Continuous Functions

McMaster University

Mathematics and Statistics

$$\int_M d\omega = \int_{\partial M} \omega$$

Mathematics 3A03
Real Analysis I

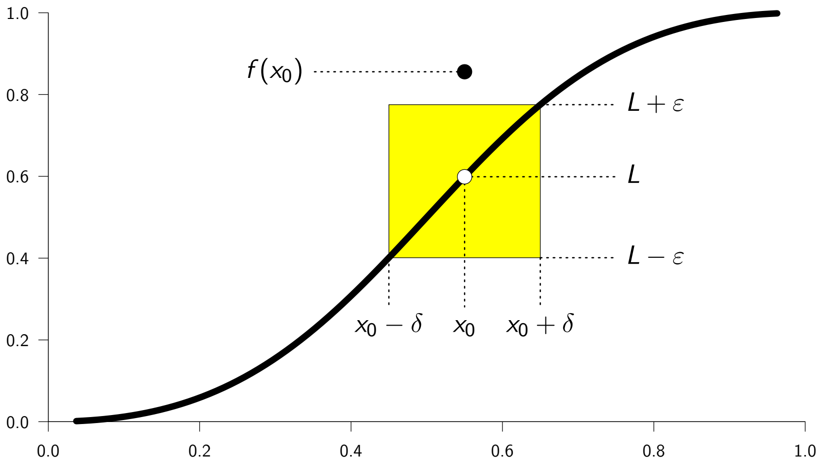
Instructor: David Earn

Lecture 19
Continuity
Wednesday 25 October 2017

Announcements

- A preliminary version of Assignment 4 has been posted on the course wiki. More problems will be added this weekend.
  The assignment is due on Friday 3 Nov 2017 at 4:25pm (remember cover sheet!)
- Today we will start a new topic, but first we'll finish the last compactness example that we discussed.

Limits of functions



Limits of functions

Definition (Limit of a function on an interval (a, b))

Let $a < x_0 < b$ and $f : (a, b) \to \mathbb{R}$. Then f is said to **approach the limit L as x approaches x_0**, often written "$f(x) \to L$ as $x \to x_0$" or

$$\lim_{x \to x_0} f(x) = L ,$$

iff for all $\varepsilon > 0$ there exists $\delta > 0$ such that if $0 < |x - x_0| < \delta$ then $|f(x) - L| < \varepsilon$.

Shorthand version:

$\forall \varepsilon > 0 \; \exists \delta > 0 \ni 0 < |x - x_0| < \delta \implies |f(x) - L| < \varepsilon.$

Limits of functions

The function f need not be defined on an entire interval.
It is enough for f to be defined on a set with at least one accumulation point.

Definition (Limit of a function with domain $E \subseteq \mathbb{R}$)

Let $E \subseteq \mathbb{R}$ and $f : E \to \mathbb{R}$. Suppose x_0 is a point of accumulation of E. Then f is said to **approach the limit L as x approaches x_0**, *i.e.*,

$$\lim_{x \to x_0} f(x) = L\,,$$

iff for all $\varepsilon > 0$ there exists $\delta > 0$ such that if $x \in E$, $x \neq x_0$, and $|x - x_0| < \delta$ then $|f(x) - L| < \varepsilon$.

Shorthand version:

$\forall \varepsilon > 0\ \exists \delta > 0 \ni \left(x \in E\ \wedge\ 0 < |x - x_0| < \delta\right) \implies |f(x) - L| < \varepsilon.$

Limits of functions

Example

Prove directly from the definition of a limit that

$$\lim_{x \to 3} (2x + 1) = 7 \,.$$

(solution on board)

Example

Prove directly from the definition of a limit that

$$\lim_{x \to 2} x^2 = 4 \,.$$

(solution on board)

Limits of functions

Rather than the ε-δ definition, we can exploit our experience with sequences to define "$f(x) \to L$ as $x \to x_0$".

Definition (Limit of a function via sequences)

Let $E \subseteq \mathbb{R}$ and $f : E \to \mathbb{R}$. Suppose x_0 is a point of accumulation of E. Then

$$\lim_{x \to x_0} f(x) = L$$

iff for every sequence $\{e_n\}$ of points in $E \setminus \{x_0\}$,

$$\lim_{n \to \infty} e_n = x_0 \quad \Longrightarrow \quad \lim_{n \to \infty} f(e_n) = L\,.$$

Limits of functions

Lemma (Equivalence of limit definitions)

The ε-δ definition of limits and the sequence definition of limits are equivalent.

(solution on board)

Note: The definition of a limit via sequences is sometimes easier to use than the ε-δ definition.

Proof of Equivalence of ε-δ definition and sequence definition of limit.

Proof (ε-$\delta \implies$ seq).

