McMaster University

Mathematics and Statistics

$$\int_M d\omega = \int_{\partial M} \omega$$

Mathematics 3A03
Real Analysis I

Instructor: David Earn

Lecture 32
Sequences of Functions
Saturday 25 November 2017 , Lecture 32
Sequences of Functions
Monday 27 November 2017

Limits of Functions

We know from calculus that it can be useful to represent functions as limits of other functions.

Example

The power series expansion

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

expresses the exponential e^x as a certain limit of the functions

$$1, \qquad 1+\frac{x}{1!}, \qquad 1+\frac{x}{1!}+\frac{x^2}{2!}, \qquad 1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}, \qquad \cdots$$

Our goal is to give meaning to the phrase "*limit of functions*", and discuss how functions behave under limits.

Pointwise Convergence

- There are multiple inequivalent ways to define the limit of a sequence of functions.
- $\therefore$ There are multiple different notions of what it means for a sequence of functions to converge.
- Some convergence notions are better behaved than others.

We will begin with the simplest notion of convergence.

Definition (Pointwise Convergence)

Suppose $\{f_n\}$ is a sequence of functions defined on a domain $D \subseteq \mathbb{R}$, and let f be another function defined on D. Then $\{f_n\}$ **converges pointwise on D to f** if, for every $x \in D$, the sequence $\{f_n(x)\}$ of real numbers converges to $f(x)$.

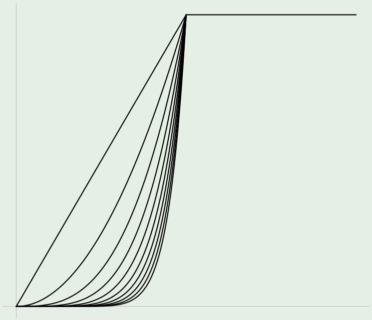
Unfortunately, *pointwise convergence does not preserve many useful properties of functions*.

Pointwise Convergence

Example

$$f_n(x) = \begin{cases} x^n & 0 \le x \le 1, \\ 1 & x \ge 1. \end{cases}$$

$$\lim_{n\to\infty} f_n(x) = \begin{cases} 0 & 0 \le x < 1 \\ 1 & x \ge 1 \end{cases}$$



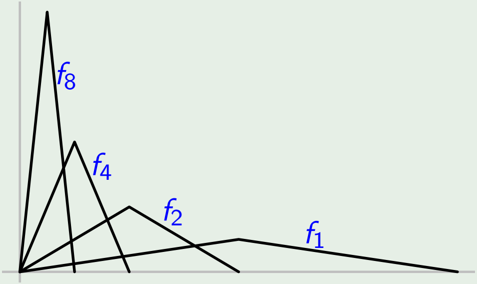
- Limit of sequence (of continuous functions) is not continuous.
- By smoothing the corner at $x = 1$, we get a sequence of differentiable functions that converge to a function that is not even continuous.

Pointwise Convergence

Example

Define $f_n(x)$ on $[0,1]$ as follows:

$$f_n(x) = \begin{cases} 2n^2x, & 0 \le x \le \frac{1}{2n} \\ 2n - 2n^2x, & \frac{1}{2n} \le x \le \frac{1}{n} \\ 0, & x \ge \frac{1}{n}. \end{cases}$$



$$\lim_{n\to\infty} f_n(x) = 0 \quad \forall\, x$$

$$\int_0^1 f_n = \frac{1}{2} \quad \forall\, n \in \mathbb{N}$$

$$\int_0^1 \lim_{n\to\infty} f_n = 0$$

Uniform Convergence

A much better behaved notion of convergence is the following.

