McMaster University

Mathematics and Statistics

$$\int_M d\omega = \int_{\partial M} \omega$$

Mathematics 3A03
Real Analysis I

Instructor: David Earn

Lecture 27
Integration
Monday 13 November 2017

Announcements

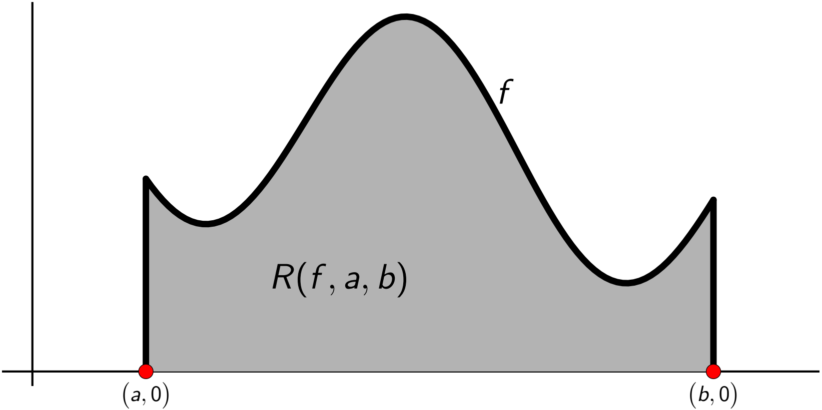
- Assignment 5 has been posted on the course wiki. The assignment is due on Monday 20 Nov 2017 at 2:25pm (remember cover sheet!)
    - Problem 1 was slightly revised on Saturday afternoon.
    - Problem 2 was slightly revised at 2:10pm today.

Last time. . .

- Proved Mean Value Theorem.
- Proved Darboux's Theorem.
- Additional slides have been added to Lecture 26: Inverse Function Theorem.

Integration

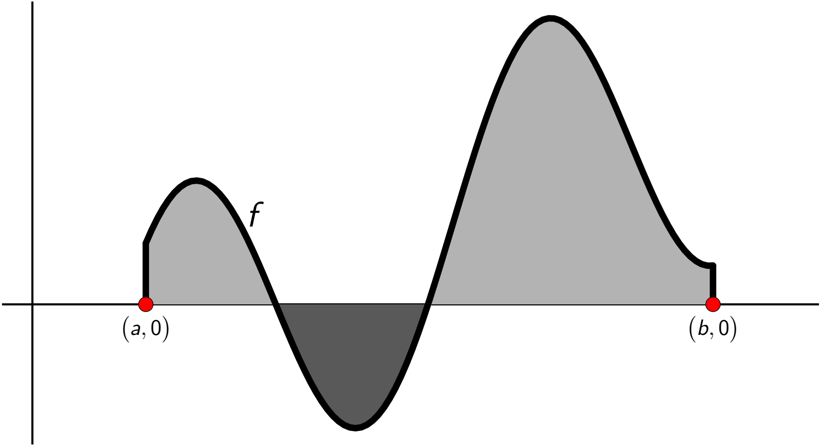
Integration



- "Area of region $R(f, a, b)$" is actually a very subtle concept.
- We will only scratch the surface of it.
- Textbook presentation of integral is different (but equivalent).

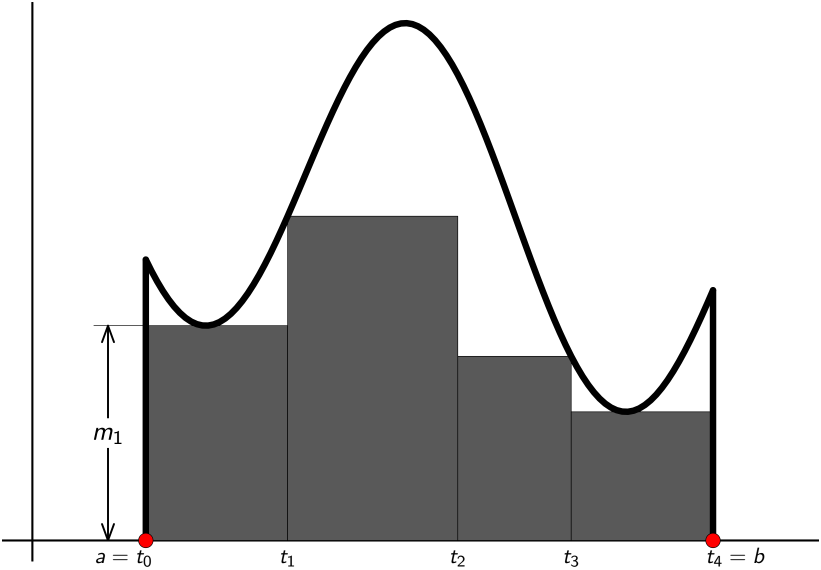
Our treatment is closer to that in M. Spivak "Calculus" (2008).

Integration

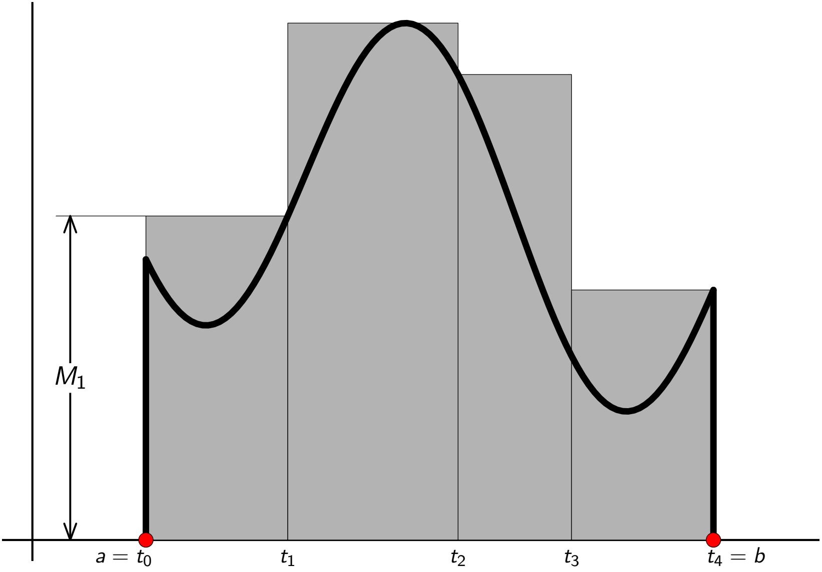


- Contribution to "area of $R(f, a, b)$" is positive or negative depending on whether f is positive or negative.

