infer the formal meaning of divergence by taking the *logical negation* of the *formal definition of convergence*. Doing so, we find that the sequence $\{s_n\}$ diverges (*i.e.*, does not converge to any $L \in \mathbb{R}$) iff

$$\forall L \in \mathbb{R}, \exists \varepsilon > 0 \text{ such that: } \forall N \in \mathbb{N} \exists n \geq N \text{ } \neg |s_n - L| \geq \varepsilon.$$

Notes:

- The n that exists will, in general, depend on L , ε and N .
- This is the meaning of not converging to any limit, but it does not tell us anything about what happens to the sequence $\{s_n\}$ as $n \rightarrow \infty$.

Divergence to $\pm\infty$

Definition (Divergence to ∞)

The sequence $\{s_n\}$ of real numbers **diverges to** ∞ if, for every real number M there is an integer N such that

$$n \geq N \implies s_n \geq M,$$

in which case we write $s_n \rightarrow \infty$ as $n \rightarrow \infty$ or $\lim_{n \rightarrow \infty} s_n = \infty$.

Definition (Divergence to $-\infty$)

The sequence $\{s_n\}$ of real numbers **diverges to** $-\infty$ if, for every real number M there is an integer N such that

$$n \geq N \implies s_n \leq M.$$

Divergence to ∞

Example

Use the [formal definition](#) to prove that

$$\left\{ \frac{n^3 - 1}{n + 1} \right\} \text{ diverges to } \infty .$$

(solution on board)

Approach: Find a lower bound for the sequence that is a simple function of n and show that that can be made bigger than any given M .

Divergence to ∞

Example (from previous slide)

Use the **formal definition** to prove that $\left\{ \frac{n^3 - 1}{n + 1} \right\}$ diverges to ∞ .

Clean proof.

Given $M \in \mathbb{R}^{>0}$, let $N = \lceil M \rceil + 1$. Then $N - 1 = \lceil M \rceil \geq M$.
 $\therefore \forall n \geq N, n - 1 \geq M$. Now observe that

$$\forall n \in \mathbb{N}, \quad n - 1 = \frac{(n - 1)(n + 1)}{n + 1} = \frac{n^2 - 1}{n + 1} \leq \frac{n^3 - 1}{n + 1}.$$

$\therefore \forall n \geq N$ we have

$$\frac{n^3 - 1}{n + 1} \geq M,$$

as required. □

Sequences of partial sums (a.k.a. Series)

Given a sequence $\{x_n\}$, we define the *sequence of partial sums of $\{x_n\}$* to be $\{s_n\}$, where

$$s_n = \sum_{k=1}^n x_k = x_1 + x_2 + \cdots + x_n.$$

Note: We can start from any integer, not necessarily $k = 1$.

Boundedness of sequences

A sequence is said to be bounded if its range is a bounded set.

Definition (Bounded sequence)

A sequence $\{s_n\}$ is **bounded** if there is a real number M such that every term in the sequence satisfies $|s_n| \leq M$.

Theorem (Every convergent sequence is bounded.)

$$L \in \mathbb{R} \wedge \lim_{n \rightarrow \infty} s_n = L \implies \exists M > 0 \ \vdash \ |s_n| \leq M \ \forall n \in \mathbb{N}.$$

(solution on board)

Note: The converse is **FALSE**.

Proof? Find a counterexample, e.g., $\{(-1)^n\}$.



Mathematics
and Statistics

$$\int_M d\omega = \int_{\partial M} \omega$$

Mathematics 3A03 Real Analysis I

Instructor: David Earn

Lecture 8
Sequences III
Thursday 19 September 2019

What we've done so far on sequences

- Definition of **convergence**.
- Definition of **divergence**.
- Definition of **divergence to $\pm\infty$** .
- Series: sequence of partial sums.
- **Bounded sequences**.
- Examples.

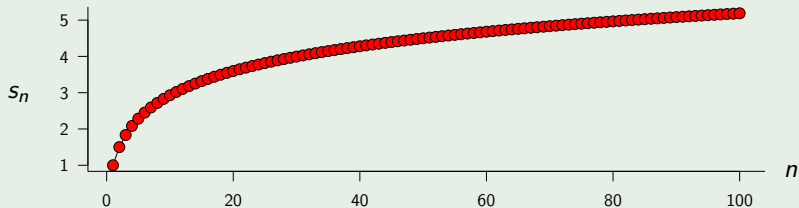
Boundedness of sequences

Corollary (Unbounded sequences diverge)

If $\{s_n\}$ is unbounded then $\{s_n\}$ *diverges*.

Example (The harmonic series diverges)

Consider the *harmonic series* $s_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$.



Prove that s_n *diverges* to ∞ .