Definition ($f_n \to f$ uniformly)

Suppose $\{f_n\}$ is a sequence of functions defined on a domain $D \subseteq \mathbb{R}$, and let f be another function defined on D. Then $\{f_n\}$ **converges uniformly on D to f** if, for every $\varepsilon > 0$, there is some $N \in \mathbb{N}$ so that, for all $x \in D$,

$$n \geq N \quad \Longrightarrow \quad |f_n(x) - f(x)| < \varepsilon.$$

Note that $\{f_n\}$ converges uniformly to f if and only if $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ such that

$$n \geq N \quad \Longrightarrow \quad \sup_{x \in D} |f_n(x) - f(x)| < \varepsilon.$$

uniform convergence $\overset{\Longrightarrow}{\not\Longleftarrow}$ pointwise convergence

Uniform Convergence

The following theorems illustrate the sense in which uniform convergence is better behaved than pointwise convergence in relation to common constructions in analysis.

Theorem (Integrability and Uniform Convergence)

Suppose $\{f_n\}$ is a sequence of functions that converges uniformly on $[a,b]$ to f. If each f_n is integrable on $[a,b]$, then f is integrable and

$$\int_a^b f = \lim_{n\to\infty} \int_a^b f_n \,.$$

(Textbook (TBB) §9.5.2, p. 571ff)

The proof that f is integrable is rather involved. We will skip it.

Uniform Convergence

Proof that $\int_a^b f = \lim_{n\to\infty} \int_a^b f_n$ given that f is integrable.

Given that f is integrable, to prove the equality, we will show that for each $\varepsilon > 0$, there is some $N \in \mathbb{N}$ so that

$$\left| \int_a^b f - \int_a^b f_n \right| < \varepsilon \qquad \text{for all } n \geq N.$$

For any $n \in \mathbb{N}$, we have

$$\left| \int_a^b f - \int_a^b f_n \right| \leq \int_a^b |f - f_n| \leq \sup_{x\in[a,b]} |f(x) - f_n(x)| \, (b-a) \, .$$

Since f_n converges uniformly to f, there is some $N \in \mathbb{N}$ so that $\sup_{x\in[a,b]} |f(x) - f_n(x)| < \varepsilon/(b-a)$ for all $n \geq N$. For such n, we have $\left| \int_a^b f - \int_a^b f_n \right| < \varepsilon$, as required. □

Please consider...

5 minute *Student Respiratory Illness Survey:*

https://surveys.mcmaster.ca/limesurvey2/index.php/893454

> *Please complete this anonymous survey to help us monitor the patterns of respiratory illness, over-the-counter drug use, and social contact within the McMaster community. There are no risks to filling out this survey, and your participation is voluntary. You do not need to answer any questions that make you uncomfortable, and all information provided will be kept strictly confidential. Thanks for participating.*
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> *–Dr. Marek Smieja (Infectious Diseases)*

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Mathematics 3A03
Real Analysis I

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Lecture 33
Sequences of Functions II
Wednesday 29 November 2017

Last time. . .

Convergence of sequences of functions:

- Pointwise convergence
- Uniform convergence
- Stated and proved theorem about integrability and uniform convergence
- Stated theorem about *continuity* and uniform convergence
  - We will prove it today.

Uniform Convergence

Theorem (Continuity and Uniform Convergence)

Suppose $\{f_n\}$ is a sequence of functions that converges uniformly on $[a,b]$ to f. If each f_n is continuous on $[a,b]$, then f is continuous on $[a,b]$.

Proof.

Fix $x \in [a,b]$ and $\varepsilon > 0$. We must show $\exists\delta > 0$ such that if $y \in [a,b]$ and $|y-x| < \delta$ then $|f(y) - f(x)| < \varepsilon$.

Since the f_n uniformly converge to f, there is some $N \in \mathbb{N}$ so that $|f_N(y) - f(y)| < \frac{\varepsilon}{3}$ for all $y \in [a,b]$. Fix such an N.

Since f_N is continuous, there is some $\delta > 0$ so that if $y \in [a,b]$ satisfies $|y-x| < \delta$, then $|f_N(y) - f_N(x)| < \frac{\varepsilon}{3}$. For such y, we then have

$$\begin{aligned} |f(y) - f(x)| &= |f(y) - f_N(y) + f_N(y) - f_N(x) + f_N(x) - f(x)| \\ &\leq |f(y) - f_N(y)| + |f_N(y) - f_N(x)| + |f_N(x) - f(x)| \\ &< \tfrac{\varepsilon}{3} + \tfrac{\varepsilon}{3} + \tfrac{\varepsilon}{3} = \varepsilon, \end{aligned}$$

as required. □

Uniform Convergence

The interaction between uniform convergence and differentiability is more subtle.