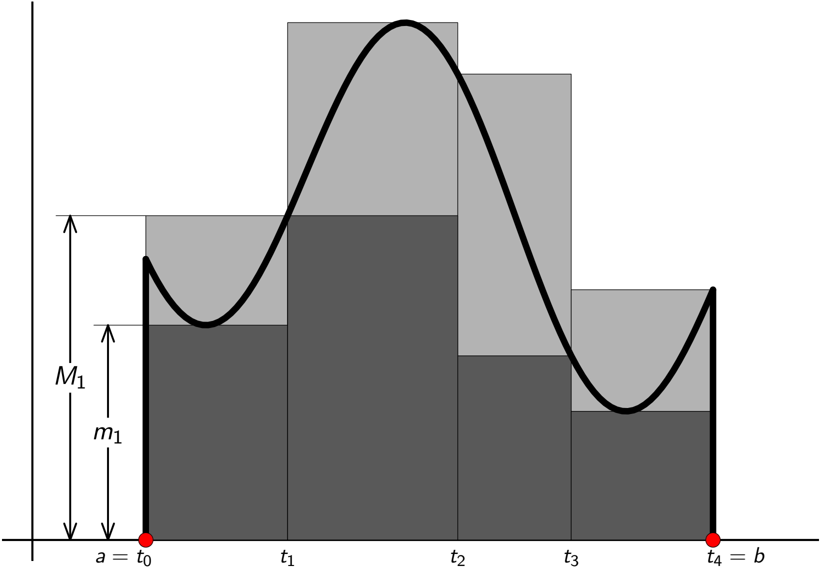
Lower sum



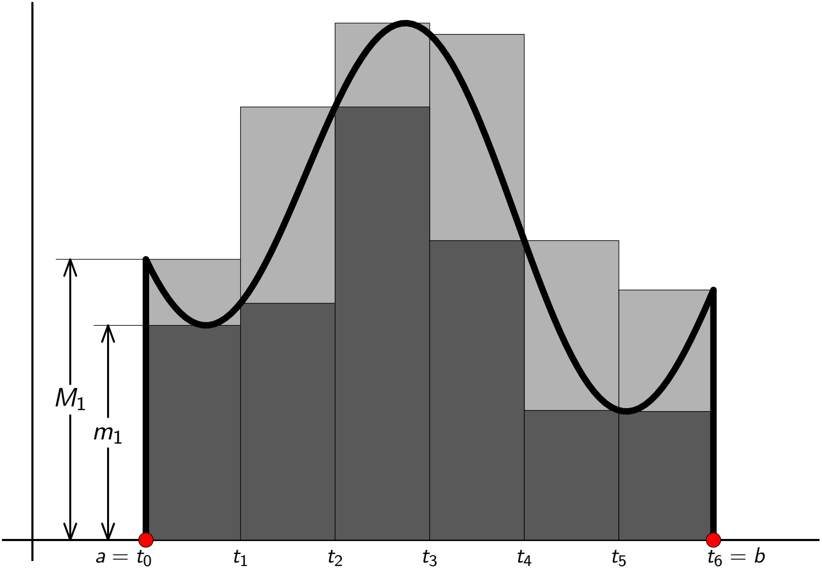
Upper sum



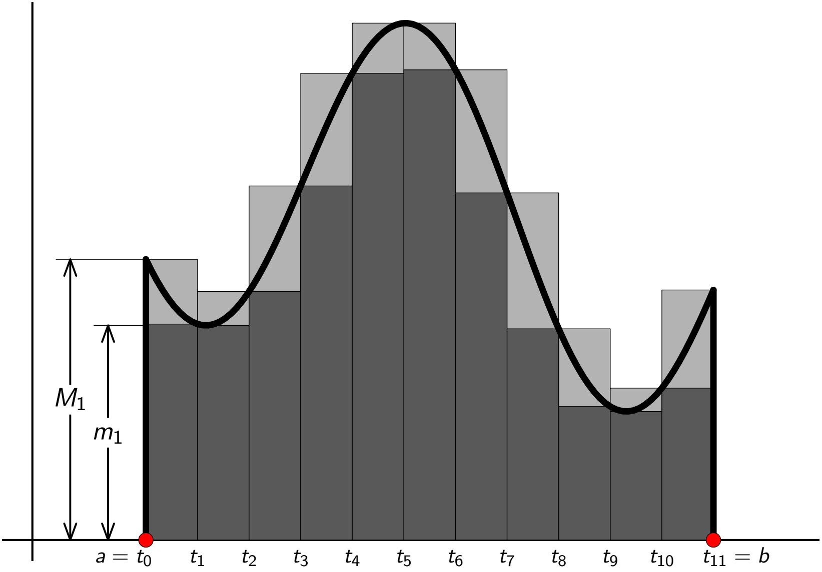
Lower and upper sums



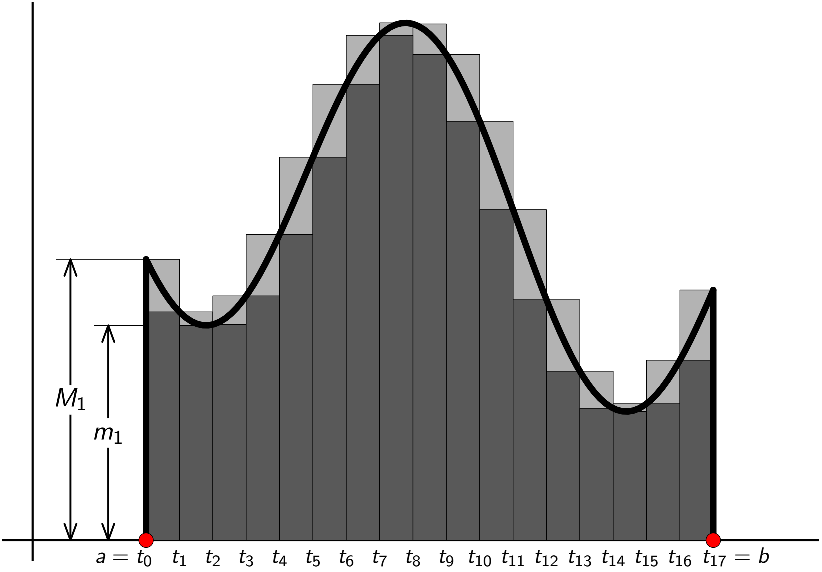
Lower and upper sums



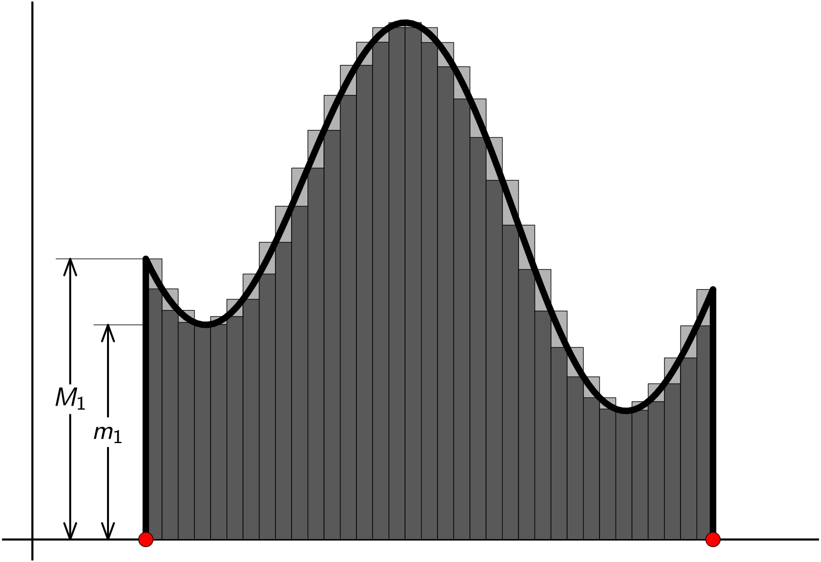
Lower and upper sums



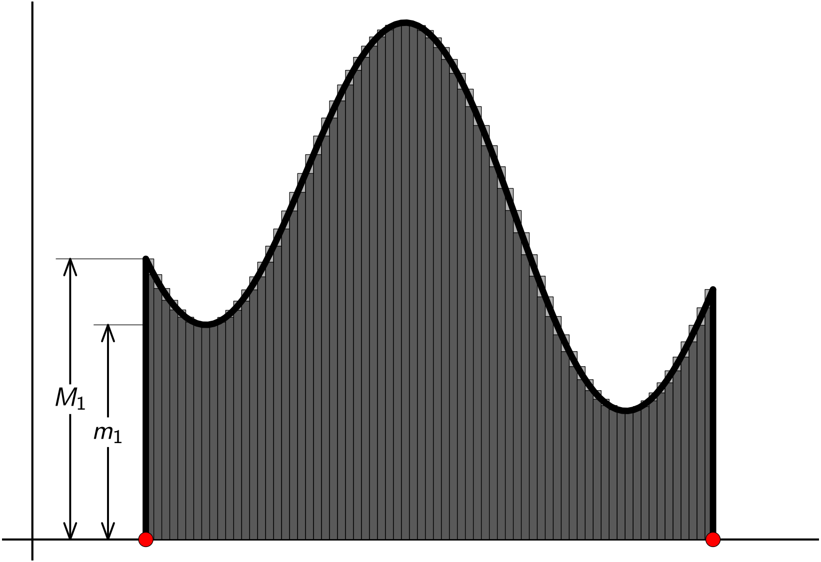
Lower and upper sums



Lower and upper sums



Lower and upper sums



Rigorous development of the integral

Definition (Partition)

Let $a < b$. A **partition** of the interval $[a, b]$ is a finite collection of points in $[a, b]$, one of which is a, and one of which is b.