Suppose the ε-δ definition holds and $\{e_n\}$ is a sequence in $E \setminus \{x_0\}$ that converges to x_0. Given $\varepsilon > 0$, there exists $\delta > 0$ such that if $0 < |x - x_0| < \delta$ then $|f(x) - L| < \varepsilon$. But since $e_n \to x_0$, given $\delta > 0$, there exists $N \in \mathbb{N}$ such that, for all $n \geq N$, $|e_n - x_0| < \delta$. This means that if $n \geq N$ then $x = e_n$ satisfies $0 < |x - x_0| < \delta$, implying that we can put $x = e_n$ in the statement $|f(x) - L| < \varepsilon$. Hence, for all $n \geq N$, $|f(e_n) - L| < \varepsilon$. Thus,

$$e_n \to x_0 \implies f(e_n) \to L\,,$$

as required. □

Mathematics and Statistics

$$\int_M d\omega = \int_{\partial M} \omega$$

Mathematics 3A03
Real Analysis I

Instructor: David Earn

Lecture 20
Continuity II
Friday 27 October 2017

Announcements

- A preliminary version of Assignment 4 has been posted on the course wiki. More problems will be added this weekend. The assignment is due on Friday 3 Nov 2017 at 4:25pm (remember cover sheet!)
- Solutions to Test 1 have been posted on the course wiki. Read them!
- We hope to return the tests in tutorials on Monday and Tuesday. Make sure to attend your tutorial in order to pick up your test.

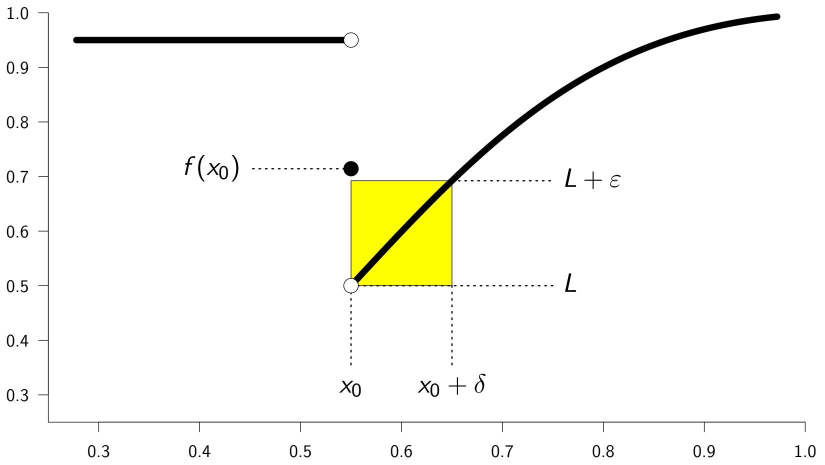
Proof of Equivalence of ε-δ definition and sequence definition of limit.

Proof (seq $\implies$ ε-δ).

Suppose that as $x \to x_0$, $f(x) \not\to L$ according to the ε-δ definition. We must show that $f(x) \not\to L$ according to the sequence definition.

Since the ε-δ criterion does not hold, $\exists \varepsilon > 0$ such that $\forall \delta > 0$ there is some $x_\delta \in E$ for which $0 < |x_\delta - x_0| < \delta$ and yet $|f(x_\delta) - L| \geq \varepsilon$. This is true, in particular, for $\delta = 1/n$, where n is any natural number. Thus, $\exists \varepsilon > 0$ such that: $\forall n \in \mathbb{N}$, there exists $x_n \in E$ such that $0 < |x_n - x_0| < 1/n$ and yet $|f(x_n) - L| \geq \varepsilon$. This demonstrates that there is a sequence $\{x_n\}$ in $E \setminus \{x_0\}$ for which $x_n \to x_0$ and yet $f(x_n) \not\to L$. Hence, $f(x) \not\to L$ as $x \to x_0$ according to the sequence criterion, as required. □

One-sided limits



One-sided limits

Definition (Right-Hand Limit)

Let $f : E \to \mathbb{R}$ be a function with domain E and suppose that x_0 is a point of accumulation of $E \cap (x_0, \infty)$. Then we write

$$\lim_{x \to x_0^+} f(x) = L$$

if for every $\varepsilon > 0$ there is a $\delta > 0$ so that

$$|f(x) - L| < \varepsilon$$

whenever $x_0 < x < x_0 + \delta$ and $x \in E$.

One-sided limits

One-sided limits can also be expressed in terms of sequence convergence.

Definition (Right-Hand Limit – sequence version)

Let $f : E \to \mathbb{R}$ be a function with domain E and suppose that x_0 is a point of accumulation of $E \cap (x_0, \infty)$. Then we write

$$\lim_{x \to x_0^+} f(x) = L$$

if for every decreasing sequence $\{e_n\}$ of points of E with $e_n > x_0$ and $e_n \to x_0$ as $n \to \infty$,

$$\lim_{n \to \infty} f(e_n) = L\,.$$

Infinite limits

Definition (Right-Hand Infinite Limit)

Let $f : E \to \mathbb{R}$ be a function with domain E and suppose that x_0 is a point of accumulation of $E \cap (x_0, \infty)$. Then we write

$$\lim_{x \to x_0^+} f(x) = \infty$$

if for every $M > 0$ there is a $\delta > 0$ such that $f(x) \geq M$ whenever $x_0 < x < x_0 + \delta$ and $x \in E$.

Properties of limits

There are theorems for limits of functions of a real variable that correspond (and have similar proofs) to the various results we proved for limits of sequences:

- Uniqueness of limits
- Algebra of limits
- Order properties of limits
- Limits of absolute values
- Limits of Max/Min

See Chapter 5 of textbook for details.