Theorem (Differentiability and Uniform Convergence)

Suppose $\{f_n\}$ is a sequence of differentiable functions on $[a, b]$ such that

1. *f_n' is continuous for each n,*
2. *the sequence $\{f_n'\}$ converges uniformly on $[a, b]$,*
3. *the sequence $\{f_n\}$ converges pointwise to a function f.*

Then f is differentiable and $\{f_n'\}$ converges uniformly to f'.

(Textbook (TBB) §9.6, p. 578ff)

Note: If we weaken the first condition to f_n' being integrable, but explicitly require in the second condition that the uniform limit is continuous, then the theorem is still true and no more difficult to prove.

Series of Real Numbers

Suppose $\{x_n\}$ is a sequence of real numbers. Recall that the **sequence of partial sums** is the sequence $\{s_n\}$ defined by

$$s_n = \sum_{k=1}^{n} x_n \,.$$

If the sequence of partial sums converges, then we write the limit as

$$\sum_{k=1}^{\infty} x_k = \lim_{n\to\infty} \sum_{k=1}^{n} x_n = \lim_{n\to\infty} s_n \,.$$

In this case, we call $\sum_{k=1}^{\infty} x_k$ a **convergent series**. A **divergent series** is a sequence of partial sums that diverges; we sometimes abuse notation and write $\sum_{k=1}^{\infty} x_k$ for divergent series as well.

A **series** is either a convergent series or a divergent series.

Our goal now is to extend this to sequences of <u>functions</u>.

Series of Functions

Suppose $\{f_n\}$ is a sequence of functions defined on a set $D \subseteq \mathbb{R}$. The **sequence of partial sums** is the sequence $\{S_n\}$ where S_n is the function defined on D by

$$S_n(x) = \sum_{k=1}^{n} f_k(x)\,.$$

When talking about limits of the S_n, we will write $\sum_{k=1}^{\infty} f_k$ and refer to this as a **series**.

Keep in mind that this is very informal, since the terminology does not specify any sense in which the S_n converge, nor does it assume that the S_n converge at all!

We will now make this more formal.

Series of Functions

Suppose $\{f_n\}$ is a sequence of functions defined on a domain D, and $\{S_n\}$ is its sequence of partial sums.

Definition (Convergence of Series)

If the sequence of partial sums $\{S_n\}$ converges pointwise on D to a function f, then we say that the series $\sum_{k=1}^{\infty} f_k$ **converges pointwise on D to f**.

If the $\{S_n\}$ converge uniformly on D to a function f, then we say that the series $\sum_{k=1}^{\infty} f_k$ **converges uniformly on D to f**.

In both cases, we will write $f = \sum_{k=1}^{\infty} f_k$ to denote that the **series converges to f**.

Series of Functions

The theorems on convergence of sequences of integrable, continuous and differentiable functions have several immediate implications for series of functions.

In the following, we assume that $\{f_n\}$ is a sequence of functions defined on an interval $[a, b]$.

Corollary (Integrals of Series)

Suppose the f_n are integrable and $\sum_{k=1}^{\infty} f_k$ converges uniformly to a function f. Then f is integrable and

$$\int_a^b f = \sum_{k=1}^{\infty} \int_a^b f_k \,.$$

Series of Functions

Corollary (Continuity of Series)

Suppose the f_n are continuous and $\sum_{k=1}^{\infty} f_k$ converges uniformly to a function f. Then f is continuous.

Corollary (Differentiability of Series)

Suppose $\{f_n\}$ is a sequence of differentiable functions on $[a, b]$ such that

- *f_n' is continuous for each n,*
- *the series $\sum_{k=1}^{\infty} f_k'$ converges uniformly on $[a, b]$,*
- *the series $\sum_{k=1}^{\infty} f_k$ converges pointwise to a function f.*

Then f is differentiable and $f' = \sum_{k=1}^{\infty} f_k'$.

