Mathematics
and Statistics

$$\int_M d\omega = \int_{\partial M} \omega$$

Mathematics 3A03
Real Analysis I

Instructor: David Earn

Lecture 6
Sequences
Friday 13 September 2019

Poll

- Go to https://www.childsmath.ca/childsa/forms/main_login.php
- Click on Math 3A03
- Click on Take Class Poll
- Fill in poll **Lecture 6: Sequence convergence**
- $\boxed{\texttt{Submit}}$.

Announcements

- Assignment 1 is due via crowdmark 5 minutes before class on Monday.
- Consider writing the Putnam competition.

Sequences

- A ***sequence*** is a list that goes on forever.
- There is a beginning (a "first term") but no end, *e.g.*,

$$\frac{1}{1},\ \frac{1}{2},\ \frac{1}{3},\ \frac{1}{4},\ \ldots,\ \frac{1}{n},\ \ldots$$

- We use the natural numbers $\mathbb{N}$ to label the terms of a sequence:

$$a_1,\ a_2,\ a_3,\ \ldots,\ a_n,\ \ldots$$

Formal definition of a sequence

Definition (Sequence of Real Numbers)

A ***sequence of real numbers*** is a function

$$f : \mathbb{N} \to \mathbb{R} \, .$$

A lot of different notation is common for sequences:

$$f(1),\ f(2),\ f(3),\ \ldots \qquad \{f(n)\}_{n=1}^{\infty}$$
$$f_1,\ f_2,\ f_3,\ \ldots \qquad \{f(n)\}$$
$$\{f(n) : n = 1, 2, 3, \ldots\} \qquad \{f_n\}_{n=1}^{\infty}$$
$$\{f(n) : n \in \mathbb{N}\} \qquad \{f_n\}$$

Specifying sequences

There are two main ways to specify a sequence:

1. Direct formula.

Specify $f(n)$ for each $n \in \mathbb{N}$. □

Example (arithmetic progression with common difference d)

Sequence is:

$$c,\ c+d,\ c+2d,\ c+3d,\ \ldots$$

$$\therefore\ f(n) = c + (n-1)d, \qquad n \in \mathbb{N}$$

$$\text{i.e.,}\quad x_n = c + (n-1)d, \qquad n = 1, 2, 3, \ldots$$

Specifying sequences

2. Recursive formula.

Specify first term and function $f(x)$ to ***iterate***. $\square$

i.e., Given x_1 and $f(x)$, we have $x_n = f(x_{n-1})$ for all $n > 1$.

$$x_2 = f(x_1), \quad x_3 = f(f(x_1)), \quad x_4 = f(f(f(x_1))), \quad \dots$$

Example (arithmetic progression with common difference d)

$$x_1 = c, \quad f(x) = x + d$$

$$\therefore \; x_n = x_{n-1} + d, \qquad n = 2, 3, 4, \dots$$

Note: f is the most typical function name for both the direct and recursive specifications. The correct interpretation of f should be clear from context.

Specifying sequences

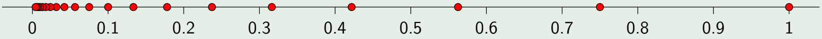
Example (geometric progression with common ratio r)

Sequence is: $c,\ cr,\ cr^2,\ cr^3,\ \ldots$

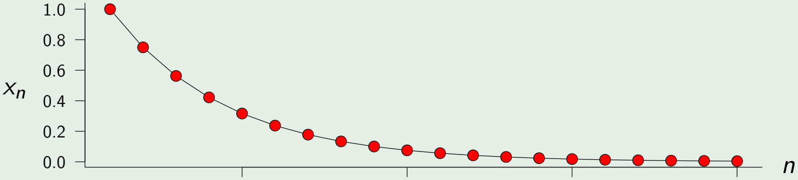
Direct formula: $x_n = f(n) = cr^{n-1}$, $n = 1, 2, 3, \ldots$

Recursive formula: $x_1 = c$, $f(x) = rx$, $x_n = f(x_{n-1})$

Number line representation of $\{x_n\}$ with $c = 1$ and $r = \frac{3}{4}$:



Graph of $f(n)$:



Specifying sequences

Example ($f(n) = 1 + \frac{1}{n^2}$)

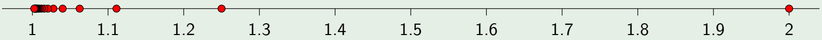
Sequence is: 2, $\frac{5}{4}$, $\frac{10}{9}$ $\frac{17}{16}$, $\ldots$