We normally label the points in a partition

$$a = t_0 < t_1 < \cdots < t_{n-1} < t_n = b\,,$$

so the ith subinterval in the partition is

$$[t_{i-1}, t_i]\,.$$

Rigorous development of the integral

Definition (Lower and upper sums)

Suppose f is bounded on $[a, b]$ and $P = \{t_0, \ldots, t_n\}$ is a partition of $[a, b]$. Let

$$m_i = \inf \{ f(x) : x \in [t_{i-1}, t_i] \} ,$$

$$M_i = \sup \{ f(x) : x \in [t_{i-1}, t_i] \} .$$

The lower sum of f for P, denoted by $L(f, P)$, is defined as

$$L(f, P) = \sum_{i=1}^{n} m_i(t_i - t_{i-1}) .$$

The upper sum of f for P, denoted by $U(f, P)$, is defined as

$$U(f, P) = \sum_{i=1}^{n} M_i(t_i - t_{i-1}) .$$

Rigorous development of the integral

Relationship between motivating sketch and rigorous definition of lower and upper sums:

- The lower and upper sums correspond to the total areas of rectangles lying below and above the graph of f in our motivating sketch.
- However, these sums have been defined precisely <u>without any appeal</u> to a concept of "area".
- The requirement that f be bounded on $[a, b]$ is <u>essential</u> in order that all the m_i and M_i be well-defined.
- It is also <u>essential</u> that the m_i and M_i be defined as inf's and sup's (rather than maxima and minima) because f was <u>not</u> assumed continuous.

Rigorous development of the integral

Relationship between motivating sketch and rigorous definition of lower and upper sums:

- Since $m_i \le M_i$ for each i, we have

$$m_i(t_i - t_{i-1}) \le M_i(t_i - t_{i-1})\,. \qquad i = 1, \dots, n.$$

$\therefore$ For <u>any</u> partition P of $[a, b]$ we have

$$L(f, P) \le U(f, P),$$

because

$$L(f, P) = \sum_{i=1}^{n} m_i(t_i - t_{i-1})\,,$$

$$U(f, P) = \sum_{i=1}^{n} M_i(t_i - t_{i-1})\,.$$

Rigorous development of the integral

Relationship between motivating sketch and rigorous definition of lower and upper sums:

- More generally, if P_1 and P_2 are any two partitions of $[a, b]$, it *ought* to be true that

$$L(f, P_1) \leq U(f, P_2),$$

because $L(f, P_1)$ should be $\leq$ area of $R(f, a, b)$, and $U(f, P_2)$ should be $\geq$ area of $R(f, a, b)$.

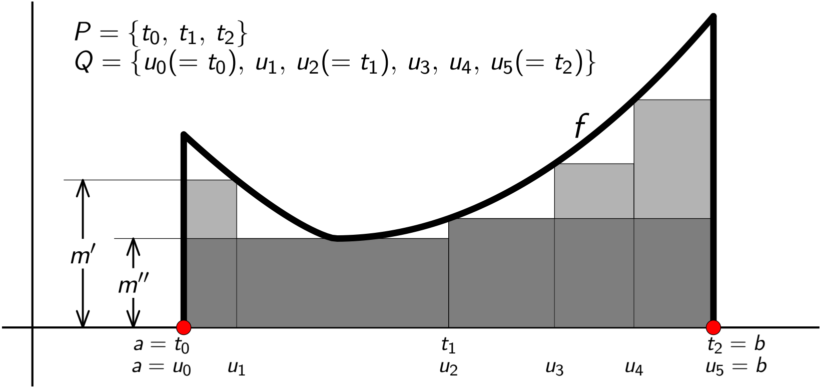
- But "ought to" and "should be" prove nothing, especially since we haven't yet even defined "area of $R(f, a, b)$".

- Before we can define "area of $R(f, a, b)$", we need to prove that $L(f, P_1) \leq U(f, P_2)$ for any partitions P_1, P_2 ...

Rigorous development of the integral

Lemma

If partition $P \subseteq$ partition Q (i.e., if every point of P is also in Q), then $L(f,P) \leq L(f,Q)$ *and* $U(f,P) \geq U(f,Q)$.



Rigorous development of the integral

Proof of Lemma

As a first step, consider the special case in which the finer partition Q contains only one more point than P:

$$\begin{aligned} P &= \{t_0, \ldots, t_n\}\,, \\ Q &= \{t_0, \ldots, t_{k-1}, u, t_k, \ldots, t_n\}\,, \end{aligned}$$

where

$$a = t_0 < t_1 < \cdots < t_{k-1} < u < t_k < \cdots < t_n = b\,.$$

Let

$$\begin{aligned} m' &= \inf\left\{\, f(x) \;:\; x \in [t_{k-1}, u] \,\right\}, \\ m'' &= \inf\left\{\, f(x) \;:\; x \in [u, t_k] \,\right\}. \end{aligned}$$

. . . continued. . .

Rigorous development of the integral

Proof of Lemma (cont.)

Then $$L(f,P)=\sum_{i=1}^{n} m_i(t_i-t_{i-1})\,,$$

and $$\begin{aligned} L(f,Q)=\sum_{i=1}^{k-1} m_i(t_i-t_{i-1}) &+ m'(u-t_{k-1}) \\ &+ m''(t_k-u)+\sum_{i=k+1}^{n} m_i(t_i-t_{i-1})\,. \end{aligned}$$

$\therefore$ To prove $L(f,P)\le L(f,Q)$, it is enough to show

$$m_k(t_k-t_{k-1})\le m'(u-t_{k-1})+m''(t_k-u)\,.$$

. . . continued. . .

Rigorous development of the integral

Proof of Lemma (cont.)

Now note that since

$$\{\, f(x) \,:\, x \in [t_{k-1}, u] \,\} \quad \subseteq \quad \{\, f(x) \,:\, x \in [t_{k-1}, t_k] \,\},$$

the RHS might contain some additional *smaller* numbers, so we must have

$$\begin{aligned} m_k &= \inf \{\, f(x) \,:\, x \in [t_{k-1}, t_k] \,\} \\ &\le \inf \{\, f(x) \,:\, x \in [t_{k-1}, u] \,\} \quad = \quad m' \,. \end{aligned}$$

Thus, $m_k \le m'$, and, similarly, $m_k \le m''$.

$$\begin{aligned} \therefore \quad m_k(t_k - t_{k-1}) &= m_k(t_k - u + u - t_{k-1}) \\ &= m_k(u - t_{k-1}) + m_k(t_k - u) \\ &\le m'(u - t_{k-1}) + m''(t_k - u)\,, \end{aligned}$$

. . . continued. . .