Proving Uniform Convergence

We have just seen that several useful conclusions can be drawn when a series converges uniformly. The following gives a practical way of proving uniform convergence.

Theorem (Weierstrass M-test)

Let $\{f_n\}$ be a sequence of functions defined on $D \subseteq \mathbb{R}$, and suppose $\{M_n\}$ is a sequence of real numbers such that

$$|f_n(x)| \leq M_n, \qquad \forall x \in D,\ \forall n \in \mathbb{N}.$$

If $\sum_n M_n$ converges, then $\sum_{k=1}^{\infty} f_k$ converges uniformly.

Proving Uniform Convergence

Approach to proving the Weierstrass M-test:

- Let $S_n = \sum_{k=1}^{n} f_k$ be the nth partial sum.
- Show that for every $\varepsilon > 0$, there is some $N \in \mathbb{N}$ so that

$$\sup_{x \in D} |S_n(x) - S_m(x)| < \varepsilon\,, \qquad \forall n, m \geq N.$$

  This condition is called the **uniform Cauchy criterion**.
- Prove that the uniform Cauchy criterion implies uniform convergence.
  - This part is an excellent exercise for you.

    Note: The proof is similar to the proof of the Cauchy criterion for real numbers that we encountered earlier this term.

Proving Uniform Convergence

Proof of the Weierstrass M-test.

Let $\varepsilon > 0$. Suppose the series $\sum M_n$ converges. By the Cauchy criterion for real numbers, there is some integer N so that

$$\left|\sum_{k=1}^{n} M_k - \sum_{k=1}^{m} M_k\right| < \varepsilon\,, \qquad \forall n, m \geq N.$$

Without loss of generality, we can assume $m < n$, so the above can be written

$$M_{m+1} + M_{m+2} + \cdots + M_n < \varepsilon.$$

Note that we have $\quad S_n - S_m = f_{m+1} + f_{m+2} + \cdots + f_n, \quad$ so the assumption that $|f_k| \leq M_k$ gives, for all $x \in D$,

$$|S_n(x) - S_m(x)| \leq M_{m+1} + M_{m+2} + \cdots + M_n < \varepsilon. \qquad \square$$

Proving Uniform Convergence

Example

Let $p > 1$, and consider the series $\displaystyle\sum_{k=1}^{\infty} \frac{\sin(kx)}{k^p}$.

This satisfies $\left|\dfrac{\sin(kx)}{k^p}\right| \le \dfrac{1}{k^p}$ for all $x \in \mathbb{R}$.

Since the series $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^p}$ converges, it follows from the Weierstrass M-test that the series $\sum_{k=1}^{\infty} \frac{\sin(kx)}{k^p}$ converges uniformly.

Hence it is a continuous function.

In fact, if $p > 2$ then the series $\sum_{k=1}^{\infty} \frac{\sin(kx)}{k^p}$ is differentiable:

Let $f_k(x) = \frac{\sin(kx)}{k^p}$. The f_k' are continuous and another application of the Weierstrass M-test shows that $\sum_{k=1}^{\infty} f_k'$ converges uniformly. Hence the series is differentiable and the derivative is $\sum_{k=1}^{\infty} f_k'$.

Please consider. . .

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> *Please complete this anonymous survey to help us monitor the patterns of respiratory illness, over-the-counter drug use, and social contact within the McMaster community. There are no risks to filling out this survey, and your participation is voluntary. You do not need to answer any questions that make you uncomfortable, and all information provided will be kept strictly confidential. Thanks for participating.*
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Mathematics 3A03
Real Analysis I

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Lecture 34
Sequences of Functions III
Friday 1 December 2017

Announcements

- Solutions to Test 2 have been posted.
  They were revised slightly on Wednesday night (10pm).
- Assignment 6 was slightly revised at 1:05pm yesterday.
  "$0 < b$" should have said "$b < 0$."
- Putnam Competition is tomorrow at 10:00am in BSB-B155.

Last time:

- Continuity and uniform convergence
- Differentiability and uniform convergence
- Convergence of series
- Theorems about uniform convergence of series of functions
- Weierstrass M-test
  - Example

Power Series

Suppose $\{a_n\}$ is a sequence of real numbers.