Limits of compositions of functions

When is $\lim_{x\to x_0} g\big(f(x)\big) = g\Big(\lim_{x\to x_0} f(x)\Big)$?

Theorem (Limit of composition)

Suppose

$$\lim_{x\to x_0} f(x) = L\,.$$

If g is a function defined in a neighborhood of the point L and

$$\lim_{z\to L} g(z) = g(L)$$

then

$$\lim_{x\to x_0} g\big(f(x)\big) = g\Big(\lim_{x\to x_0} f(x)\Big) = g(L)\,.$$

Note: It is a little more complicated to generalize the statement of this theorem so as to minimize the set on which g must be defined (see next slide) but the proof is no more difficult.

Limits of compositions of functions – more generally

Theorem (Limit of composition)

Let $A, B \subseteq \mathbb{R}$, $f : A \to \mathbb{R}$, $f(A) \subseteq B$, and $g : B \to \mathbb{R}$. Suppose x_0 is an accumulation point of A and

$$\lim_{x \to x_0} f(x) = L\,.$$

Suppose further that g is defined at L. If L is an accumulation point of B and

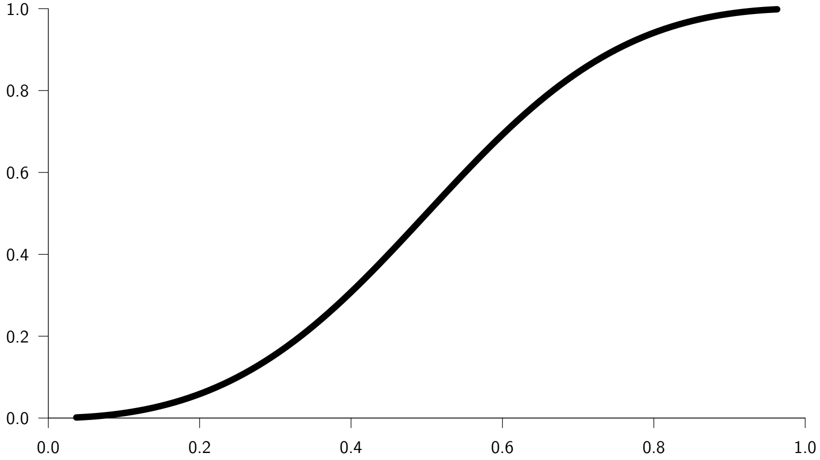
$$\lim_{z \to L} g(z) = g(L)\,,$$

or $\exists \delta > 0$ *such that $f(x) = L$ for all $x \in (x_0 - \delta, x_0 + \delta) \cap A$, then*

$$\lim_{x \to x_0} g\big(f(x)\big) = g\Big(\lim_{x \to x_0} f(x)\Big) = g(L)\,.$$

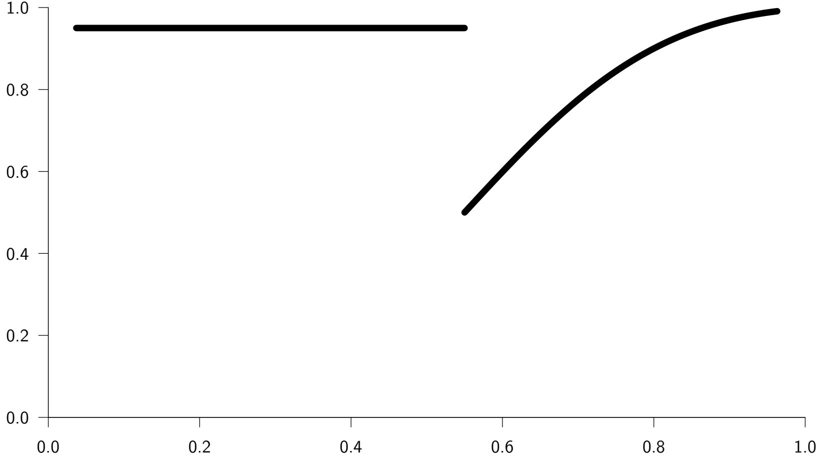
Continuity

Intuitively, a function f is **continuous** if you can draw its graph without lifting your pencil from the paper...



Continuity

and **discontinuous** otherwise. . .



Continuity

In order to develop a rigorous foundation for the theory of functions, we need to be more precise about what we mean by "continuous".

The main challenge is to define "continuity" in a way that works consistently on sets other than intervals (and generalizes to spaces that are more abstract than $\mathbb{R}$).

We will define:

- continuity at a single point;
- continuity on an open interval;
- continuity on a closed interval;
- continuity on more general sets.