(solution on board)

Harmonic series – idea for proof of divergence

Approach: Group terms and use the [corollary above](#).

$$\begin{array}{c}
 \underbrace{\left(1 + \frac{1}{2}\right)}_{> 1 \times \frac{1}{2}} + \underbrace{\left(\frac{1}{3} + \frac{1}{4}\right)}_{> 2 \times \frac{1}{4}} + \underbrace{\left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right)}_{> 4 \times \frac{1}{8}} + \dots \\
 \underbrace{s_2}_{> 1 \times \frac{1}{2}} \\
 \underbrace{s_4}_{> 2 \times \frac{1}{2}} \\
 \underbrace{s_8}_{> 3 \times \frac{1}{2}} \\
 \\
 \implies s_{2^n} > n \times \frac{1}{2}
 \end{array}$$

Note: These sorts calculations are just “rough work”, not a formal proof. A proof must be a clearly presented coherent argument from beginning to end.

Harmonic series – clean proof of divergence

Proof.

Part (i). Prove (e.g., by induction) that $s_{2^n} > n/2 \quad \forall n \in \mathbb{N}$.

Part (ii). Suppose we are given $M \in \mathbb{R}$.

- If $M \leq 0$ then note that $s_n > 0 \quad \forall n \in \mathbb{N}$.
- If $M > 0$, let $\tilde{N} = 2 \lceil M \rceil$ and $N = 2^{\tilde{N}}$. Then, $\forall n \geq N$, we have $s_n \geq s_N = s_{2^{\tilde{N}}} > \tilde{N}/2 = \lceil M \rceil \geq M$, as required.



Poll

- Go to https://www.childsmath.ca/childsa/forms/main_login.php
- Click on [Math 3A03](#)
- Click on [Take Class Poll](#)
- Fill in poll **Lecture 8: Harmonic series of primes**
- .

Algebra of limits

Theorem (Algebraic operations on limits)

Suppose $\{s_n\}$ and $\{t_n\}$ are *convergent sequences* and $C \in \mathbb{R}$.

$$1 \quad \lim_{n \rightarrow \infty} C s_n = C \left(\lim_{n \rightarrow \infty} s_n \right) ;$$

$$2 \quad \lim_{n \rightarrow \infty} (s_n + t_n) = \left(\lim_{n \rightarrow \infty} s_n \right) + \left(\lim_{n \rightarrow \infty} t_n \right) ;$$

$$3 \quad \lim_{n \rightarrow \infty} (s_n - t_n) = \left(\lim_{n \rightarrow \infty} s_n \right) - \left(\lim_{n \rightarrow \infty} t_n \right) ;$$

$$4 \quad \lim_{n \rightarrow \infty} (s_n t_n) = \left(\lim_{n \rightarrow \infty} s_n \right) \left(\lim_{n \rightarrow \infty} t_n \right) ;$$

5 *if $t_n \neq 0$ for all n and $\lim_{n \rightarrow \infty} t_n \neq 0$ then*

$$\lim_{n \rightarrow \infty} \left(\frac{s_n}{t_n} \right) = \frac{\lim_{n \rightarrow \infty} s_n}{\lim_{n \rightarrow \infty} t_n} .$$

(solution on board)

Revisit example

Example (previously proved directly from definition)

Use the algebraic properties of limits to prove that

$$\frac{n^5 - n^3 + 1}{n^8 - n^5 + n + 1} \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

(solution on board)



Mathematics
and Statistics

$$\int_M d\omega = \int_{\partial M} \omega$$

Mathematics 3A03 Real Analysis I

Instructor: David Earn

Lecture 9
Sequences IV
Friday 20 September 2019

Announcements

- [Assignment 2](#) is posted.
Due 1 Oct 2019, at 2:25pm.

What we've done so far on sequences

- Definition of **convergence**.
- Definition of **divergence**.
- Definition of **divergence to $\pm\infty$** .
- Examples.
- **Every convergent sequence is bounded**.
- **Harmonic series diverges**.
- **Algebra of limits** (more today).

Product Rule for Limits

The 4th item in the [algebra of limits](#) theorem was:

Theorem (Product Rule for Limits)

If $s_n \rightarrow S$ and $t_n \rightarrow T$ as $n \rightarrow \infty$ then $s_n t_n \rightarrow ST$ as $n \rightarrow \infty$.

Proof.

$$\begin{aligned} \text{For any } n \in \mathbb{N}, \quad |s_n t_n - ST| &= |s_n t_n - ST + s_n T - s_n T| \\ &= |s_n(t_n - T) + T(s_n - S)| \\ &\leq |s_n| |t_n - T| + |T| |s_n - S| \end{aligned}$$

Now, $\{s_n\}$ [converges, so it is bounded](#) by some $M > 0$, i.e., $|s_n| \leq M \forall n \in \mathbb{N}$. Therefore, given $\varepsilon > 0$, choose $N \in \mathbb{N}$ such that

$$|t_n - T| < \frac{\varepsilon}{2M} \quad \text{and} \quad |s_n - S| < \frac{\varepsilon}{2(1 + |T|)}.$$

Then $|s_n t_n - ST| < \varepsilon/2 + \varepsilon/2 = \varepsilon$, as required. □

Quotient Rule for Limits

Quotient Rule was the 5th item in the **algebra of limits** theorem.