Rigorous development of the integral

Proof of Lemma (cont.)

which proves (in this special case where Q contains only one more point than P) that $L(f,P) \leq L(f,Q)$.

We can now prove the general case by adding one point at a time.

If Q contains ℓ more points than P, define a sequence of partitions

$$P = P_0 \subset P_1 \subset \cdots \subset P_\ell = Q$$

such that P_{j+1} contains exactly one more point that P_j. Then

$$L(f,P) = L(f,P_0) \leq L(f,P_1) \leq \cdots \leq L(f,P_\ell) = L(f,Q)\,,$$

so $L(f,P) \leq L(f,Q)$.

(Proving $U(f,P) \geq U(f,Q)$ is similar: check!) $\square$

Rigorous development of the integral

Theorem (Partition Theorem)

Let P_1 and P_2 be any two partitions of $[a, b]$. If f is bounded on $[a, b]$ then

$$L(f, P_1) \le U(f, P_2)\,.$$

Proof.

This is a straightforward consequence of the partition lemma.

Let $P = P_1 \cup P_2$, *i.e.*, the partition obtained by combining all the points of P_1 and P_2.

Then

$$L(f, P_1) \le L(f, P) \le U(f, P) \le U(f, P_2)\,.$$

□

Mathematics
and Statistics

$$\int_M d\omega = \int_{\partial M} \omega$$

Mathematics 3A03
Real Analysis I

Instructor: David Earn

Lecture 28
Integration II
Wednesday 15 November 2017

Announcements

- Assignment 5 is due on Monday 20 Nov 2017 at 2:25pm (remember cover sheet!)
  - Assignment was slightly revised at 2:10pm on Monday.

Rigorous development of the integral

Important inferences that follow from the partition theorem:

- For <u>any</u> partition P', the upper sum $U(f,P')$ is an upper bound for the set of <u>all</u> lower sums $L(f,P)$.

  $$\therefore \quad \sup\left\{L(f,P) : P \text{ a partition of } [a,b]\right\} \le U(f,P') \qquad \forall P'$$

  $$\therefore \quad \sup\left\{L(f,P)\right\} \le \inf\left\{U(f,P)\right\}$$

  $\therefore$ For <u>any</u> partition P',

  $$L(f,P') \le \sup\left\{L(f,P)\right\} \le \inf\left\{U(f,P)\right\} \le U(f,P')$$

- If $\sup\left\{L(f,P)\right\} = \inf\left\{U(f,P)\right\}$ then we can define "area of $R(f,a,b)$" to be this number.
  - Is it possible that $\sup\left\{L(f,P)\right\} < \inf\left\{U(f,P)\right\}$?

Rigorous development of the integral

Example

$\exists ?\ f : [a,b] \to \mathbb{R}$ such that $\sup\left\{L(f,P)\right\} < \inf\left\{U(f,P)\right\}$

Let

$$f(x) = \begin{cases} 1 & x \in \mathbb{Q} \cap [a,b], \\ 0 & x \in \mathbb{Q}^{\mathrm{c}} \cap [a,b]. \end{cases}$$

If $P = \{t_0, \ldots, t_n\}$ then $m_i = 0 \qquad (\because \quad [t_{i-1}, t_i] \cap \mathbb{Q}^{\mathrm{c}} \neq \varnothing)$,
and $M_i = 1 \qquad (\because \quad [t_{i-1}, t_i] \cap \mathbb{Q} \neq \varnothing)$.

$\therefore \quad L(f,P) = 0$ and $U(f,P) = b - a$ for <u>any</u> partition P.

$\therefore \quad \sup\left\{L(f,P)\right\} = 0 < b - a = \inf\left\{U(f,P)\right\}$. □

Can we define "area of $R(f,a,b)$" for such a weird function?
Yes, but not in this course!

Rigorous development of the integral

Definition (Integrable)

A function $f : [a, b] \to \mathbb{R}$ is said to be **integrable** on $[a, b]$ if it is bounded on $[a, b]$ and

$$\sup\left\{L(f,P) : P \text{ a partition of } [a,b]\right\} = \inf\left\{U(f,P) : P \text{ a partition of } [a,b]\right\}.$$

In this case, this common number is called the **integral** of f on $[a, b]$ and is denoted

$$\int_a^b f$$

Note: If f is integrable then for any partition P we have

$$L(f,P) \leq \int_a^b f \leq U(f,P),$$

and $\int_a^b f$ is the unique number with this property.

Rigorous development of the integral

- *Notation:*

$$\int_a^b f(x)\,dx \qquad \text{means precisely the same as} \qquad \int_a^b f\,.$$

  - The symbol "dx" has no meaning in isolation just as "$x \to$" has no meaning except in $\lim_{x\to a} f(x)$.

- It is not clear from the definition which functions are integrable.

- The definition of the integral does not itself indicate how to compute the integral of any given integrable function. So far, without a lot more effort we can't say much more than these two things:
  1. If $f(x) \equiv c$ then f is integrable on $[a, b]$ and $\int_a^b f = c \cdot (b - a)$.
  2. The weird example function is <u>not</u> integrable.

Rigorous development of the integral

- A function that is integrable according to our definition is usually said to be **Riemann integrable**, to distinguish this definition from other definitions of integrability.
- In Math 4A03 you will define "Lebesgue integrable", a more subtle concept that makes it possible to attach meaning to "area of $R(f, a, b)$" for the weird example function (among others), and to precisely characterize functions that are Riemann integrable.

Rigorous development of the integral

Theorem (Equivalent condition for integrability)

A bounded function $f : [a, b] \to \mathbb{R}$ is integrable on $[a, b]$ iff for all $\varepsilon > 0$ there is a partition P of $[a, b]$ such that

$$U(f, P) - L(f, P) < \varepsilon\,.$$

Proof.

2016 Assignment 5. □

Note: This theorem is just a restatement of the definition of integrability. It is often more convenient to work with $\varepsilon > 0$ than with sup's and inf's.

Integral theorems

Theorem

If f is continuous on $[a, b]$ then f is integrable on $[a, b]$.

Rough work to prepare for proof:

$$U(f, P) - L(f, P) = \sum_{i=1}^{n} (M_i - m_i)(t_i - t_{i-1})$$

Given $\varepsilon > 0$, choose a partition P that is so fine that $M_i - m_i < \varepsilon$ for all i. Then

$$U(f, P) - L(f, P) < \varepsilon \sum_{i=1}^{n} (t_i - t_{i-1}) = \varepsilon(b - a)\,.$$

Not quite what we want. So choose the partition P such that $M_i - m_i < \varepsilon/(b - a)$ for all i. To get that, choose P such that

$$|f(x) - f(y)| < \frac{\varepsilon}{2(b-a)} \qquad \text{if } |x - y| < \max_{1 \le i \le n} (t_i - t_{i-1}),$$

which we can do because f is uniformly continuous on $[a, b]$.

Integral theorems

Proof that continuous $\implies$ integrable

Since f is continuous on the compact set $[a, b]$, it is bounded on $[a, b]$ (which is the first requirement to be integrable on $[a, b]$).

Also, since f is continuous on the compact set $[a, b]$, it is uniformly continuous on $[a, b]$. $\therefore \forall \varepsilon > 0 \; \exists \delta > 0$ such that $\forall x, y \in [a, b]$,

$$|x - y| < \delta \quad \implies \quad |f(x) - f(y)| < \frac{\varepsilon}{2(b-a)} \, .$$

Now choose a partition of $[a, b]$ such that the length of each subinterval $[t_{i-1}, t_i]$ is less than δ, *i.e.*, $t_i - t_{i-1} < \delta$. Then, for any $x, y \in [t_{i-1}, t_i]$ we have $|x - y| < \delta$ and therefore

. . . continued. . .