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Mathematics and Statistics

$$\int_M d\omega = \int_{\partial M} \omega$$

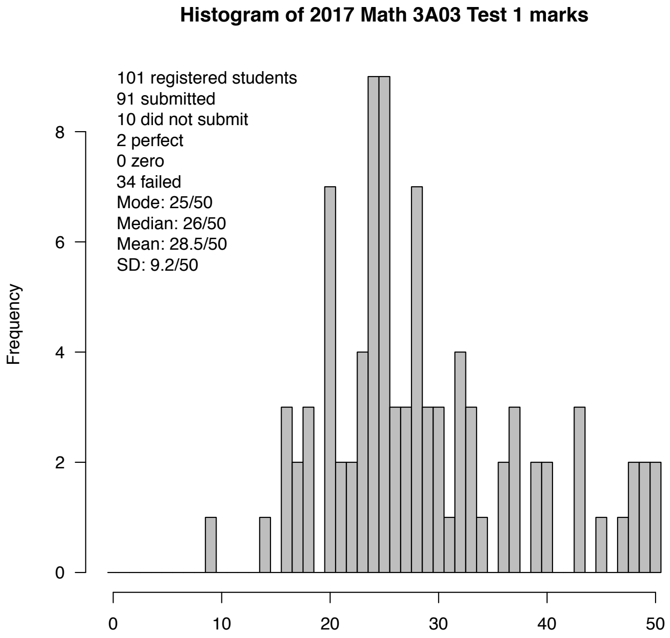
Mathematics 3A03
Real Analysis I

Instructor: David Earn

Lecture 21
Continuity III
Monday 30 October 2017

Announcements

- A preliminary version of Assignment 4 has been posted on the course wiki. More problems will be added tomorrow.
  The assignment is due on Monday 6 Nov 2017 at 2:25pm (remember cover sheet!)
- Solutions to Test 1 have been posted on the course wiki. Read them!
- Tests will be returned in tutorials today and tomorrow. Make sure to attend your tutorial in order to pick up your test.
- Test results (next slide).
  - *Note:* It seems the test was more difficult than we anticipated. The average is very low. We will adjust marks before submitting final grades.


Histogram of 2017 Math 3A03 Test 1 marks
101 registered students
91 submitted
10 did not submit
2 perfect
0 zero
34 failed
Mode: 25/50
Median: 26/50
Mean: 28.5/50
SD: 9.2/50
Frequency
0
2
4
6
8
0
10
20
30
40
50

Last time. . .

- Equivalence of ε-δ definition and sequence definition of limit.
- One-sided limit from the right.
- Limit of composition.
- Intuition for notion of continuity.

Pointwise continuity

Definition (Continuous at an interior point of the domain of f)

If the function f is defined in a neighbourhood of the point x_0 then we say f is **continuous at x_0** iff

$$\lim_{x \to x_0} f(x) = f(x_0)\,.$$

This definition works more generally provided x_0 is a point of accumulation of the domain of f (notation: $\mathrm{dom}(f)$).

We will also consider a function to be continuous at any isolated point in its domain.

Pointwise continuity

Definition (Continuous at any $x_0 \in \mathrm{dom}(f)$ – limit version)

If $x_0 \in \mathrm{dom}(f)$ then f is **continuous at x_0** iff x_0 is either an isolated point of $\mathrm{dom}(f)$ or x_0 is an accumulation point of $\mathrm{dom}(f)$ and $\lim_{x\to x_0} f(x) = f(x_0)$.

Definition (Continuous at any $x_0 \in \mathrm{dom}(f)$ – sequence version)

If $x_0 \in \mathrm{dom}(f)$ then f is **continuous at x_0** iff for any sequence $\{x_n\}$ in $\mathrm{dom}(f)$, if $x_n \to x_0$ then $f(x_n) \to f(x_0)$.

Definition (Continuous at any $x_0 \in \mathrm{dom}(f)$ – ε-δ version)

If $x_0 \in \mathrm{dom}(f)$ then f is **continuous at x_0** iff for any $\varepsilon > 0$ there exists $\delta > 0$ such that if $x \in \mathrm{dom}(f)$ and $|x - x_0| < \delta$ then $|f(x) - f(x_0)| < \varepsilon$.

Pointwise continuity

Example

Suppose $f : A \to \mathbb{R}$. In which cases is f continuous on A?

- $A = (0,1) \cup \{2\}, \quad f(x) = x;$
- $A = \{0\} \cup \{\frac{1}{n} : n \in \mathbb{N}\} \cup \{2\}, \quad f(x) = x;$
- $A = \{\frac{1}{n} : n \in \mathbb{N}\} \cup \{2\}, \quad f(x) =$ whatever you like.

Example

Is it possible for a function f to be discontinuous at every point of $\mathbb{R}$ and yet for its restriction to the rational numbers ($f\big|_{\mathbb{Q}}$) to be continuous at every point in $\mathbb{Q}$?

Extra Challenge Problem:

Prove or disprove: There is a function $f : \mathbb{R} \to \mathbb{R}$ that is continuous at every irrational number and discontinuous at every rational number.

Continuity on an interval

Definition (Continuous on an open interval)

The function f is said to be **continuous on** (a, b) iff

$$\lim_{x \to x_0} f(x) = f(x_0) \qquad \text{for all } x_0 \in (a, b)\,.$$

Definition (Continuous on a closed interval)

The function f is said to be **continuous on** $[a, b]$ iff it is continuous on the open interval (a, b), and

$$\lim_{x \to a^+} f(x) = f(a) \qquad \text{and} \qquad \lim_{x \to b^-} f(x) = f(b)\,.$$

Continuity on an arbitrary set $E \subseteq \mathbb{R}$

Definition (Continuous on a set E)

The function f is said to be **continuous on** E iff f is continuous at each point $x \in E$.

Example

- Every polynomial is continuous on $\mathbb{R}$.
- Every rational function is continuous on its domain.