Definition (Power Series)

A **power series (centred at 0)** is a series of the form

$$\sum_{k=0}^{\infty} a_k x^k \,.$$

More generally, a **power series centred at c** has the form

$$\sum_{k=0}^{\infty} a_k (x-c)^k \,.$$

Power Series

Corollary (Convergence of Power Series)

Suppose that the series $f(x_0) = \sum_{k=0}^{\infty} a_k x_0^k$ converges for some $x_0 > 0$, and $0 < a < x_0$. Then on $[-a, a]$, the series

$$f(x) = \sum_{k=0}^{\infty} a_k x^k$$

converges uniformly. Moreover, f is continuous and

$$\int_c^d f = \sum_{k=0}^{\infty} a_k \int_c^d x^k \qquad \forall c, d \in [-a, a].$$

Finally, f is differentiable and $\sum_{k=1}^{\infty} k a_k x^{k-1}$ converges uniformly on $[-a, a]$ to f'.

Power Series

Sketch of proof of convergence of power series $f(x) = \sum_{k=0}^{\infty} a_k x^k$ on $[-a, a]$

- Weierstrass M-test with $M_k = a_k x_0^k$ $\implies$ uniform convergence to f.
- Uniform convergence to f $\implies$ f is continuous and
$$\int_c^d f = \sum_{k=0}^{\infty} a_k \int_c^d x^k \, .$$
- That the derivative $\sum_{k=1}^{\infty} k a_k x^{k-1}$ converges uniformly on $[-a, a]$ can be proved via the ratio test (Textbook (TBB) Theorem 3.28) or the root test (Textbook (TBB) Theorem 3.30), which we have not formally discussed.
- Uniform convergence of the derivative series $\implies$ uniform limit f is differentiable.

Power Series

Example

Consider the series $\sum_{k=0}^{\infty} x^k$. If $0 < x_0 < 1$, then the series $\sum_{k=0}^{\infty} x_0^k$ converges. Consequently, for any $0 < a < 1$, the series $\sum_{k=0}^{\infty} x^k$ converges uniformly on $[-a, a]$ to a differentiable function. In fact, the function it converges to is $1/(1-x)$. The derivative is

$$\frac{1}{(1-x)^2} = \sum_{k=1}^{\infty} kx^{k-1},$$

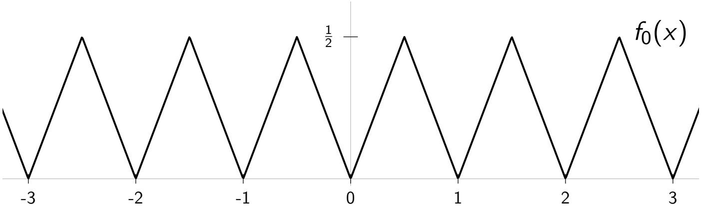
and the integral (from 0 to x) is

$$-\log(1-x) = \sum_{k=0}^{\infty} \frac{x^{k+1}}{k+1}.$$

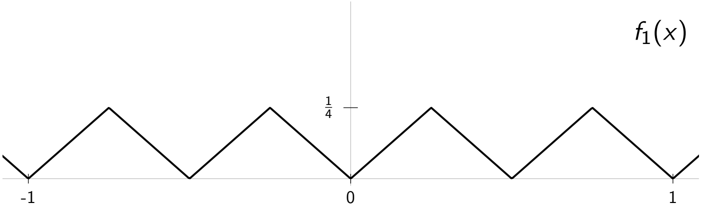
These are all valid for $x \in (-1, 1)$.

How bad can a continuous function be?

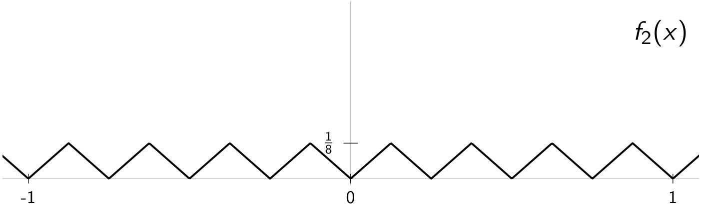
Let $f_0(x) =$ the distance from x to the nearest integer.

