McMaster University

Mathematics and Statistics

$$\int_M d\omega = \int_{\partial M} \omega$$

Mathematics 3A03
Real Analysis I

Instructor: David Earn

Lecture 6
Sequences
Monday 18 September 2017

Announcements

- Solutions to Assignment 1 have been posted. **Study them!**
- Assignment 2: first few problems are posted; more to come.
- Remember that solutions to assignments and tests from the 2015 and 2016 versions of the course are available on the course wiki. Take advantage of these problems and solutions. They provide many useful examples that should help you prepare for tests and the final exam. (However, note that while most of the content of the course is the same this year, there are some differences.)
- No late submission of assignments. No exceptions. However, best 5 of 6 assignments will be counted. *Always due 5 minutes before class on the due date.*
- Consider writing the Putnam competition.

THINKING ABOUT GRADUATE SCHOOL?

JOIN US TO FIND OUT MORE AT THE GRAD INFO SESSION!

WHEN: TUESDAY SEPTEMBER 26, 2017
TIME: 5:30 PM – 6:30 PM
WHERE: HH/305 AND THE MATH CAFÉ

Nicholas Kevlahan and Shui Feng will talk about graduate programs in Math and Stats, Computational Science and Engineering at McMaster and elsewhere. David Lozinski will talk about graduate opportunities for AFM students, including M-Phimac.

Miroslav Lovric will give tips about applying to teachers' college.

PIZZA will be served! See you there!

Sequences

- A **sequence** is a list that goes on forever.
- There is a beginning (a "first term") but no end, *e.g.*,

$$\frac{1}{1},\ \frac{1}{2},\ \frac{1}{3},\ \frac{1}{4},\ \ldots,\ \frac{1}{n},\ \ldots$$

- We use the natural numbers $\mathbb{N}$ to label the terms of a sequence:

$$a_1,\ a_2,\ a_3,\ \ldots,\ a_n,\ \ldots$$

Formal definition of a sequence

Definition (Sequence of Real Numbers)

A **sequence of real numbers** is a function

$$f : \mathbb{N} \to \mathbb{R} .$$

A lot of different notation is common for sequences:

$f(1),\ f(2),\ f(3),\ \ldots$	$\{f(n)\}_{n=1}^{\infty}$
$f_1,\ f_2,\ f_3,\ \ldots$	$\{f(n)\}$
$\{f(n) : n = 1, 2, 3, \ldots\}$	$\{f_n\}_{n=1}^{\infty}$
$\{f(n) : n \in \mathbb{N}\}$	$\{f_n\}$

Specifying sequences

There are two main ways to specify a sequence:

1. Direct formula.

Specify $f(n)$ for each $n \in \mathbb{N}$. □

Example (arithmetic progression with common difference d)

Sequence is:

$$c,\ c+d,\ c+2d,\ c+3d,\ \ldots$$

$$\therefore\ f(n) = c + (n-1)d, \qquad n \in \mathbb{N}$$

$$\text{i.e.,} \quad x_n = c + (n-1)d, \qquad n = 1, 2, 3, \ldots$$

Specifying sequences

2. Recursive formula.

Specify first term and function $f(x)$ to **iterate**. □

i.e., Given x_1 and $f(x)$, we have $x_n = f(x_{n-1})$ for all $n > 1$.

$$x_2 = f(x_1), \quad x_3 = f(f(x_1)), \quad x_4 = f(f(f(x_1))), \quad \dots$$

Example (arithmetic progression with common difference d)

$$x_1 = c, \quad f(x) = x + d$$

$$\therefore \; x_n = x_{n-1} + d, \qquad n = 2, 3, 4, \dots$$

Note: f is the most typical function name for both the direct and recursive specifications. The correct interpretation of f should be clear from context.

Specifying sequences

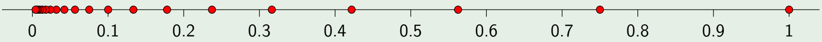
Example (geometric progression with common ratio r)

Sequence is: $c, \ cr, \ cr^2, \ cr^3, \ \dots$

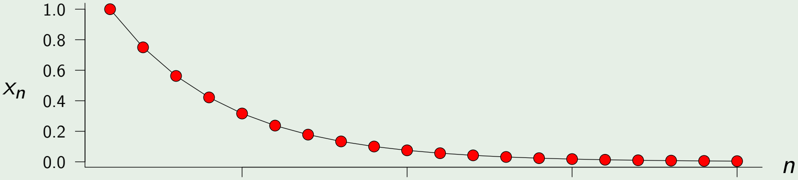
Direct formula: $x_n = f(n) = cr^{n-1}$, $n = 1, 2, 3, \dots$

Recursive formula: $x_1 = c$, $f(x) = rx$, $x_n = f(x_{n-1})$

Number line representation of $\{x_n\}$ with $c = 1$ and $r = \frac{3}{4}$:



Graph of $f(n)$:



Specifying sequences

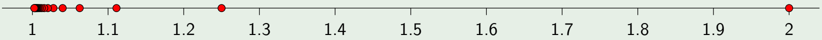
Example ($f(n) = 1 + \frac{1}{n^2}$)