These facts are painful to prove directly from the definition. But they follow easily if we know that the composition of continuous functions is continuous (which is true under natural conditions) and we have the theorem on the algebra of limits.

Continuity of compositions of functions

Theorem (Continuity of $f \circ g$ at a point)

If g is continuous at x_0 and f is continuous at $g(x_0)$ then $f \circ g$ is continuous at x_0.

Consequently, if g is continuous at x_0 and f is continuous at $g(x_0)$ then

$$\lim_{x\to x_0} f\big(g(x)\big) = f\Big(\lim_{x\to x_0} g(x)\Big)\,.$$

Theorem (Continuity of $f \circ g$ on a set)

If g is continuous on $A \subseteq \mathbb{R}$ and f is continuous on $g(A)$ then $f \circ g$ is continuous on A.

Continuity of compositions of functions

Example

Use the theorem on continuity of $f \circ g$, and the theorem on the algebra of limits, to prove that

1. the polynomial $x^8 + x^3 + 2$ is continuous on $\mathbb{R}$;
2. the rational function $\dfrac{x^2+2}{x^2-2}$ is continuous on $\mathbb{R} \setminus \{-\sqrt{2}, \sqrt{2}\}$.
3. the function $\sqrt{\dfrac{x^2+2}{x^2-2}}$ is continuous on its domain.

Uniform continuity

In the ε-δ definition of continuity, the δ that must exist depends on ε **AND** on the point x_0, *i.e.,* $\delta = \delta(f, \varepsilon, x_0)$.

Definition (Uniformly continuous)

If $f : A \to \mathbb{R}$ then f is said to be **uniformly continuous on A** iff for every $\varepsilon > 0$ there exists $\delta > 0$ such that if $x, y \in A$ and $|x - y| < \delta$ then $|f(x) - f(y)| < \varepsilon$.

Note: This is a stronger form of continuity: Given any $\varepsilon > 0$, there is a single $\delta > 0$ that works for the entire set A.
(δ still depends on f and ε.)

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Mathematics and Statistics

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Mathematics 3A03
Real Analysis I

Instructor: David Earn

Lecture 22
Continuity IV
Wednesday 1 November 2017

Announcements

- Assignment 4 is now COMPLETE on the course wiki.
  Due Monday 6 Nov 2017 at 2:25pm
  (remember cover sheet!)
- Solutions to Test 1 have been posted on the course wiki.
  Read them!
- On Friday you will be asked to do a 5 minute online survey.
  Please make sure to have your laptop or phone or some device you like for filling in web forms.

Last time. . .

- Limits of compositions.
- Continuity at a point and on a set.
- Continuity of compositions.
- Uniform continuity.

Uniform continuity

Theorem (Unif. cont. on a bounded interval $\implies$ bounded)

If f is uniformly continuous on a bounded interval I then f is bounded on I.

(solution on board)

Clean proof.

Suppose f is uniformly continuous on the interval I with endpoints a, b (where $a < b$). Then, given $\varepsilon > 0$ we can find $\delta > 0$ such that if $x, y \in I$ and $|x - y| < \delta$ then $|f(x) - f(y)| < \varepsilon$.

Moreover, given any $\delta > 0$ and any $c > 0$, we can find $n \in \mathbb{N}$ such that $0 < \frac{c}{n} < \delta$.

Choose $n \in \mathbb{N}$ such that if $x, y \in I$ and $|x - y| < 2\left(\frac{b-a}{n}\right)$ then $|f(x) - f(y)| < 1$.

Continued...

Uniform continuity

Clean proof (continued).

Divide I into n subintervals with endpoints

$$x_i = a + i\left(\frac{b-a}{n}\right), \qquad i = 0, 1, \ldots, n.$$

For $0 \le i \le n-1$, define $I_i = [x_i, x_{i+1}] \cap I$ (we intersect with I in case $a \notin I$ or $b \notin I$), and note that $\forall x, y \in I_i$ we have $|x - y| \le \frac{b-a}{n} < 2\left(\frac{b-a}{n}\right)$ and hence $|f(x) - f(y)| < 1$ $\forall x, y \in I_i$.

Let $\overline{x}_i = (x_i + x_{i+1})/2$ (the midpoint of interval I_i). Then, in particular, we have $|f(x) - f(\overline{x}_i)| < 1$ $\forall x \in I_i$, *i.e.*,

$$f(\overline{x}_i) - 1 < f(x) < f(\overline{x}_i) + 1 \qquad \forall x \in I_i\,.$$

Thus, f is bounded on I_i and therefore has a LUB and GLB on I_i.

Continued...

Uniform continuity

Clean proof (continued).

Therefore, for $i = 0, 1, \ldots, n-1$, define

$$m_i = \inf\{f(x) : x \in I_i\}\,,$$
$$M_i = \sup\{f(x) : x \in I_i\}\,,$$

and let

$$m = \min\{m_i : i = 0, 1, \ldots, n-1\}\,,$$
$$M = \max\{M_i : i = 0, 1, \ldots, n-1\}\,.$$

Then

$$m \le f(x) \le M \qquad \forall x \in I = \bigcup_{i=1}^{n-1} I_i\,,$$

i.e., f is bounded on the entire interval I. □

Uniform continuity

Theorem (Cont. on a closed interval $\implies$ unif. cont.)