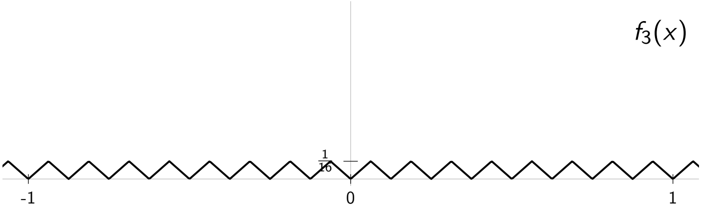


Let $f_n(x) = \dfrac{1}{2^n} f_0(2^n x)$.



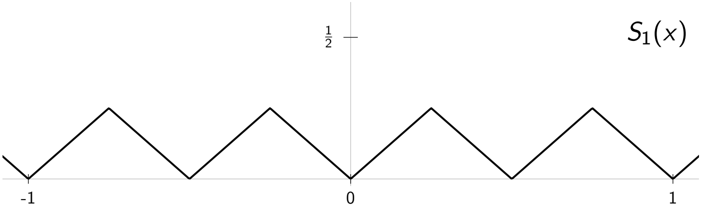
How bad can a continuous function be?

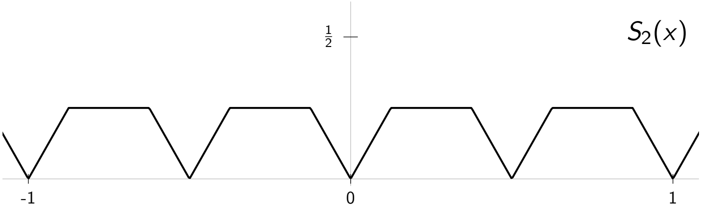




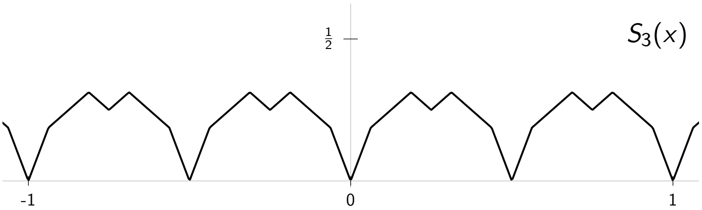
How bad can a continuous function be?

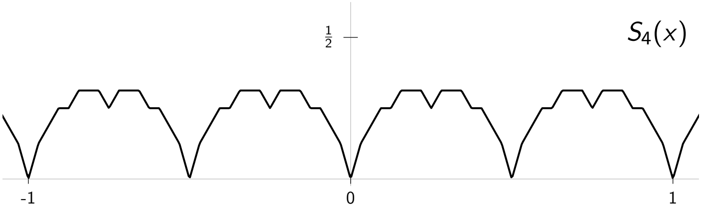
Now define $S_n(x) = \sum_{k=1}^{n} f_n(x)$.



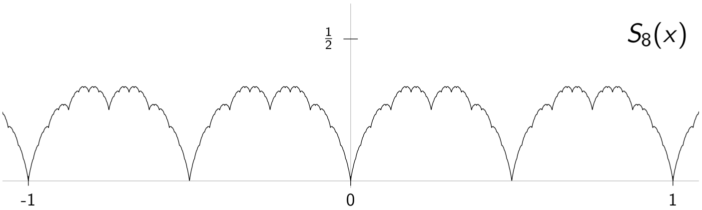


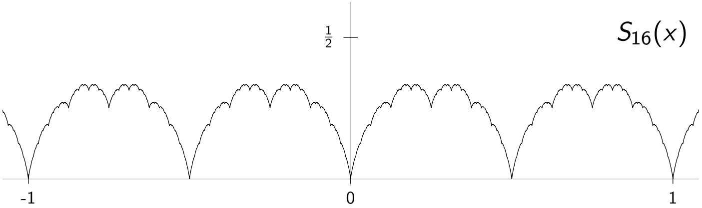
How bad can a continuous function be?