Sequence is: 2, $\frac{5}{4}$, $\frac{10}{9}$ $\frac{17}{16}$, $\ldots$

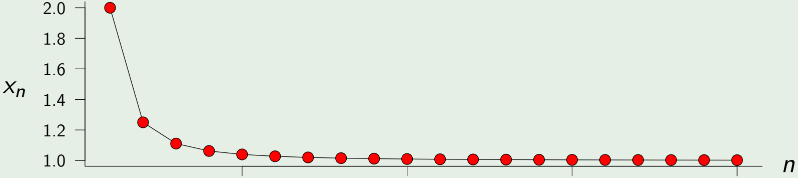
Direct formula: $x_n = f(n) = 1 + \frac{1}{n^2}$, $n = 1, 2, 3, \ldots$

Recursive formula: $x_1 = 2$, $\quad f(x) = 1 + [1 + (x-1)^{-1/2}]^{-2}$

Number line representation of $\{x_n\}$:



Graph of $f(n)$:



Convergence of sequences

We know from previous experience that:

- $cr^{n-1} \to 0$ as $n \to \infty$ (if $|r| < 1$).
- $1 + \frac{1}{n^2} \to 1$ as $n \to \infty$.

How do we make our intuitive notion of **convergence** *mathematically rigorous*?

Informal definition: "$x_n \to L$ as $n \to \infty$" means "we can make the difference between x_n and L as small as we like by choosing n big enough".

More careful informal definition: "$x_n \to L$ as $n \to \infty$" means "given any *error tolerance*, say ε, we can make the distance between x_n and L smaller than ε by choosing n big enough".

Convergence of sequences

Definition (Limit of a sequence)

A sequence $\{s_n\}$ **converges to L** if, given any $\varepsilon > 0$ there is some integer N such that

$$\text{if } n \geq N \qquad \text{then} \qquad |s_n - L| < \varepsilon\,.$$

In this case, we write $\lim_{n\to\infty} s_n = L$ or $s_n \to L$ as $n \to \infty$ and we say that L is the **limit** of the sequence $\{s_n\}$.

Note: To use this definition to prove that the limit of a sequence is L, we start by imagining that we are given some error tolerance $\varepsilon > 0$. Then we have to find a suitable N, which will depend on ε. This means that *the N that we find will be a function of ε*.

Shorthand:

$$\lim_{n\to\infty} s_n = L \quad \stackrel{\text{def}}{=} \quad \forall\varepsilon > 0 \quad \exists N \in \mathbb{N} \quad \ni \quad n \geq N \implies |s_n - L| < \varepsilon.$$

Convergence of sequences

Convergence terminology:

- A sequence that converges is said to be **convergent**.
- A sequence that is not convergent is said to be **divergent**.

Remark (Sequences in spaces other than $\mathbb{R}$)

The formal definition of a limit of a sequence works in any space where we have a notion of distance if we replace $|s_n - L|$ with $d(s_n, L)$.

Convergence of sequences

Example

Use the formal definition of a limit of a sequence to prove that

$$\frac{n^2+1}{n^2} \to 1 \quad \text{as} \quad n \to \infty \,.$$

(solution on board)

Note: Our strategy here was to solve for n in the inequality $|s_n - L| < \varepsilon$. From this we were able to infer how big N has to be in order to ensure that $|s_n - L| < \varepsilon$ for all $n \geq N$. That much was "rough work". Only after this rough work did we have enough information to be able to write down a rigorous proof.

McMaster University

Mathematics and Statistics

$$\int_M d\omega = \int_{\partial M} \omega$$

Mathematics 3A03
Real Analysis I

Instructor: David Earn

Lecture 7
Sequences II
Wednesday 20 September 2017

Announcements

- Solutions to Assignment 1 have been posted. **Study them!**
- Assignment 2: first few problems are posted; more to come. Due Friday 29 Sep 2017 at 4:25pm.
  - Typo in question 1(a): 2^{k+1} should say 2^{n+1} (now corrected on course wiki)

Convergence of sequences

Example

Use the formal definition of a limit of a sequence to prove that

$$\frac{n^5 - n^3 + 1}{n^8 - n^5 + n + 1} \to 0 \quad \text{as} \quad n \to \infty\,.$$

(solution on board)

Note: In this example, it was not possible to solve for n in the inequality $|s_n - L| < \varepsilon$. Instead, we first needed to bound $|s_n - L|$ by a much simpler expression that is always greater than $|s_n - L|$. If that bound is less than ε then so is $|s_n - L|$.

Uniqueness of limits

Theorem (Uniqueness of limits)

If $\lim_{n\to\infty} s_n = L_1$ *and* $\lim_{n\to\infty} s_n = L_2$ *then* $L_1 = L_2$.

(solution on board)

So, we are justified in referring to "the" limit of a convergent sequence.