If $f : [a,b] \to \mathbb{R}$ is continuous then f is uniformly continuous.

(solution on board)

Corollary (Continuous on a closed interval $\implies$ bounded)

If $f : [a,b] \to \mathbb{R}$ is continuous then f is bounded.

Proof.

Combine the above two theorems. $\square$

Mathematics
and Statistics

$$\int_M d\omega = \int_{\partial M} \omega$$

Mathematics 3A03
Real Analysis I

Instructor: David Earn

Lecture 23
Continuity V
Friday 3 November 2017

Note:

Lecture 23 was given by David Duncan using the blackboard only.

The following slides summarize the content of that lecture.

Last time and today

Last time:

Theorem (UC $\Longrightarrow$ C)

Suppose $f : D \to \mathbb{R}$ is uniformly continuous. Then f is continuous.

Theorem (CptUC $\Longrightarrow$ Bdd)

Suppose D is a compact set and $f : D \to \mathbb{R}$ is uniformly continuous. Then f is bounded.

Today:

Theorem (CptC $\Longrightarrow$ UC)

Suppose D is a compact set and $f : D \to \mathbb{R}$ is continuous. Then f is uniformly continuous.

Theorems UC $\Longrightarrow$ C and CptC $\Longrightarrow$ UC say that on a compact domain, continuity is equivalent to uniform continuity. The converse is also true:

Theorem

If every continuous function on a set E is uniformly continuous then E is compact.

Recall that compactness is associated with global properties (as opposed to local properties). Uniform continuity is a global property in that a single δ is sufficient for an entire set.

Proof of Theorem CptC $\implies$ UC

Suppose D is compact and $f : D \to \mathbb{R}$ is continuous. To show that f is uniformly continuous, fix $\epsilon > 0$. By the continuity of f, for each $x \in D$, there is some $\delta_x > 0$ so that if $y \in D \setminus \{x\}$ satisfies $|x - y| < \delta_x$, then $|f(x) - f(y)| < \epsilon/2$. Consider the collection

$$\mathcal{U} = \{(x - \delta_x/2, x + \delta_x/2) \ : \ x \in D\}$$

of open sets (in class, David D forgot to include the "/2" here). This clearly is an open cover of D, since $x \in (x - \delta_x/2, x + \delta_x/2)$ for all $x \in D$. Since D is compact, $\mathcal{U}$ has a finite subcover $\mathcal{U}'$, which we can write as

$$\mathcal{U}' = \{(x_1 - \delta_{x_1}/2, x_1 + \delta_{x_1}/2), \ldots, (x_N - \delta_{x_N}/2, x_N + \delta_{x_N}/2)\}$$

for some natural number N. Set $\delta = \min(\delta_1/2, \ldots, \delta_N/2)$.

Proof of Theorem CptC $\Longrightarrow$ UC (cont'd)

To verify the uniform continuity property, suppose $x, y \in D$ satisfy $|x - y| < \delta$. Since $\mathcal{U}'$ is a cover of D, there is some $1 \le n \le N$ so that $x \in (x_n - \delta_{x_n}/2, x_n + \delta_{x_n}/2)$. This implies $|x - x_n| < \delta_n$ and so $|f(x) - f(x_n)| < \epsilon/2$. Note that we also have

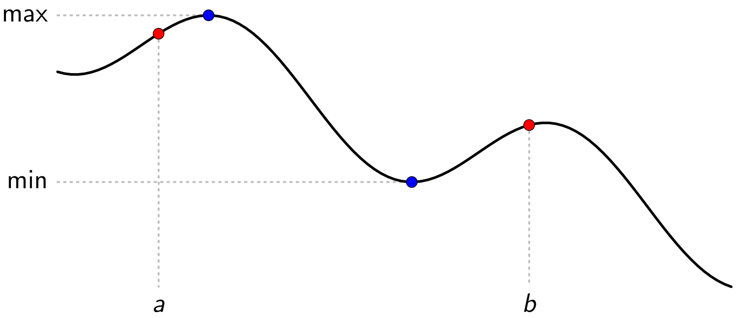
$$|y - x_n| \le |y - x| + |x - x_n| < \delta + \delta_n/2 < \delta_n$$

and so $|f(y) - f(x_n)| < \epsilon/2$. The triangle inequality then gives

$$|f(x) - f(y)| \le |f(x) - f(x_n)| + |f(y) - f(x_n)| < \epsilon.$$

□

Extreme Value Theorem



Theorem (Extreme value theorem)

Suppose D is compact and non-empty, and $f : D \to \mathbb{R}$ is continuous. Then f achieves its maximum and its minimum. That is, there are some $x_m, x_M \in D$ so that

$$f(x_m) \leq f(x) \leq f(x_M), \forall x \in D.$$

Proof 1 (by contradiction).

We will prove that f attains its maximum; the proof that it attains its minimum is similar. Since f is continuous on the compact set D, it is bounded on D. This means that the **range** of f, *i.e.*, the set

$$f(D) \stackrel{\textbf{def}}{=} \{f(x) : x \in D\}$$

is bounded. This set is not $\varnothing$, so it has a LUB α. Since $\alpha \geq f(x)$ for $x \in D$, it suffices to show that $\alpha = f(x_M)$ for some $x_M \in D$.