Direct formula: $x_n = f(n) = 1 + \frac{1}{n^2}$, $n = 1, 2, 3, \ldots$

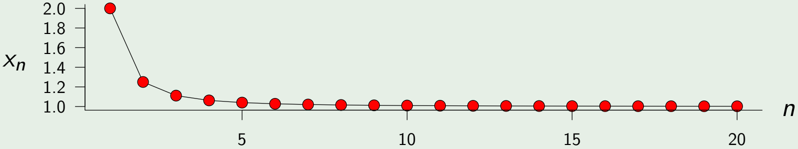
Recursive formula: $x_1 = 2, \quad f(x) = 1 + [1 + (x-1)^{-1/2}]^{-2}$

Get this formula by solving for n in terms of x in
$x = 1 + 1/(n-1)^2 \quad (= x_{n-1})$.
Such an inversion will NOT always be possible.

Number line representation of $\{x_n\}$:



Graph of $f(n)$:



Convergence of sequences

We know from previous experience that:

- $cr^{n-1} \to 0$ as $n \to \infty$ (if $|r| < 1$).
- $1 + \frac{1}{n^2} \to 1$ as $n \to \infty$.

How do we make our intuitive notion of ***convergence*** *mathematically rigorous*?

Informal definition: "$x_n \to L$ as $n \to \infty$" means "we can make the difference between x_n and L as small as we like by choosing n big enough".

More careful informal definition: "$x_n \to L$ as $n \to \infty$" means "given any *error tolerance*, say ε, we can make the distance between x_n and L smaller than ε by choosing n big enough".

Convergence of sequences

Definition (Limit of a sequence)

A sequence $\{s_n\}$ ***converges to*** L if, given any $\varepsilon > 0$ there is some integer N such that

$$\text{if } n \geq N \qquad \text{then} \qquad |s_n - L| < \varepsilon\,.$$

In this case, we write $\lim\limits_{n\to\infty} s_n = L$ or $s_n \to L$ as $n \to \infty$ and we say that L is the ***limit*** of the sequence $\{s_n\}$.

Note: To use this definition to prove that the limit of a sequence is L, we start by imagining that we are given some error tolerance $\varepsilon > 0$. Then we have to find a suitable N, which will depend on ε. This means that *the N that we find will be a function of ε*.

Shorthand:

$$\lim_{n\to\infty} s_n = L \quad \stackrel{\text{def}}{=} \quad \forall \varepsilon > 0 \quad \exists N \in \mathbb{N} \quad \ni \quad n \geq N \implies |s_n - L| < \varepsilon.$$

Convergence of sequences

Convergence terminology:

- A sequence that converges is said to be ***convergent***.
- A sequence that is not convergent is said to be ***divergent***.

Remark (Sequences in spaces other than $\mathbb{R}$)

The formal definition of a limit of a sequence works in any space where we have a notion of distance if we replace $|s_n - L|$ with $d(s_n, L)$.

Convergence of sequences

Example

Use the formal definition of a limit of a sequence to prove that

$$\frac{n^2+1}{n^2} \to 1 \quad \text{as} \quad n \to \infty\,.$$

(solution on board)

Note: Our strategy here was to solve for n in the inequality $|s_n - L| < \varepsilon$. From this we were able to infer how big N has to be in order to ensure that $|s_n - L| < \varepsilon$ for all $n \geq N$. That much was "rough work". Only after this rough work did we have enough information to be able to write down a rigorous proof.

Convergence of sequences

Example

Use the formal definition of a limit of a sequence to prove that

$$\frac{n^5 - n^3 + 1}{n^8 - n^5 + n + 1} \to 0 \quad \text{as} \quad n \to \infty\,.$$

(solution on board)

Note: In this example, it was not possible to solve for n in the inequality $|s_n - L| < \varepsilon$. Instead, we first needed to bound $|s_n - L|$ by a much simpler expression that is always greater than $|s_n - L|$. If that bound is less than ε then so is $|s_n - L|$.

McMaster University

Mathematics and Statistics

$$\int_M d\omega = \int_{\partial M} \omega$$

Mathematics 3A03
Real Analysis I

Instructor: David Earn

Lecture 7
Sequences II
Tuesday 17 September 2019

Poll

- Go to https://www.childsmath.ca/childsa/forms/main_login.php
- Click on Math 3A03
- Click on Take Class Poll
- Fill in poll **Lecture 7: Sequence divergence**
- Submit.