How bad can a continuous function be?





How bad can a continuous function be?

Now consider:

- Each f_n is continuous, so each $S_n = \sum_{k=1}^{n} f_n$ is continuous.
- $|f_n(x)| \le \dfrac{1}{2^n} \quad \forall x \in \mathbb{R}$.
- $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n}$ converges.
- $\therefore$ Weierstrass M-test $\implies \displaystyle\sum_{k=1}^{\infty} f_n$ converges uniformly.
- $\therefore$ The uniform limit, say f, is continuous.
- Is f uniformly continuous?
- Is f differentiable?

How bad can a continuous function be?

Extra Challenge Problem:

Prove that the uniform limit function,

$$f = \sum_{k=1}^{\infty} f_n \,,$$

which is continuous on $\mathbb{R}$, is in fact

1. uniformly continuous
2. differentiable **nowhere**

Note: Proving uniform continuity should be really really easy.

What <u>exactly</u> is $\mathbb{R}$?

Construction of the Real Numbers

- Recall that we defined the natural numbers $\mathbb{N}$ as sets: $0 \equiv \varnothing$, $1 \equiv \{0\}$, $2 \equiv \{0, 1\}$, *etc.*
- For $m, n \in \mathbb{N}$ we defined $m < n$ to mean $m \subset n$.
- We defined the rational numbers $\mathbb{Q}$ to be ordered pairs of integers (more precisely, $\mathbb{Q}$ is a set of equivalence classes of $\mathbb{Z} \times \mathbb{N}$).
- In the same spirit, we can define real numbers not by determining what they "really are" but instead by settling for a definition that determines their mathematical properties completely.
- So, just as $\mathbb{Z}$ can be built from $\mathbb{N}$, and $\mathbb{Q}$ can be built from $\mathbb{Z}$, we can build $\mathbb{R}$ from $\mathbb{Q}$.
- Richard Dedekind's idea was to construct a real number α as a set of rational numbers, in a way that naturally yields the one property of $\mathbb{R}$ that $\mathbb{Q}$ does not have: least upper bounds...

Construction of the Real Numbers

Dedekind's stroke of genius (on 24 Nov 1858) was to define α as "*the set of rational numbers less than α*" in a way that is not circular.

Definition (Real number)

A **real number** is a set $\alpha \subseteq \mathbb{Q}$, with the following four properties:

1. $\forall x \in \alpha$, if $y \in \mathbb{Q}$ and $y < x$, then $y \in \alpha$;
2. $\alpha \neq \varnothing$;
3. $\alpha \neq \mathbb{Q}$;
4. there is no greatest element in α,
   i.e., if $x \in \alpha$ then $\exists\, y \in \alpha$ such that $y > x$.

The set of all real numbers is denoted by $\mathbb{R}$.

Historical note: Dedekind originally defined a real number α as the pair of sets (L, R) where L is the set of rationals $< \alpha$ and R is the set of rationals $\geq \alpha$. A real number is then described as a **Dedekind cut**.

Construction of the Real Numbers

Example: $\sqrt{2} = \{q \in \mathbb{Q} : q^2 < 2 \text{ or } q < 0\}$.

With real numbers defined, we can easily define an ordering on $\mathbb{R}$.

Definition (Order of real numbers)

If $\alpha, \beta \in \mathbb{R}$ then $\alpha < \beta$ iff $\alpha \subset \beta$. (Similarly for $>$, $\leq$, and $\geq$.)

We now have enough to prove:

Theorem

If $A \subset \mathbb{R}$, $A \neq \varnothing$, and A is bounded above, then A has a least upper bound.

We also need to define $+$, $\cdot$, 1 and α^{-1}.

Then we can prove that $\mathbb{R}$ *is a complete ordered field* and, moreover, it is the *unique* such field (up to isomorphism).

Course Evaluations

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Next time. . .

An alternative construction of $\mathbb{R}$. . .

And much much more. . .

Surreal numbers. . .

. . . Guest lecture by Dr. Jonathan Dushoff