Integral theorems

Proof that continuous $\implies$ integrable (cont.)

$$|f(x) - f(y)| < \frac{\varepsilon}{2(b-a)} \qquad \forall x, y \in [t_{i-1}, t_i]\,.$$

$$\therefore \qquad M_i - m_i \le \frac{\varepsilon}{2(b-a)} < \frac{\varepsilon}{b-a} \qquad i = 1, \ldots, n.$$

Since this is true for all i, it follows that

$$U(f,P) - L(f,P) = \sum_{i=1}^{n} (M_i - m_i)(t_i - t_{i-1})$$

$$< \frac{\varepsilon}{b-a} \sum_{i=1}^{n} (t_i - t_{i-1}) = \frac{\varepsilon}{b-a}(b-a) = \varepsilon\,.$$

□

Properties of the integral

Theorem (Integral segmentation)

Let $a < c < b$. If f is integrable on $[a, b]$, then f is integrable on $[a, c]$ and on $[c, b]$. Conversely, if f is integrable on $[a, c]$ and $[c, b]$ then f is integrable on $[a, b]$. Finally, if f is integrable on $[a, b]$ then

$$\int_a^b f = \int_a^c f + \int_c^b f \,. \qquad (\heartsuit)$$

(a good exercise)

This theorem motivates these definitions:

$$\int_a^a f = 0 \qquad \text{and} \qquad \int_a^b f = -\int_b^a f \quad \text{if } a > b.$$

Then $(\heartsuit)$ holds for any $a, b, c \in \mathbb{R}$.

McMaster University

Mathematics and Statistics

$$\int_M d\omega = \int_{\partial M} \omega$$

Mathematics 3A03
Real Analysis I

Instructor: David Earn

Lecture 29
Integration III
Friday 17 November 2017

Announcements

- Assignment 5 is due on Monday 20 Nov 2017 at 2:25pm (remember cover sheet!)
  - Assignment was slightly revised at 2:10pm on Monday.

Last time. . .

Rigorous development of integral:

- Definition: integrable.
- Example: non-integrable function.
- Theorem: Equivalent "ε-P" definition of integrable.
- Theorem: continuous $\implies$ integrable.
- Theorem: Integral segementation.

Properties of the integral

Theorem (Algebra of integrals – a.k.a. $\int_a^b$ is a linear operator)

If f and g are integrable on $[a, b]$ and $c \in \mathbb{R}$ then $f + g$ and cf are integrable on $[a, b]$ and

1. $\displaystyle\int_a^b (f+g) = \int_a^b f + \int_a^b g\,;$
2. $\displaystyle\int_a^b cf = c\int_a^b f\,.$

(proofs are relatively easy; good exercises)

Theorem (Integral of a product)

If f and g are integrable on $[a, b]$ then fg is integrable on $[a, b]$.

(proof is much harder; tough exercise)

Properties of the integral

Lemma (Integral bounds)

Suppose f is integrable on $[a, b]$. If $m \le f(x) \le M$ for all $x \in [a, b]$ then

$$m(b-a) \le \int_a^b f \le M(b-a)\,.$$

Proof.

For any partition P, we must have $m \le m_i \ \forall i$ and $M \ge M_i \ \forall i$.

$$\therefore \qquad m(b-a) \le L(f,P) \le U(f,P) \le M(b-a) \qquad \forall P$$

$$\therefore \quad m(b-a) \le \sup\{L(f,P)\} = \int_a^b f = \inf\{U(f,P)\} \le M(b-a)\,.$$

□

Properties of the integral

Theorem (Integrals are continuous)

If f is integrable on $[a,b]$ and F is defined on $[a,b]$ by

$$F(x) = \int_a^x f\,,$$

then F is continuous on $[a,b]$.

Proof

Let's first consider $x_0 \in [a,b)$ and show F is continuous from above at x_0, *i.e.*, $\lim_{x\to x_0^+} F(x) = F(x_0)$. If $x \in (x_0, b]$ then

$$(\heartsuit) \quad \Longrightarrow \quad F(x) - F(x_0) = \int_a^x f - \int_a^{x_0} f = \int_{x_0}^x f\,. \qquad (*)$$

...continued...

Properties of the integral

Proof (cont.)

Since f is integrable on $[a, b]$, it is bounded on $[a, b]$, so $\exists M > 0$ such that

$$-M \leq f(x) \leq M \qquad \forall x \in [a, b],$$

from which the integral bounds lemma implies

$$-M(x - x_0) \leq \int_{x_0}^{x} f \leq M(x - x_0)\,,$$

$$\therefore \quad (*) \implies -M(x - x_0) \leq F(x) - F(x_0) \leq M(x - x_0)\,.$$

$\therefore$ For any $\varepsilon > 0$ we can ensure $|F(x) - F(x_0)| < \varepsilon$ by requiring $0 \leq x - x_0 < \varepsilon/M$, which proves $\lim_{x \to x_0^+} F(x) = F(x_0)$.

A similar argument starting from $x_0 \in (a, b]$ and $x \in [a, x_0)$ yields $\lim_{x \to x_0^-} F(x) = F(x_0)$. Thus, "integrals are continuous". □

Fundamental Theorem of Calculus

Theorem (First Fundamental Theorem of Calculus)

Let f be integrable on $[a,b]$, and define F on $[a,b]$ by

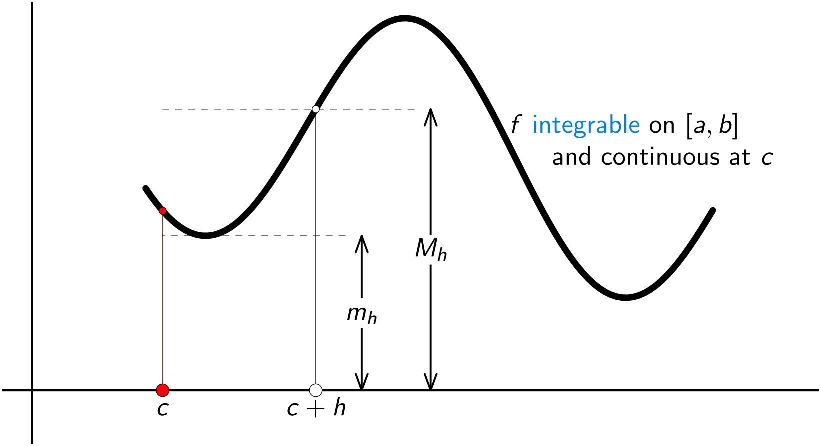
$$F(x) = \int_a^x f\,.$$

If f is continuous at $c \in [a,b]$, then F is differentiable at c, and

$$F'(c) = f(c)\,.$$

Note: If $c = a$ or b, then $F'(c)$ is understood to mean the right- or left-hand derivative of F.

Fundamental Theorem of Calculus



$$\lim_{h\to 0} f(c+h) = f(c) \quad \Longrightarrow \quad F'(c) = \lim_{h\to 0} \frac{F(c+h)-F(c)}{h} = f(c)$$

Fundamental Theorem of Calculus

Proof of First Fundamental Theorem of Calculus

Suppose $c \in [a, b)$, and $0 < h \le b - c$. Then the integral segmentation theorem implies

$$F(c+h) - F(c) = \int_c^{c+h} f\,.$$

Motivated by the sketch, define

$$m_h = \inf\left\{\, f(x) \,:\, x \in [c, c+h] \,\right\},$$
$$M_h = \sup\left\{\, f(x) \,:\, x \in [c, c+h] \,\right\}.$$

Then the integral bounds lemma implies

$$m_h \cdot h \le \int_c^{c+h} f \le M_h \cdot h\,,$$

...continued...

