1 Introduction

Mathematics
and Statistics

$$\int_M d\omega = \int_{\partial M} \omega$$

Mathematics 3A03
Real Analysis I

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Lecture 1
Introduction
Tuesday 3 September 2019

Where to find course information

- The course web site: `http://ms.mcmaster.ca/earn/3A03`
- Click on Course information to download course information as pdf file. *You are expected to read and pay attention to every word of this file.*
- Let's have a look now...

What is a "real" number?

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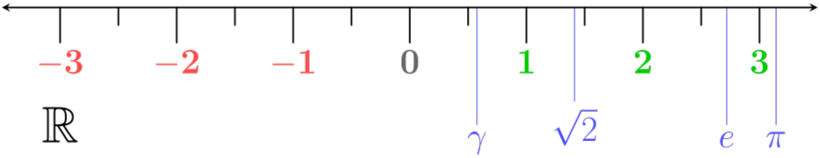
- The "Reals" ($\mathbb{R}$) are all the numbers that are needed to fill in the "number line" (so it has no "gaps" or "holes").
- Why aren't the rational numbers ($\mathbb{Q}$) sufficient?



- How do we know that $\sqrt{2}$ is not rational?
- How can we *prove* this?
  Approach: "Proof by contradiction."

$\sqrt{2}$ is irrational

Theorem

$\sqrt{2} \notin \mathbb{Q}$.

Proof.

Suppose $\sqrt{2} \in \mathbb{Q}$. Then there exist two positive integers m and n with $\gcd(m,n) = 1$ such that $m/n = \sqrt{2}$.

$$\therefore \left(\frac{m}{n}\right)^2 = (\sqrt{2})^2 \quad \Longrightarrow \quad \frac{m^2}{n^2} = 2 \quad \Longrightarrow \quad m^2 = 2n^2.$$

$\therefore m^2$ is even $\Longrightarrow$ m is even ($\because$ odd numbers have odd squares).

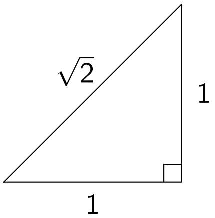
$\therefore m = 2k$ for some $k \in \mathbb{N}$.

$$\therefore 4k^2 = m^2 = 2n^2 \quad \Longrightarrow \quad 2k^2 = n^2 \quad \Longrightarrow \quad n \text{ is even}.$$

$\therefore$ 2 is a factor of both m and n. **Contradiction!** $\therefore \sqrt{2} \notin \mathbb{Q}$. $\square$

Does $\sqrt{2}$ exist?

- We have established that $\sqrt{2}$ is not rational.
- But do we really know it exists?
- Can we do without it?
- No. Objects with side length $\sqrt{2}$ exist!



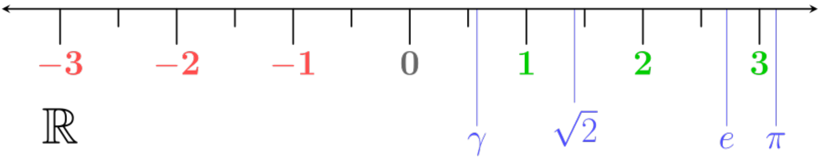
- So irrational numbers are "real".

Poll on rationality

- Please log in (right now) to this web site: https://www.childsmath.ca/childsa/forms/main_login.php
- Click on Math 3A03.
- Click on Take Class Poll.
- After selecting the numbers you think are rational, click the `Submit` button.
- Everybody done?
- Let's Deactivate the poll and View Results

What exactly *are* non-rational real numbers?

- We have solid intuition for what rational numbers are. (Ratios of integers.)
- The number line contains numbers that are not rational.



- Can we *construct* irrational numbers? (Just as we construct rationals as ratios of integers?)
- Do we need to *construct* integers first?
- Maybe we should start with 0, 1, 2, . . .
- But <u>what</u> exactly are we supposed to *construct* numbers <u>from</u>?

Informal introduction to construction of numbers ($\mathbb{N}$)

- Assume we know what a ***set*** is.
- Define $0 \equiv \varnothing = \{\}$ (the empty set)
- Define $1 \equiv \{0\} = \{\varnothing\} = \{\{\}\}$
- Define $2 \equiv \{0, 1\} = \{\{\}, \{\{\}\}\}$
- Define $n + 1 \equiv n \cup \{n\}$ (successor function)
- Define ***natural numbers*** $\mathbb{N} = \{1, 2, 3, \ldots\}$
  - Some books define $\mathbb{N} = \{0, 1, 2, \ldots\}$ and $\mathbb{N}^+ = \{1, 2, 3, \ldots\}$.
  - It is more common to define $\mathbb{N}$ to start with 1.
- Thus, n is defined to be a set containing n elements.

Informal introduction to construction of numbers ($\mathbb{N}$)

Historical note:

- We have defined n to be a set containing n elements.
- Logicians first tried to define n as "the set of all sets containing n elements".
- The earlier definition possibly better captures our intuitive notion of what n "really is", but such "sets" are unweildy and create serious challenges for development of mathematical foundations.

Informal introduction to construction of numbers ($\mathbb{N}$)

Order of natural numbers:

- Natural numbers defined as above have the right order:

$$m \leq n \iff m \subseteq n$$

Note: we *define* "$\leq$" on natural numbers via "$\subseteq$" on sets.

Addition and multiplication of natural numbers:

- Still possible to define in terms of sets, but trickier.
- We'll defer this for later, after gaining more experience with rigorous mathematical concepts.
- If you can't wait, see this free e-book:

  "Transition to Higher Mathematics"
  http://openscholarship.wustl.edu/books/10/.

Informal introduction to construction of numbers ($\mathbb{Z}$)

Integers:

- Need additive inverses for all natural numbers.
- Need to define $\cdot, +, -,$ for all pairs of integers.
- Again, possible to define everything via set theory.
- Again, we'll defer this for later.

- For now, we'll assume we "know" what the naturals $\mathbb{N}$ and the integers $\mathbb{Z}$ "are".
- We can then *construct* the rationals $\mathbb{Q}$...