Divergence of sequences

Divergence is the logical opposite (negation) of convergence.
We can infer the formal meaning of divergence by taking the *logical negation* of the formal definition of convergence.
Doing so, we find that the sequence $\{s_n\}$ diverges (*i.e.*, does not converge to any $L \in \mathbb{R}$) iff

$$\forall L \in \mathbb{R}, \;\; \exists \varepsilon > 0 \;\text{ such that: } \forall N \in \mathbb{N} \;\; \exists n \geq N \;\; \ni \;\; |s_n - L| \geq \varepsilon .$$

Notes:

- The n that exists will, in general, depend on L, ε and N.
- This is the meaning of not converging to any limit, but it does not tell us anything about what happens to the sequence $\{s_n\}$ as $n \to \infty$.

Divergence to $\pm\infty$

Definition (Divergence to ∞)

The sequence $\{s_n\}$ of real numbers **diverges to ∞** if, for every real number M there is an integer N such that

$$n \ge N \quad \Longrightarrow \quad s_n \ge M \,,$$

in which case we write $s_n \to \infty$ as $n \to \infty$ or $\lim\limits_{n\to\infty} s_n = \infty$.

Definition (Divergence to $-\infty$)

The sequence $\{s_n\}$ of real numbers **diverges to $-\infty$** if, for every real number M there is an integer N such that

$$n \ge N \quad \Longrightarrow \quad s_n \le M \,.$$

Divergence to ∞

Example

Use the formal definition to prove that

$$\left\{\frac{n^3-1}{n+1}\right\} \quad \text{diverges to } \infty\,.$$

(solution on board)

Approach: Find a lower bound for the sequence that is a simple function of n and show that that can be made bigger than any given M.

Mathematics
and Statistics

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Mathematics 3A03
Real Analysis I

Instructor: David Earn

Lecture 8
Sequences III
Friday 22 September 2017

An additional online resource

A sequence of 15 short (3–7 minute) videos covering the very basics of mathematical logic and theorem proving has been posted associated with a course at the University of Toronto:

- Go to `http://uoft.me/MAT137`, click on the **Videos** tab and then on **Playlist 1**.

These videos go at a slower pace that we have, and may be very helpful to you to get your head around the idea of a rigorous mathematical proof.

What we've done so far on sequences

- Definition of convergence.
- Definition of divergence.
- Definition of divergence to $\pm\infty$.
- Examples.

Divergence to ∞

Example (Example from last time)

Use the formal definition to prove that $\left\{\frac{n^3-1}{n+1}\right\}$ diverges to ∞.

Clean proof.

Given $M \in \mathbb{R}^{>0}$, let $N = \lceil M \rceil + 1$. Then $N - 1 = \lceil M \rceil \geq M$. $\therefore \forall n \geq N,\ n - 1 \geq M$. Now observe that

$$\forall n \in \mathbb{N}, \quad n - 1 = \frac{(n-1)(n+1)}{n+1} = \frac{n^2-1}{n+1} \leq \frac{n^3-1}{n+1}\,.$$

$\therefore \forall n \geq N$ we have

$$\frac{n^3-1}{n+1} \geq M\,,$$

as required. □

Sequences of partial sums (a.k.a. Series)

Given a sequence $\{x_n\}$, we define the **sequence of partial sums of $\{x_n\}$** to be $\{s_n\}$, where

$$s_n = \sum_{k=1}^{n} x_k = x_1 + x_2 + \cdots + x_n \,.$$

Note: We can start from any integer, not necessarily $k = 1$.

Boundedness of sequences

A sequence is said to be bounded if its range is a bounded set.

Definition (Bounded sequence)

A sequence $\{s_n\}$ is **bounded** if there is a real number M such that every term in the sequence satisfies $|s_n| \leq M$.

Theorem (Every convergent sequence is bounded.)

$$L \in \mathbb{R} \ \wedge \ \lim_{n\to\infty} s_n = L \quad \Longrightarrow \quad \exists M > 0 \ \ni \ |s_n| \leq M \ \forall n \in \mathbb{N}.$$

(solution on board)

Note: The converse is FALSE.
Proof? Find a counterexample, *e.g.*, $\{(-1)^n\}$.

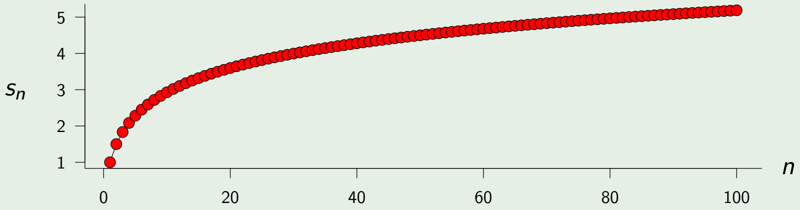
Boundedness of sequences

Corollary (Unbounded sequences diverge)

If $\{s_n\}$ is unbounded then $\{s_n\}$ diverges.

Example (The harmonic series diverges)

Consider the **harmonic series** $s_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$.



Prove that s_n diverges to ∞.