Fundamental Theorem of Calculus

Proof of First Fundamental Theorem of Calculus (cont.)

and hence

$$m_h \leq \frac{F(c+h) - F(c)}{h} \leq M_h \,.$$

This inequality is true for <u>any</u> integrable function. However, because f is continuous at c, we have

$$\lim_{h\to 0^+} m_h = \lim_{h\to 0^+} M_h = f(c)\,,$$

so the squeeze theorem implies

$$F'_+(c) = \lim_{h\to 0^+} \frac{F(c+h) - F(c)}{h} = f(c)\,.$$

A similar argument for $c \in (a,b]$ and $c - a \leq h < 0$ yields $F'_-(c) = f(c)$. □

Fundamental Theorem of Calculus

Corollary

If f is continuous on $[a,b]$ and $f = g'$ for some function g, then

$$\int_a^b f = g(b) - g(a)\,.$$

Proof.

Let $F(x) = \int_a^x f$. Then throughout $[a,b]$ we have $F' = f = g'$.

$\therefore$ $\exists c \in \mathbb{R}$ such that $F = g + c$ (2016 Assignment 5).

$\therefore$ $F(a) = g(a) + c$.

But $F(a) = \int_a^a f = 0$, so $c = -g(a)$.

$\therefore$ $F(x) = g(x) - g(a)$.

This is true, in particular, for $x = b$, so $\int_a^b f = g(b) - g(a)$. $\square$

Fundamental Theorem of Calculus

Theorem (Second Fundamental Theorem of Calculus)

If f is integrable on $[a, b]$ and $f = g'$ for some function g, then

$$\int_a^b f = g(b) - g(a)\,.$$

Notes:

- This looks like the corollary to the first fundamental theorem, except that f is only assumed integrable, <u>not</u> continuous.
- Recall from Darboux's theorem that if $f = g'$ for some g then f has the intermediate value property, but f need not be continuous.
- The proof of the second fundamental theorem is completely different from the corollary to the first, because we cannot use the first fundamental theorem (which assumed f is continuous).

McMaster University

Mathematics and Statistics

$$\int_M d\omega = \int_{\partial M} \omega$$

Mathematics 3A03
Real Analysis I

Instructor: David Earn

Lecture 30

Integration IV

Monday 20 November 2017

Please consider. . .

5 minute *Student Respiratory Illness Survey:*

https://surveys.mcmaster.ca/limesurvey2/index.php/893454

> *Please complete this anonymous survey to help us monitor the patterns of respiratory illness, over-the-counter drug use, and social contact within the McMaster community. There are no risks to filling out this survey, and your participation is voluntary. You do not need to answer any questions that make you uncomfortable, and all information provided will be kept strictly confidential. Thanks for participating.*
>
> *–Dr. Marek Smieja (Infectious Diseases)*

Announcements

- Assignment 5 was due at 2:25pm today
- Assignment 6 **will be due @ 2:25pm on Mon 4 Dec 2017.**
- Math 3A03 **Test #2**: **ONE WEEK FROM TODAY!**
  Mon 27 Nov 2017, 7:00–8:30pm in MDCL 1102
- Math 3A03 **Final Exam**: Thurs 21 Dec 2017, 4:00–6:30pm,
  Location: IWC 1

Last time...

Rigorous development of integral:

- Algebra of integrals.
- Integral bounds lemma.
- Integrals are continuous.
- First Fundamental Theorem of Calculus.
- Second Fundamental Theorem of Calculus.

Fundamental Theorem of Calculus

Proof of Second Fundamental Theorem of Calculus

Let $P = \{t_0, \dots, t_n\}$ be any partition of $[a, b]$. By the Mean Value Theorem, for each $i = 1, \dots, n$, $\exists\, x_i \in [t_{i-1}, t_i]$ such that

$$g(t_i) - g(t_{i-1}) = g'(x_i)(t_i - t_{i-1}) = f(x_i)(t_i - t_{i-1})\,.$$

Define m_i and M_i as usual. Then $m_i \le f(x_i) \le M_i\ \forall i$, so

$$m_i(t_i - t_{i-1}) \le f(x_i)(t_i - t_{i-1}) \le M_i(t_i - t_{i-1})\,,$$

$$\text{i.e.,} \quad m_i(t_i - t_{i-1}) \le g(t_i) - g(t_{i-1}) \le M_i(t_i - t_{i-1})\,.$$

$$\therefore \quad \sum_{i=1}^{n} m_i(t_i - t_{i-1}) \le \sum_{i=1}^{n} \Big(g(t_i) - g(t_{i-1})\Big) \le \sum_{i=1}^{n} M_i(t_i - t_{i-1})$$

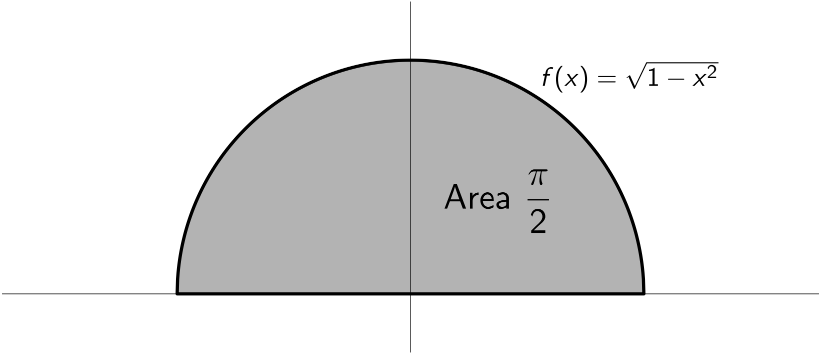
$$\text{i.e.,} \quad L(f,P) \le \quad g(b) - g(a) \quad \le U(f,P)$$

for <u>any</u> partition P. $\quad \therefore \quad g(b) - g(a) = \int_a^b f\,.$ □

What useful things can we do with integrals?

- Compute areas of complicated shapes: find anti-derivatives and use the second fundamental theorem of calculus.
- Define trigonometric functions (rigorously).
- Define logarithm and exponential functions (rigorously).

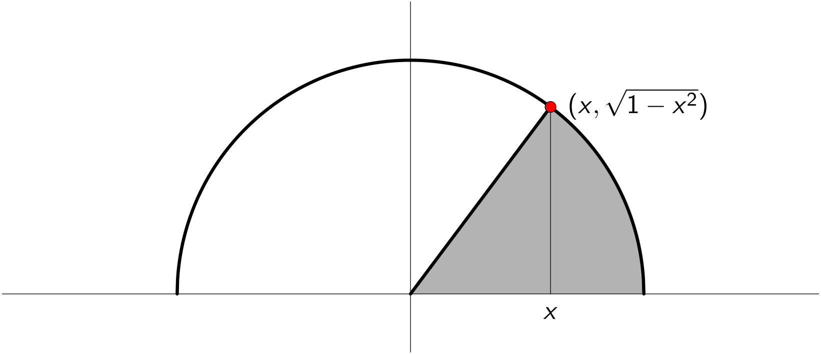
What is π ?



Definition

$$\pi \equiv 2\int_{-1}^{1} \sqrt{1-x^2}\, dx\,.$$

What are cos and sin ?