Suppose instead that $\alpha \neq f(y)$ for any $y \in D$, *i.e.*, $\alpha > f(y)$ for all $y \in D$. Then the function g defined by ...

Extreme Value Theorem

Proof 1 of Extreme Value Theorem (continued).

$$g(x) = \frac{1}{\alpha - f(x)}, \qquad x \in D,$$

is positive and continuous on D, since the denominator of the RHS is always positive. Since D is compact, it follows from Theorems CptC $\Longrightarrow$ UC and CptUC $\Longrightarrow$ Bdd that g is bounded. We will show that g is unbounded, which will be the contradiction. Since α is a LUP of $f(D)$, it follows from Problem 9 of the Midterm that α is contained in the closure $\overline{f(D)}$. Hence, we can find a sequence of points x_n in D with $f(x_n) \to \alpha$. But then

$$\lim_{n\to\infty} g(x_n) = +\infty,$$

which shows g is unbounded. $\Rightarrow\Leftarrow$

Extreme Value Theorem

Proof 2 (Sketch).

Once again, we will only prove that x_M exists; the existence of x_m is similar. Suppose D is compact and $f : D \to \mathbb{R}$ is continuous. Then $f(D)$ is compact (try to prove this on your own). In particular, $f(D)$ is closed, bounded and non-empty, so by Problem 9 of the Midterm, $f(D)$ contains its own supremum. That is, we can write the supremum as $f(x_M)$ for some $x_M \in D$. Since the supremum is an upper bound, we have $f(x) \leq f(x_M)$ for all $x \in D$. □

Mathematics
and Statistics

$$\int_M d\omega = \int_{\partial M} \omega$$

Mathematics 3A03
Real Analysis I

Instructor: David Earn

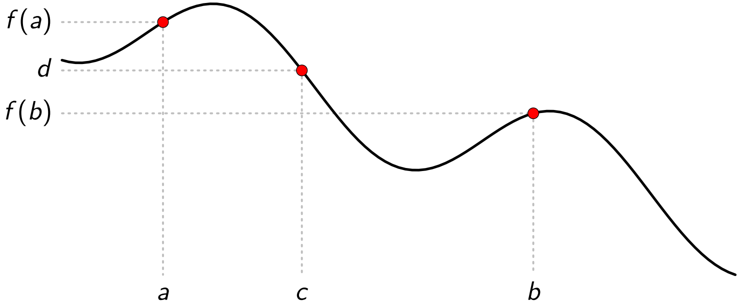
Lecture 24
Continuity VI
Monday 6 November 2017

Last time. . .

- Online survey: Thanks if you did it! Please do it if you haven't already. Link is on Course information page of course wiki.
- Continuous on a compact set $\Longrightarrow$ uniformly continuous.
- Mentioned that a continuous image of a compact set is compact.
- Stated and proved Extreme Value Theorem.

Note: Assignment 4 was due at 2:25pm today.

Intermediate Value Theorem



Definition (Intermediate Value Property (IVP))

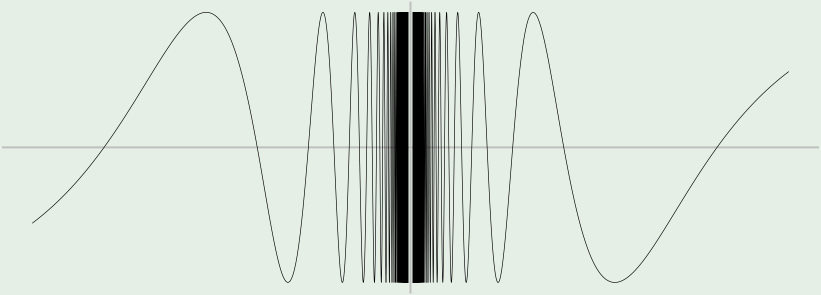
A function f defined on an interval I is said to have the **intermediate value property (IVP)** on I iff for each $a, b \in I$ with $f(a) \neq f(b)$, and for each d between $f(a)$ and $f(b)$, there exists c between a and b for which $f(c) = d$.

Intermediate Value Theorem

Question: If a function has the IVP on an interval I, must it be continuous on I?

Example

$$f(x) = \begin{cases} \sin \frac{1}{x} & x \neq 0, \\ 0 & x = 0. \end{cases}$$



Intermediate Value Theorem

Theorem (Intermediate Value Theorem (IVT))

If f is continuous on an interval I then f has the intermediate value property (IVP) on I.

(solution after proving the neighbourhood sign lemma)

Note: The interval I in the statement of the IVT does not have to be closed and it does not have to be bounded.
Unlike the extreme value theorem, the IVT is not a theorem about functions defined on compact sets.

Intermediate Value Theorem

Lemma (Neighbourhood sign)

Suppose I is an interval and $f : I \to \mathbb{R}$ is continuous at $a \in I$. If $f(a) > 0$ then f is positive in a neighbourhood of a. Similarly, if $f(a) < 0$, then f is negative in a neighbourhood of a.

Proof.