(solution on board)

Harmonic series – idea for proof of divergence

Approach: Group terms and use the corollary above.

$$\underbrace{\underbrace{\underbrace{\underbrace{\left(1+\frac{1}{2}\right)}_{>\ 1\times\frac{1}{2}}}_{s_2\ >\ 1\times\frac{1}{2}}+\underbrace{\left(\frac{1}{3}+\frac{1}{4}\right)}_{>\ 2\times\frac{1}{4}}}_{s_4\ >\ 2\times\frac{1}{2}}+\underbrace{\left(\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}\right)}_{>\ 4\times\frac{1}{8}}}_{s_8\ >\ 3\times\frac{1}{2}}+\cdots$$

$$\Longrightarrow \qquad s_{2^n} \ > \ n\times\frac{1}{2}$$

Note: These sorts calculations are just "rough work", not a formal proof.
A proof must be a clearly presented coherent argument from beginning to end.

Harmonic series – clean proof of divergence

Proof.

Part (i). Prove (e.g., by induction) that $\quad s_{2^n} > n/2 \quad \forall n \in \mathbb{N}$.

Part (ii). Suppose we are given $M \in \mathbb{R}$.

- If $M \leq 0$ then note that $s_n > 0 \; \forall n \in \mathbb{N}$.
- If $M > 0$, let $\tilde{N} = 2\lceil M \rceil$ and $N = 2^{\tilde{N}}$. Then, $\forall n \geq N$, we have $s_n \geq s_N = s_{2^{\tilde{N}}} > \tilde{N}/2 = \lceil M \rceil \geq M$, as required.

□

Algebra of limits

Theorem (Algebraic operations on limits)

Suppose $\{s_n\}$ *and* $\{t_n\}$ *are convergent sequences and* $C \in \mathbb{R}$.

1. $\lim_{n\to\infty} C\,s_n = C\big(\lim_{n\to\infty} s_n\big)$;
2. $\lim_{n\to\infty} (s_n + t_n) = \big(\lim_{n\to\infty} s_n\big) + \big(\lim_{n\to\infty} t_n\big)$;
3. $\lim_{n\to\infty} (s_n - t_n) = \big(\lim_{n\to\infty} s_n\big) - \big(\lim_{n\to\infty} t_n\big)$;
4. $\lim_{n\to\infty} (s_n t_n) = \big(\lim_{n\to\infty} s_n\big)\big(\lim_{n\to\infty} t_n\big)$;
5. *if* $t_n \neq 0$ *for all* n *and* $\lim_{n\to\infty} t_n \neq 0$ *then*
$$\lim_{n\to\infty} \left(\frac{s_n}{t_n}\right) = \frac{\lim_{n\to\infty} s_n}{\lim_{n\to\infty} t_n} .$$

(solution on board)

Revisit example

Example (previously proved directly from definition)

Use the algebraic properties of limits to prove that

$$\frac{n^5 - n^3 + 1}{n^8 - n^5 + n + 1} \to 0 \quad \text{as} \quad n \to \infty\,.$$

(solution on board)

McMaster University

Mathematics and Statistics

$$\int_M d\omega = \int_{\partial M} \omega$$

Mathematics 3A03
Real Analysis I

Instructor: David Earn

Lecture 9
Sequences IV
Monday 25 September 2017

Announcements

- Assignment 2: all problems are now posted.
  Due this Friday, 29 Sep 2017, at 4:25pm.
- I need to leave around 4:40pm on Friday...
  - Friday's lecture will be given by Dr. David Duncan (in HH-109) to both sections of the course.

What we've done so far on sequences

- Definition of convergence.
- Definition of divergence.
- Definition of divergence to $\pm\infty$.
- Examples.
- Every convergent sequence is bounded.
- Harmonic series diverges.
- Algebra of limits (more today).

Product Rule for Limits

The 4th item in the algebra of limits theorem was:

Theorem (Product Rule for Limits)

If $s_n \to S$ and $t_n \to T$ as $n \to \infty$ then $s_n t_n \to ST$ as $n \to \infty$.

Proof.

For any $n \in \mathbb{N}$,

$$\begin{aligned} |s_n t_n - ST| &= |s_n t_n - ST + s_n T - s_n T| \\ &= |s_n(t_n - T) + T(s_n - S)| \\ &\le |s_n||t_n - T| + |T||s_n - S| \end{aligned}$$

Now, $\{s_n\}$ converges, so it is bounded by some $M > 0$, *i.e.*, $|s_n| \le M \; \forall n \in \mathbb{N}$. Therefore, given $\varepsilon > 0$, choose $N \in \mathbb{N}$ such that

$$|t_n - T| < \frac{\varepsilon}{2M} \qquad \text{and} \qquad |s_n - S| < \frac{\varepsilon}{2(1 + |T|)} .$$

Then $|s_n t_n - ST| < \varepsilon/2 + \varepsilon/2 = \varepsilon$, as required. □

Quotient Rule for Limits

Quotient Rule was the 5th item in the algebra of limits theorem.

Lemma (Reciprocal Rule for Limits)

If $t_n \neq 0 \ \forall n$ and $t_n \to T \neq 0$ then $1/t_n \to 1/T$.