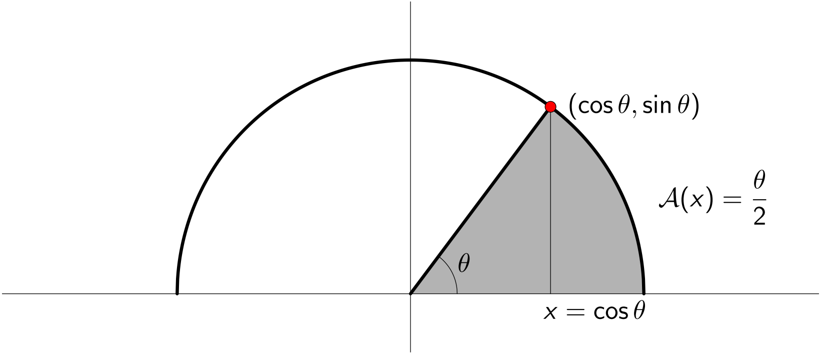


Definition (Sectoral area)

If $x \in [-1,1]$ then $\quad \mathcal{A}(x) = \dfrac{x\sqrt{1-x^2}}{2} + \displaystyle\int_x^1 \sqrt{1-t^2}\,dt\,.$

Note: $\mathcal{A}(-1) = \pi/2, \quad \mathcal{A}(1) = 0.$

What are cos and sin ?



Length of circular arc swept out by angle θ: θ

Area of sectoral region swept out by angle θ: $\theta/2$

So, if $\theta \in [0, \pi]$ then we define $\cos\theta$ to be the unique number in $[-1, 1]$ such that $\mathcal{A}(\cos\theta) = \theta/2$, and we define $\sin\theta$ to be $\sqrt{1 - (\cos\theta)^2}$.

We must prove: given $x \in [0, \pi]$ $\exists!\, y \in [-1, 1]$ such that $\mathcal{A}(y) = x/2$.

What are cos and sin ?

Proof that $\forall\, x \in [0,\pi]\ \exists!\, y \in [-1,1]$ *such that* $\mathcal{A}(y) = x/2$*:*

Existence: $\mathcal{A}(1) = 0$, $\mathcal{A}(-1) = \pi/2$, and $\mathcal{A}$ is continuous. Hence by the intermediate value theorem $\exists\, y \in [-1,1]$ such that $\mathcal{A}(y) = x/2$.

Uniqueness: $\mathcal{A}$ is differentiable on $(-1,1)$ and $\mathcal{A}'(x) < 0$ on $(-1,1)$.
$\therefore$ On $(-1,1)$, $\mathcal{A}$ is decreasing, and hence one-to-one.

Definition (cos and sin)

If $x \in [0,\pi]$ then $\cos x$ is the unique number in $[-1,1]$ such that $\mathcal{A}(\cos x) = x/2$, and $\sin x = \sqrt{1-(\cos x)^2}$.

These definitions are easily extended to all of $\mathbb{R}$:

- For $x \in [\pi, 2\pi]$, define $\cos x = \cos(2\pi - x)$ and $\sin x = -\sin(2\pi - x)$.
- Then, for $x \in \mathbb{R} \setminus [0,2\pi]$ define $\cos x = \cos(x \mod 2\pi)$ and $\sin x = \sin(x \mod 2\pi)$.

Trigonometric theorems

Given the rigorous definition of cos and sin, we can prove:

1. cos and sin are differentiable on $\mathbb{R}$. Moreover, $\cos' = -\sin$ and $\sin' = \cos$.
2. sec, tan, csc and cot can all be defined in the usual way and have all the usual properties.
3. The inverse function theorem allows us to define and compute the derivatives of all the inverse trigonometric functions.
4. If f is twice differentiable on $\mathbb{R}$, $f'' + f = 0$, $f(0) = a$ and $f'(0) = b$, then $f = a\cos + b\sin$.
5. For all $x, y \in \mathbb{R}$,

$$\sin(x+y) = \sin x \cos y + \cos x \sin y\,,$$
$$\cos(x+y) = \cos x \cos y - \sin x \sin y\,.$$

Something deep that you know enough to prove

Extra Challenge Problem:

Prove that π is irrational.

What are log and exp ?

Consider the function

$$f(x) = 10^x \,.$$

What exactly is this function?

In our mathematically naïve previous life, we just assumed that $f(x)$ is well-defined $\forall\, x \in \mathbb{R}$, and that f has a well-defined inverse function,

$$f^{-1}(x) = \log_{10}(x)\,.$$

But how are 10^x and $\log_{10}(x)$ defined for irrational x ?

Let's review what we know. . .

What are log and exp ?

$$n \in \mathbb{N} \quad \Longrightarrow \quad 10^n = \underbrace{10 \cdots 10}_{n \text{ times}}$$

$$n, m \in \mathbb{N} \quad \Longrightarrow \quad 10^n \cdot 10^m = 10^{n+m}$$

When we extend 10^x to $x \in \mathbb{Q}$, we want this product rule to be preserved:

$$10^0 \cdot 10^n = 10^{0+n} = 10^n \quad \Longrightarrow \quad 10^0 = 1$$

$$10^{-n} \cdot 10^n = 10^0 = 1 \quad \Longrightarrow \quad 10^{-n} = \frac{1}{10^n}$$

$$\underbrace{10^{1/n} \cdots 10^{1/n}}_{n \text{ times}} = 10^{\overbrace{}^{}\underbrace{1/n \cdots 1/n}_{n \text{ times}}} = 10^1 = 10 \quad \Longrightarrow \quad 10^{1/n} = \sqrt[n]{10}$$

McMaster University

Mathematics and Statistics

$$\int_M d\omega = \int_{\partial M} \omega$$

Mathematics 3A03
Real Analysis I

Instructor: David Earn

Lecture 31
Integration V
Wednesday 22 November 2017

Please consider. . .

5 minute *Student Respiratory Illness Survey:*

https://surveys.mcmaster.ca/limesurvey2/index.php/893454

> *Please complete this anonymous survey to help us monitor the patterns of respiratory illness, over-the-counter drug use, and social contact within the McMaster community. There are no risks to filling out this survey, and your participation is voluntary. You do not need to answer any questions that make you uncomfortable, and all information provided will be kept strictly confidential. Thanks for participating.*
>
> *–Dr. Marek Smieja (Infectious Diseases)*

Announcements

- Assignment 5 solutions posted.
- Assignment 6 **due @ 2:25pm on Mon 4 Dec 2017.** (Incomplete version) has been posted.
- Math 3A03 **Test #2**: **ONE WEEK FROM TODAY!** Mon 27 Nov 2017, 7:00–8:30pm in MDCL 1102
- Math 3A03 **Final Exam**: Thurs 21 Dec 2017, 4:00–6:30pm, **Location:** IWC 1

Last time. . .

- Second Fundamental Theorem of Calculus.
- Rigorous definition of trig functions.
- Working towards rigorous definition of 10^x for $x \in \mathbb{R}$.

What are log and exp ?

Finally,

$$\underbrace{10^{1/n}\cdots 10^{1/n}}_{m \text{ times}} = 10^{\overbrace{}^{}\underbrace{1/n\cdots 1/n}_{m \text{ times}}} = 10^{m/n} \quad\Longrightarrow\quad 10^{m/n} = \left(\sqrt[n]{10}\right)^m$$

Now we're stuck. *How do we extend this scheme to irrational x?*

We need a more sophisticated idea.

Let's try to find a function on all of $\mathbb{R}$ that satisfies

$$f(x+y) = f(x)\cdot f(y)\,, \qquad \forall x,y\in\mathbb{R},$$
$$\text{and} \qquad f(1) = 10\,.$$

It then follows that $f(0) = 1$ and, $\forall\, x \in \mathbb{Q}$, $f(x) = [f(1)]^x$.

What additional properties can we impose on $f(x)$ that will lead us to a sensible definition of $f(x)$ for all $x \in \mathbb{R}$?