Consider the case $f(a) > 0$. Since f is continuous at a, given $\varepsilon > 0$ $\exists \delta > 0$ such that if $|x - a| < \delta$ then $|f(x) - f(a)| < \varepsilon$. Since $f(a) > 0$ we can take $\varepsilon = f(a)$. Thus, $\exists \delta > 0$ such that if $|x - a| < \delta$ then $|f(x) - f(a)| < f(a)$, *i.e.*,

$$|x - a| < \delta \implies -f(a) < f(x) - f(a) < f(a) \implies 0 < f(x) < 2f(a)\,.$$

In particular, $f(x) > 0$ in a neighbourhood* of radius δ about a.

The case $f(a) < 0$ is similar: take $\varepsilon = -f(a)$. □

*The neighbourhood is $(a - \delta, a + \delta)$, unless a is an endpoint of the set on which f is defined, in which case the neighbourhood is either $[a, a + \delta)$ or $(a - \delta, a]$.

Intermediate Value Theorem

The Intermediate Value Theorem follows directly from the following lemma, which is what we'll prove:

Lemma (Existence of roots)

If f is continuous on $[a, b]$ and $f(a) < 0 < f(b)$ then there exists $x \in [a, b]$ such that $f(x) = 0$.

How does Intermediate Value Property follow?

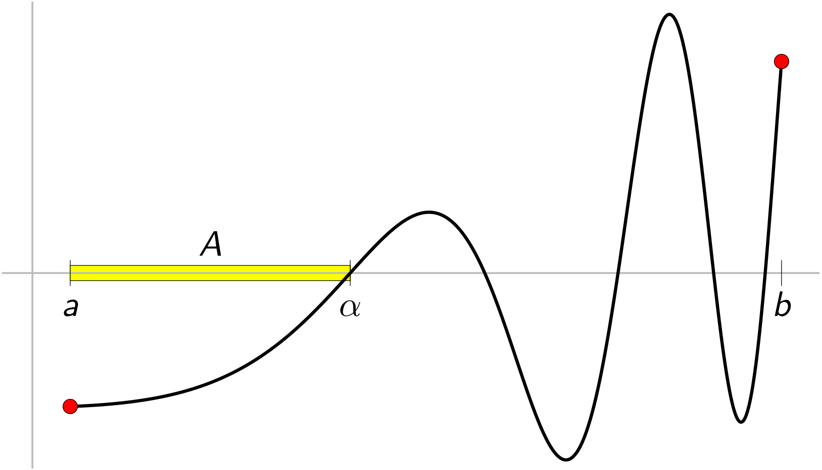
If $f(a) < M < f(b)$ for some $M \in \mathbb{R}$, then apply the lemma to $g(x) = f(x) - M$.

If $f(a) > M > f(b)$ for some $M \in \mathbb{R}$, then apply the lemma to $g(x) = M - f(x)$.

What if the interval I on which f is continuous is not a closed interval?

Intermediate Value Theorem

Idea for proof of root existence lemma:



Intermediate Value Theorem

Sketch of proof of root existence lemma:

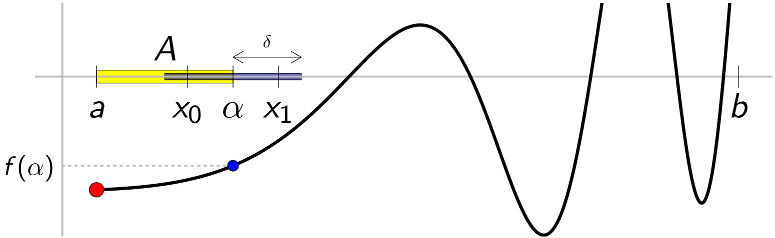
1. (i) $A = \{x : a \leq x \leq b, \text{and } f \text{ is negative on the interval } [a, x]\}$;
   (ii) $\alpha = \sup(A)$ exists;
   (iii) neighbourhood sign lemma $\implies a < \alpha < b$.

2. Prove by contradiction that $f(\alpha) < 0$ is impossible.
   To guide this argument, it helps to draw a picture that is consistent with the assumption that $f(\alpha) < 0$. This picture is not really correct because it represents an assumption that we will prove to be false.

3. Prove by contradiction that $f(\alpha) > 0$ is impossible.

Intermediate Value Theorem

Picture to guide proof by contradiction that it is impossible that $f(\alpha) < 0$:



- Given $f(\alpha) < 0$, the neighbourhood sign lemma implies $\exists\delta > 0$ such that $f(x) < 0$ on $(\alpha - \delta, \alpha + \delta)$.
- For any $x_0 \in (\alpha - \delta, \alpha)$, since $x_0 < \alpha$, we must have $x_0 \in A$, *i.e.*, $f(x) < 0$ on $[a, x_0]$. Otherwise, α would not be the least upper bound of A.
- Now pick any $x_1 \in (\alpha, \alpha + \delta)$. We know $x_1 \notin A$ because $\alpha < x_1$. But $f(x) < 0$ on $[x_0, x_1]$ since $[x_0, x_1] \subset (\alpha - \delta, \alpha + \delta)$ and $f(x) < 0$ on $[a, x_0]$ because $x_0 \in A$. Hence $f(x) < 0$ on $[a, x_1]$, *i.e.*, $x_1 \in A$. $\Rightarrow\Leftarrow$