Proof.

For any $n \in \mathbb{N}$, $\left|\frac{1}{t_n} - \frac{1}{T}\right| = \left|\frac{t_n - T}{t_n T}\right| = |t_n - T| \cdot \frac{1}{|t_n|} \cdot \frac{1}{|T|}$.
Since $\{t_n\}$ converges, $\exists N_1 \in \mathbb{N}$ such that $\forall n \geq N_1$, $|t_n| > |T|/2$ (details on next slide) and hence $1/|t_n| < 2/|T|$.
Now choose $N \geq N_1$ such that $|t_n - T| < \varepsilon |T|^2/2$. Then

$$\left|\frac{1}{t_n} - \frac{1}{T}\right| = |t_n - T| \cdot \frac{1}{|t_n|} \cdot \frac{1}{|T|} < \frac{\varepsilon |T|^2}{2} \cdot \frac{2}{|T|} \cdot \frac{1}{|T|} = \varepsilon\,,$$

as required. □

Quotient Rule for Limits

Details missing on previous slide:

Since $t_n \to T$, $\exists N_1 \in \mathbb{N}$ such that $\forall n \geq N_1$, $|t_n - T| < \dfrac{|T|}{2}$,

i.e., $-\dfrac{|T|}{2} < t_n - T < \dfrac{|T|}{2}$, i.e., $T - \dfrac{|T|}{2} < t_n < T + \dfrac{|T|}{2}$.

If $T > 0$ this says

$$0 < \frac{T}{2} < t_n < \frac{3T}{2},$$

whereas if $T < 0$ it says

$$-\frac{3T}{2} < t_n < \frac{T}{2} < 0.$$

In either case, $\forall n \geq N_1$, we have $0 < \dfrac{|T|}{2} < |t_n|$.

Order properties of limits (§2.8)

Theorem (Limits retain order)

If $\{s_n\}$ and $\{t_n\}$ are convergent sequences then

$$s_n \le t_n \quad \forall n \in \mathbb{N} \qquad \Longrightarrow \qquad \lim_{n\to\infty} s_n \le \lim_{n\to\infty} t_n \,.$$

(solution on board)

Note: If $s_n < t_n$ for all $n \in \mathbb{N}$, can we conclude that

$$\lim_{n\to\infty} s_n < \lim_{n\to\infty} t_n \qquad ?$$

No! No! No! No! No! No!! NO!!!!!!!!!!

Order properties of limits (§2.8)

Theorem (Limits retain bounds)

If $\{s_n\}$ is a convergent sequence then

$$\alpha \leq s_n \leq \beta \quad \forall n \in \mathbb{N} \qquad \Longrightarrow \qquad \alpha \leq \lim_{n\to\infty} s_n \leq \beta\,.$$

(solution on board)

Order properties of limits (§2.8)

Theorem (Squeeze Theorem)

If $\{s_n\}$ and $\{t_n\}$ are convergent sequences such that

(i) $s_n \le x_n \le t_n \quad \forall n \in \mathbb{N}$, *($x_n$ is always between them)*

(ii) $\lim_{n\to\infty} s_n = \lim_{n\to\infty} t_n = L$. *(both approach the same limit)*

Then $\{x_n\}$ is convergent and $\lim_{n\to\infty} x_n = L$.

Proof? (What's **WRONG**?).

$\{s_n\}$ and $\{t_n\}$ are both bounded since they both converge. $\{x_n\}$ is therefore bounded (by the lower bound of $\{s_n\}$ and the upper bound of $\{t_n\}$). $\{x_n\}$ therefore converges, say $x_n \to X$. Hence, by order retension, $L \le X \le L \implies X = L$. □

(solution on board)

McMaster University

Mathematics and Statistics

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Mathematics 3A03
Real Analysis I

Instructor: David Earn

Lecture 10
Sequences V
Wednesday 27 September 2017

Announcements

- Typo alert in Assignment 2:
  $$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} s_n = L \quad \text{should have said} \quad \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} s_k = L.$$
  Now corrected on the course wiki.
  Due this Friday, 29 Sep 2017, at 4:25pm.
- I need to leave around 4:40pm on Friday...
  - Friday's lecture will be given by Dr. David Duncan (in HH-109) to both sections of the course.

What we've done so far on sequences

- Definition of convergence.
- Definition of divergence.
- Definition of divergence to $\pm\infty$.
- Every convergent sequence is bounded.
- Harmonic series diverges.
- Algebra of limits (sums, products, quotients).
- Order properties of limits (squeeze theorem).

Today:

- Absolute value and max/min of limits.
- Monotone convergence.