What are log and exp ?

One approach is to insist that f is *differentiable*.

Then we can compute

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{f(x)\cdot f(h) - f(x)}{h}$$
$$= f(x)\cdot \lim_{h\to 0} \frac{f(h) - 1}{h} \quad = \quad f(x)\cdot f'(0) \quad \equiv \quad \alpha\, f(x)$$

So $f'(x) = \alpha\, f(x)$, *i.e.*, we have f' in terms of unknowns f and α.
So what?!?

Let's look at the inverse function, f^{-1} (think "$\log_{10}$"):

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))} = \frac{1}{\alpha\, f(f^{-1}(x))} = \frac{1}{\alpha\, x}$$

Holy $\#@%! We have a simple formula for the derivative of f^{-1}!

What are log and exp ?

Since we want $\log_{10} 1 = 0$, we should define $\log_{10} x$ as $(1/\alpha) \int_1^x t^{-1}\,dt$. *Great idea, but we don't know what α is.*

So, let's ignore α ...

(and hope that what we end up with is log to some "natural" base).

Definition (Logarithm function)

If $x > 0$ then

$$\log x = \int_1^x \frac{1}{t}\,dt\,.$$

This function is strictly increasing ($\log'(x) > 0$ for all $x > 0$) so we can now define:

Definition (Exponential function)

$$\exp = \log^{-1}\,.$$

What are log and exp ?

With these rigorous defintions of log and exp, we can prove the following as theorems:

1. If $x, y > 0$ then $\log(xy) = \log x + \log y$.
2. If $x, y > 0$ then $\log(x/y) = \log x - \log y$.
3. If $n \in \mathbb{N}$ and $x > 0$ then $\log(x^n) = n \log x$.
4. For all $x \in \mathbb{R}$, $\exp'(x) = \exp(x)$.
5. For all $x, y \in \mathbb{R}$, $\exp(x + y) = \exp(x) \cdot \exp(y)$.
6. For all $x \in \mathbb{Q}$, $\exp(x) = [\exp(1)]^x$.

The last theorem above motivates:

Definition

$$\begin{aligned} e &= \exp(1)\,, \\ e^x &= \exp(x) \qquad \text{for all } x \in \mathbb{R}. \end{aligned}$$

What are log and exp ?

We can now give a rigorous definition of 10^x for any $x \in \mathbb{R}$.
In fact, we can do this for any $a > 0$.

Definition (a^x)

If $a > 0$ and x is any real number then

$$a^x = e^{x \log a} .$$

We then have the following theorems for any $a > 0$:

1. $(a^x)^y = a^{xy}$ for all $x, y \in \mathbb{R}$;
2. $a^0 = 1; \quad a^1 = a$;
3. $a^{x+y} = a^x \cdot a^y$ for all $x, y \in \mathbb{R}$;
4. $a^{-x} = 1/a^x$ for all $x \in \mathbb{R}$;
5. if $a > 1$ then a^x is increasing on $\mathbb{R}$;
6. if $0 < a < 1$ then a^x is decreasing on $\mathbb{R}$.

Using the integral to define useful functions rigorously

- Just as we defined 10^x via the definition of $\log x = \int_1^x \frac{1}{t}\,dt$, we could have defined the trigonometric functions starting from

$$\arcsin x = \int_0^x \frac{1}{\sqrt{1-t^2}}\,dt\,, \qquad -1 < x < 1,$$

  rather than the more complicated definition of cos via $\mathcal{A}(x)$. Many common functions are defined as integrals of rational functions of square roots.

- Any compositions of trig functions, log, exp, rational functions and radicals, are called **elementary functions**.

- Most functions that turn up a lot in applications can be defined rigorously via integrals of elementary functions. Such functions are collectively called **special functions**.

Approximation by Polynomial Functions

Definition (Taylor polynomial)

If f is n times differentiable at a then the **Taylor polynomial of degree n for f at a** is

$$P_{n,a}(x) = f(a) + f'(a)(x-a) + \cdots + \frac{f^{(n)(a)}}{n!}(x-a)^n \,.$$

Theorem (Taylor's theorem)

Suppose $f', \ldots, f^{(n+1)}$ are defined on $[a,x]$, and that $R_{n,a}(x)$ is defined by $f(x) = P_{n,a}(x) + R_{n,a}(x)$. *Then*

$$R_{n,a}(x) = \frac{f^{(n+1)}(t)}{(n+1)!}(x-a)^{n+1} \,, \qquad \textit{for some } t \in (a,x)\,.$$

Note: The form of the remainder term here is known as the **Lagrange form** of the remainder.

Approximation by Polynomial Functions

Example (Approximating e)

Use Taylor's theorem to show that e can be approximated to within $\frac{3}{(n+1)!}$ for any given n. Also show that $2 < e < 3$.

Recall $e = e^1 = \exp(1)$. $\therefore\ e = 1 + 1 + \frac{1}{2!} + \cdots + \frac{1}{n!} + R_n$, where $R_n = \frac{e^t}{(n+1)!}$ for some $t \in (0,1)$. Since e^x is increasing on $(0,1)$, we must have $e^t < e$, so $\frac{1}{(n+1)!} < R_n < \frac{e}{(n+1)!}$. But we can't estimate e using e.

Recall $1 = \log e = \int_1^e \frac{1}{t}\,dt$, and note $\log 4 = \int_1^4 \frac{1}{t}\,dt > 1$, since $\frac{1}{2}(2-1) + \frac{1}{4}(4-2) = 1$ is a lower sum for $f(t) = 1/t$ on $[1,4]$.

$\therefore\ \log e < \log 4$, *i.e.*, $e < 4$. (*Similarly:* Use $\int_1^2 \frac{1}{t}\,dt$ to get $e > 2$.)

$\therefore\ 2 < e < 4$ and $R_n < \frac{4}{(n+1)!}$. Great, but what we actually want is $2 < e < 3$ and $R_n < \frac{3}{(n+1)!}$.

...continued...

Approximation by Polynomial Functions

Example (Approximating e (cont.))

Given $R_n < \frac{4}{(n+1)!}$, note that for $n = 4$ we have

$$0 < R_n < \frac{4}{5!} = \frac{1}{30},$$

so applying Taylor's theorem with $n = 4$ we get

$$e = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + R_n = \left(2 + \frac{17}{24}\right) + R_n$$
$$< \left(2 + \frac{17}{24}\right) + \frac{1}{30} < 3.$$

Thus $e < 3$, and consequently

$$R_n < \frac{e}{(n+1)!} \implies R_n < \frac{3}{(n+1)!}.$$

□

e is irrational

Theorem (e is irrational)

$\nexists\, k, m \in \mathbb{N}$ such that $e = k/m$.

Proof.

Suppose $e = k/m$ with $k, m \in \mathbb{N}$. Then, for any $n \in \mathbb{N}$, we have

$$\frac{k}{m} = e^1 = 1 + 1 + \frac{1}{2!} + \cdots + \frac{1}{n!} + R_n\,, \qquad 0 < R_n < \frac{3}{(n+1)!}\,.$$

$$\therefore \quad \frac{n!k}{m} = n! + n! + \frac{n!}{2!} + \cdots + \frac{n!}{n!} + n!R_n\,, \qquad n \in \mathbb{N}.$$

This is true, in particular, for $n > 3$ and $n > m$, in which case every term in this equation other than $n!R_n$ is an integer. So $n!R_n$ is also an integer! But $0 < R_n < 3/(n+1)!$, so since $n > 3$ we have

$$0 < n!R_n < \frac{3}{n+1} < \frac{3}{4} < 1,$$

which is impossible for an integer. □