Lemma (Reciprocal Rule for Limits)

If $t_n \neq 0 \forall n$ and $t_n \rightarrow T \neq 0$ then $1/t_n \rightarrow 1/T$.

Proof.

For any $n \in \mathbb{N}$, $\left| \frac{1}{t_n} - \frac{1}{T} \right| = \left| \frac{t_n - T}{t_n T} \right| = |t_n - T| \cdot \frac{1}{|t_n|} \cdot \frac{1}{|T|}$.

Since $\{t_n\}$ **converges**, $\exists N_1 \in \mathbb{N}$ such that $\forall n \geq N_1$, $|t_n| > |T|/2$ (details on **next slide**) and hence $1/|t_n| < 2/|T|$.

Now choose $N \geq N_1$ such that $|t_n - T| < \varepsilon|T|^2/2$. Then

$$\left| \frac{1}{t_n} - \frac{1}{T} \right| = |t_n - T| \cdot \frac{1}{|t_n|} \cdot \frac{1}{|T|} < \frac{\varepsilon|T|^2}{2} \cdot \frac{2}{|T|} \cdot \frac{1}{|T|} = \varepsilon,$$

as required. □

Quotient Rule for Limits

Details missing on previous slide: (consider $\varepsilon = \frac{|T|}{2}$)

Since $t_n \rightarrow T$, $\exists N_1 \in \mathbb{N}$ such that $\forall n \geq N_1$, $|t_n - T| < \frac{|T|}{2}$,

$$\text{i.e., } -\frac{|T|}{2} < t_n - T < \frac{|T|}{2}, \quad \text{i.e., } T - \frac{|T|}{2} < t_n < T + \frac{|T|}{2}.$$

If $T > 0$ this says

$$0 < \frac{T}{2} < t_n < \frac{3T}{2},$$

whereas if $T < 0$ it says

$$\frac{3T}{2} < t_n < \frac{T}{2} < 0.$$

In either case, $\forall n \geq N_1$, we have $0 < \frac{|T|}{2} < |t_n|$.

Order properties of limits (§2.8)

Theorem (Limits retain order)

If $\{s_n\}$ and $\{t_n\}$ are *convergent sequences* then

$$s_n \leq t_n \quad \forall n \in \mathbb{N} \quad \implies \quad \lim_{n \rightarrow \infty} s_n \leq \lim_{n \rightarrow \infty} t_n.$$

Proof.

Given $\varepsilon > 0$, choose $N \in \mathbb{N}$ s.t. $|s_n - S| < \frac{\varepsilon}{2}$ and $|t_n - T| < \frac{\varepsilon}{2}$. Then

$$\begin{aligned} S - T &= S - T + s_n - s_n + t_n - t_n \\ &= (S - s_n) + (t_n - T) + s_n - t_n \\ &\leq (S - s_n) + (t_n - T) && (\because s_n - t_n \leq 0) \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \end{aligned}$$

Hence $S - T \leq 0$, i.e., $S \leq T$. □

Poll

- Go to https://www.childsmath.ca/childsa/forms/main_login.php
- Click on [Math 3A03](#)
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- .

Order properties of limits (§2.8)

Question: If $s_n < t_n$ for all $n \in \mathbb{N}$, can we conclude that

$$\lim_{n \rightarrow \infty} s_n < \lim_{n \rightarrow \infty} t_n \quad ?$$

No! No! No! No! No! No!! NO!!!!!!!!!!!!!!

Theorem (Limits retain bounds)

If $\{s_n\}$ is a *convergent sequence* then

$$\alpha \leq s_n \leq \beta \quad \forall n \in \mathbb{N} \quad \implies \quad \alpha \leq \lim_{n \rightarrow \infty} s_n \leq \beta.$$

Proof.

Apply [previous theorem](#) with $\alpha_n = \alpha \forall n$ and $\beta_n = \beta \forall n$. □

Order properties of limits (§2.8)

Theorem (Squeeze Theorem)

If $\{s_n\}$ and $\{t_n\}$ are *convergent sequences* such that

(i) $s_n \leq x_n \leq t_n \quad \forall n \in \mathbb{N},$ *(x_n is always between them)*

(ii) $\lim_{n \rightarrow \infty} s_n = \lim_{n \rightarrow \infty} t_n = L.$ *(both approach the same limit)*

Then $\{x_n\}$ is *convergent* and $\lim_{n \rightarrow \infty} x_n = L.$

Proof? (What's **WRONG?**).

$\{s_n\}$ and $\{t_n\}$ are both bounded since they both converge. $\{x_n\}$ is therefore bounded (by the lower bound of $\{s_n\}$ and the upper bound of $\{t_n\}$). $\{x_n\}$ therefore converges, say $x_n \rightarrow X$. Hence, by *order retention*, $L \leq X \leq L \implies X = L.$ □