Announcements

- If you are interested in becoming a volunteer notetaker to support students with disabilities, please go to `https://sas.mcmaster.ca/volunteer-notetaking/`.
- Solutions to Assignment 1 will be posted soon. **Study them!**
- Assignment 2 will be posted soon. Due in two weeks.
- No late submission of assignments. No exceptions. However, best 5 of 6 assignments will be counted. *Always due 5 minutes before class on the due date.*
- **Note as stated on course info sheet:** *Only a selection of problems on each assignment will be marked; your grade on each assignment will be based only on the problems selected for marking. Problems to be marked will be selected after the due date.*

Announcements continued...

- Remember that solutions to assignments and tests from previous years are available on the course web site. Take advantage of these problems and solutions. They provide many useful examples that should help you prepare for tests and the final exam. (However, note that while most of the content of the course is the same this year, there are some differences.)

Uniqueness of limits

Theorem (Uniqueness of limits)

If $\lim_{n\to\infty} s_n = L_1$ *and* $\lim_{n\to\infty} s_n = L_2$ *then* $L_1 = L_2$.

(solution on board)

So, we are justified in referring to "the" limit of a convergent sequence.

Divergence of sequences

Divergence is the logical opposite (negation) of convergence.
We can infer the formal meaning of divergence by taking the *logical negation* of the formal definition of convergence.
Doing so, we find that the sequence $\{s_n\}$ diverges (*i.e.*, does not converge to any $L \in \mathbb{R}$) iff

$$\forall L \in \mathbb{R}, \;\; \exists \varepsilon > 0 \;\text{ such that: } \forall N \in \mathbb{N} \;\; \exists n \geq N \;\ni\; |s_n - L| \geq \varepsilon.$$

Notes:

- The n that exists will, in general, depend on L, ε and N.
- This is the meaning of not converging to any limit, but it does not tell us anything about what happens to the sequence $\{s_n\}$ as $n \to \infty$.

Divergence to $\pm\infty$

Definition (Divergence to ∞)

The sequence $\{s_n\}$ of real numbers ***diverges to*** ∞ if, for every real number M there is an integer N such that

$$n \geq N \quad \Longrightarrow \quad s_n \geq M\,,$$

in which case we write $s_n \to \infty$ as $n \to \infty$ or $\lim\limits_{n\to\infty} s_n = \infty$.

Definition (Divergence to $-\infty$)

The sequence $\{s_n\}$ of real numbers ***diverges to*** $-\infty$ if, for every real number M there is an integer N such that

$$n \geq N \quad \Longrightarrow \quad s_n \leq M\,.$$

Divergence to ∞

Example

Use the formal definition to prove that

$$\left\{ \frac{n^3 - 1}{n + 1} \right\} \quad \text{diverges to } \infty \,.$$

(solution on board)

Approach: Find a lower bound for the sequence that is a simple function of n and show that that can be made bigger than any given M.

Divergence to ∞

Example (from previous slide)

Use the formal definition to prove that $\left\{ \dfrac{n^3-1}{n+1} \right\}$ diverges to ∞.

Clean proof.

Given $M \in \mathbb{R}^{>0}$, let $N = \lceil M \rceil + 1$. Then $N - 1 = \lceil M \rceil \geq M$.
$\therefore \forall n \geq N$, $n - 1 \geq M$. Now observe that

$$\forall n \in \mathbb{N}, \quad n - 1 = \frac{(n-1)(n+1)}{n+1} = \frac{n^2-1}{n+1} \leq \frac{n^3-1}{n+1}\,.$$

$\therefore \forall n \geq N$ we have

$$\frac{n^3-1}{n+1} \geq M\,,$$

as required. $\square$

Sequences of partial sums (a.k.a. Series)

Given a sequence $\{x_n\}$, we define the ***sequence of partial sums of*** $\{x_n\}$ to be $\{s_n\}$, where

$$s_n = \sum_{k=1}^{n} x_k = x_1 + x_2 + \cdots + x_n \,.$$

Note: We can start from any integer, not necessarily $k = 1$.

Boundedness of sequences

A sequence is said to be bounded if its range is a bounded set.

Definition (Bounded sequence)

A sequence $\{s_n\}$ is ***bounded*** if there is a real number M such that every term in the sequence satisfies $|s_n| \leq M$.

Theorem (Every convergent sequence is bounded.)

$$L \in \mathbb{R} \ \wedge \ \lim_{n\to\infty} s_n = L \quad \Longrightarrow \quad \exists M > 0 \ \ni \ |s_n| \leq M \ \forall n \in \mathbb{N}.$$

(solution on board)

Note: The converse is FALSE.
Proof?