Order properties of limits (§2.8)

Theorem (Limits of Absolute Values)

If $\{s_n\}$ *converges then so does* $\{|s_n|\}$*, and*

$$\lim_{n\to\infty} |s_n| = \left|\lim_{n\to\infty} s_n\right| .$$

(solution on board)

Order properties of limits (§2.8)

Corollary (Max/Min of Limits)

If $\{s_n\}$ *and* $\{t_n\}$ *converge then* $\{\max\{s_n, t_n\}\}$ *and* $\{\min\{s_n, t_n\}\}$ *both converge and*

$$\lim_{n\to\infty} \max\{s_n, t_n\} = \max\left\{\lim_{n\to\infty} s_n, \ \lim_{n\to\infty} t_n\right\},$$

$$\lim_{n\to\infty} \min\{s_n, t_n\} = \min\left\{\lim_{n\to\infty} s_n, \ \lim_{n\to\infty} t_n\right\}.$$

Idea for proof:

$$\forall x, y \in \mathbb{R} \quad \max\{x, y\} = \frac{x+y}{2} + \frac{|x-y|}{2}$$

$$\forall x, y \in \mathbb{R} \quad \min\{x, y\} = \frac{x+y}{2} - \frac{|x-y|}{2}$$

Prove these facts, then use theorems on sums and absolute values of limits.

Monotone convergence (§2.9)

Definition (Monotonic sequence)

The sequence $\{s_n\}$ is **monotonic** iff it satisfies any of the following conditions:

(i) **Increasing**: $s_1 < s_2 < s_3 < \cdots < s_n < s_{n+1} < \cdots$;

(ii) **Decreasing**: $s_1 > s_2 > s_3 > \cdots > s_n > s_{n+1} > \cdots$;

(iii) **Non-decreasing**: $s_1 \leq s_2 \leq s_3 \leq \cdots \leq s_n \leq s_{n+1} \leq \cdots$;

(iv) **Non-increasing**: $s_1 \geq s_2 \geq s_3 \geq \cdots \geq s_n \geq s_{n+1} \geq \cdots$.

Monotone convergence (§2.9)

Theorem (Monotone Convergence Theorem)

A monotonic sequence $\{s_n\}$ is convergent iff it is bounded.
In particular,

(i) $\{s_n\}$ *non-decreasing and unbounded* $\implies s_n \to \infty$;
(ii) $\{s_n\}$ *non-decreasing and bounded* $\implies s_n \to \sup\{s_n\}$;
(iii) $\{s_n\}$ *non-increasing and unbounded* $\implies s_n \to -\infty$;
(iv) $\{s_n\}$ *non-increasing and bounded* $\implies s_n \to \inf\{s_n\}$.

(solution on board)

Subsequences

Definition (Subsequence)

Let $\{s_1, s_2, s_3, \ldots\}$ be a sequence. If $\{n_1, n_2, n_3, \ldots\}$ is an increasing sequence of natural numbers then $\{s_{n_1}, s_{n_2}, s_{n_3}, \ldots\}$ is a **subsequence** of $\{s_1, s_2, s_3, \ldots\}$.

Example (Subsequences)

Consider the sequence $\{s_n\}$ defined by $s_n = n^2$ for all $n \in \mathbb{N}$. What are the first few terms of these subsequences?

- $\{s_n : n \text{ even}\}$ $\{2^2, 4^2, 6^2, \ldots\}$
- $\{s_n : n = 2k+1,\ \exists k \in \mathbb{N}\}$ $\{3^2, 5^2, 7^2, \ldots\}$
- $\{s_{2n+1}\}$ Same as line above
- $\{s_{2^n}\}$ $\{2^2, 4^2, 8^2, \ldots\}$
- $\{s_{n^2}\}$ $\{1^2, 4^2, 9^2, \ldots\}$

Subsequences

Given any sequence $\{s_n\}$, can you always find a subsequence that is monotonic?

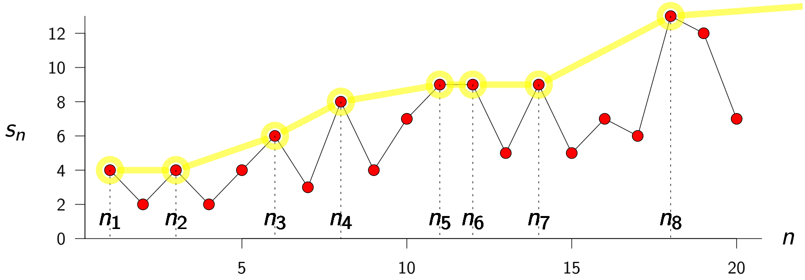
Theorem

Every sequence contains a monotonic subsequence.

Let's draw some pictures to help us visualize how we might construct a proof. . .

Idea for proof that every sequence contains a monotonic subsequence ("point of no return")

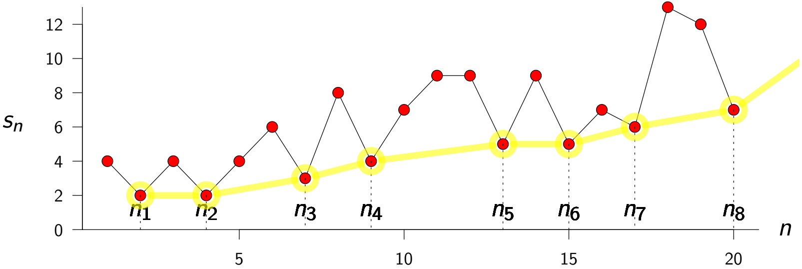
Given a sequence $\{s_1, s_2, s_3, \ldots\}$, try to build a subsequence $\{s_{n_1}, s_{n_2}, s_{n_3}, \ldots\}$ that is non-decreasing ($s_{n_1} \le s_{n_2} \le s_{n_3} \le \cdots$) by discarding any terms that are less than the running maximum:



If this works indefinitely then we have a non-decreasing subsequence. But if we can find only finitely many such terms then we're stuck because our subsequence is defined using earlier terms.

Better idea for proof that every sequence contains a monotonic subsequence ("turn-back point")

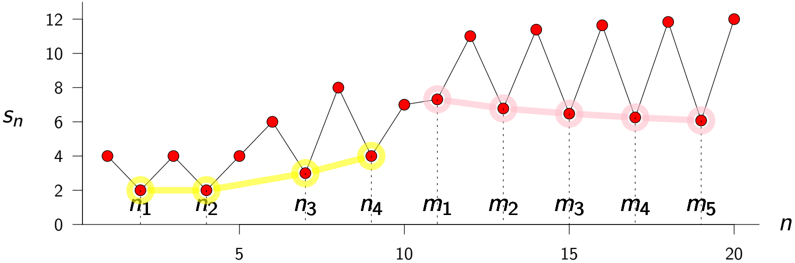
Given a sequence $\{s_1, s_2, s_3, \ldots\}$, try to build a subsequence $\{s_{n_1}, s_{n_2}, s_{n_3}, \ldots\}$ that is non-decreasing ($s_{n_1} \le s_{n_2} \le s_{n_3} \le \cdots$) by identifying terms that are less than or equal to all later terms.



If this works indefinitely then we have a non-decreasing subsequence. What if there are only finitely many such terms? (There might not be any at all!)

Better idea for proof that every sequence contains a monotonic subsequence ("turn-back point")

If there are only finitely many s_{n_i} such that $s_{n_i} \leq s_n \ \forall n > n_i$ ...



... then after the last "turn-back point" (s_{n_4} above) there must be some $m_1 > n_4$ such that s_{m_1} is **not** $\leq$ all later terms, *i.e.*, $\exists m_2 > m_1$ with $s_{m_2} < s_{m_1}$, and similarly for m_2, so there must be a decreasing subsequence $s_{m_1} > s_{m_2} > s_{m_3} > \cdots$

McMaster University

Mathematics and Statistics

$$\int_M d\omega = \int_{\partial M} \omega$$

Mathematics 3A03
Real Analysis I

Instructor: David Earn

Lecture 11
Sequences VI
Friday 29 September 2017

Announcements

- This lecture was given by David Duncan, using the board only. The slides that follow summarize what was covered in this class.

Subsequences

Theorem (Monotone Convergence Theorem)

Every bounded monotone sequence converges.

(last time)

Theorem

Every sequence contains a monotonic subsequence.

(last time)

Subsequences

Theorem (Bolzano-Weierstrass theorem)

Every bounded sequence contains a convergent subsequence.

Proof.

Suppose $\{x_n\}$ is a bounded sequence. It follows from the previous theorem that $\{x_n\}$ contains a subsequence $\{x_{m_k}\}$ that is monotone. Since $\{x_n\}$ is bounded, the subsequence $\{x_{m_k}\}$ is bounded as well (by the same bound). Then $\{x_{m_k}\}$ is a subsequence of $\{x_n\}$ that is bounded and monotone. Hence, it converges by the Monotone Convergence Theorem. □

Cauchy sequences

Definition (Cauchy sequence)

A sequence $\{s_n\}$ is said to be a **Cauchy sequence** iff for all $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that if $m \geq N$ and $n \geq N$ then $|s_n - s_m| < \varepsilon$.

Theorem (Cauchy criterion)

A sequence of real numbers $\{s_n\}$ is convergent iff it is a Cauchy sequence.

Remark: The proof of the "only if" direction is easy. The proof of the "if" direction contains only one tricky feature: showing that every Cauchy sequence $\{s_n\}$ is bounded.

Cauchy sequences

Proof of Cauchy criterion

"only if": If $\{s_n\}$ is converges then, given $\varepsilon > 0$, choose $N \in \mathbb{N}$ such that for all $n \geq N$, $|s_n - L| < \varepsilon/2$. Then, for any $m, n > N$ we have $|s_m - s_n| = |s_m - L + L - s_n| \leq |s_m - L| + |s_n - L| < (\varepsilon/2) + (\varepsilon/2) = \varepsilon$.

"if": If we take $\varepsilon = 1$ in the definition of a Cauchy sequence, we find that there is some N such that $|s_m - s_n| < 1$ for all $m, n > N$. In particular, this means that $|s_m - s_{N+1}| < 1$ for all $m > N$. Thus $\{s_m : m > N\}$ is bounded; since there are only finitely many other s_i's the whole sequence is bounded. Hence, by the Bolzano-Weierstrass theorem, some subsequence of s_n converges; let's write this subsequence as s_{m_k}, and its limit as L.

. . . continued on next slide. . .

Cauchy sequences

Proof of Cauchy criterion (cont'd).

We will show that $\{s_n\}$ converges to L. To prove this, let $\epsilon > 0$. Since the sequence $\{s_n\}$ is Cauchy, there is some N so that

$$|s_n - s_m| < \epsilon/2$$

for all $n, m \geq N$. Since the subsequence converges to L, there is some N' so that

$$|s_{m_k} - L| < \epsilon/2$$

for all $k \geq N'$. Fix an integer k satisfying $k \geq N'$ and $m_k \geq N$. Then if $n \geq N$, we have

$$|s_n - L| \leq |s_n - s_{m_k} + s_{m_k} - L| \leq |s_n - s_{m_k}| + |s_{m_k} - L| < \epsilon/2 + \epsilon/2 = \epsilon.$$

□

Cauchy sequences

Notes:

- The Cauchy criterion is sometimes easier to use in proofs than the original definition of convergence.
- Its significance is more evident in spaces other than $\mathbb{R}$, where Cauchy sequences do not necessarily converge.

Bijectivity

Definition (Bijection)

Let $f : A \to B$ be a function. Then

(i) f is **injective** (or **one-to-one**) if $\forall a_1, a_2 \in A$, $f(a_1) = f(a_2) \implies a_1 = a_2$.

(ii) f is **surjective** (or **onto**) if $\forall b \in B, \exists a \in A$ so that $f(a) = b$.

(iii) Then f is **bijective** if it is both **injective** and **surjective**.

A bijective function is said to be a **bijection**.

Note: A bijection is sometimes called a **one-to-one correspondence**. This termniology can be confusing because it means one-to-one **and** onto.

Countability

Definition (Countable set)

A set S is **countable** if S is finite, or if there is a bijective function $f : \mathbb{N} \to S$.
A set is **uncountable** if it is not countable.

Sequences $\{s_n\}_{n=1}^{\infty}$ are generalizations of the intuitive notion of sets whose elements can be counted.

Example

Suppose S is a subset of $\mathbb{R}$. Then S is countable if and only if S is the range of a sequence.

Countability

Theorem

The natural numbers $\mathbb{N}$ are countable.

(solution on board)

Theorem

The integers $\mathbb{Z}$ are countable.

(solution on board)

McMaster University

Mathematics and Statistics

$$\int_M d\omega = \int_{\partial M} \omega$$

Mathematics 3A03
Real Analysis I

Instructor: David Earn

Lecture 12
Sequences VII
Monday 2 October 2017

Announcements

- Slides corresponding to the blackboard lecture given by Dr. Duncan on Friday have been posted.
- Assignment 3 is posted.
  Due Friday 20 Oct 2017 at 4:25pm.

  Note:
  - Starting with Assignment 3, you are required to have a cover sheet on your assignments.
  - You can use the posted template, or you can do it yourself, but all the information must be given in large clear text.
  - If the information is complete and you have stapled the cover sheet to your assignment then you will receive an additional 2 marks (all or nothing).

Sequences Finale!

Countability

Sequences $\{s_n\}_{n=1}^{\infty}$ are generalizations of the intuitive notion of sets whose elements can be counted.

Key notions defined last time:

- countable
- uncountable

Key results discussed last time:

- $\mathbb{N}$ is countable
- $\mathbb{Z}$ is countable
- A set $S \neq \varnothing$ is countable $\iff$ S is the range of a sequence.

Theorem

The rational numbers $\mathbb{Q}$ are countable.

(solution on board)

What about $\mathbb{R}$?...

Countability

Theorem (Cantor)

The real numbers $\mathbb{R}$ are uncountable.

(solution on board)

Notes:

- The main argument in the proof is known as "Cantor's diagonal argument".
- We can infer that not only are some real numbers not rational, but there are "many more" real numbers than rational numbers.
- Cantor's proof depends on there being a binary expansion for any real number number...

Countability

Theorem (Existence and uniqueness of binary expansions)

If $x \in [0,1)$ then there is a sequence $\{a_n\}$ such that $a_n \in \{0,1\}$ $\forall n$ and

$$x = \sum_{n=1}^{\infty} \frac{a_n}{2^n}\,.$$

Specifically,
$$a_n = \left\lfloor \left(x - \sum_{i=1}^{n-1} \frac{a_i}{2^i} \right) 2^n \right\rfloor\,.$$

Moreover, this **binary representation** *is unique unless $x = m/2^k$ for some $k \in \mathbb{N}$ and $m \in \mathbb{N}$, in which case there are exactly two binary representations, the second being given by $\{b_n\}$ where*

$$b_n = \begin{cases} a_n & n < k, \\ 0 & n = k, \\ 1 & n > k. \end{cases}$$

Countability

Theorem (Properties of countable sets)

(i) *Any subset of a countable set is countable.*

(ii) *The union of a sequence of countable sets is countable.*

(iii) *No (non-degenerate*) interval is countable.*

* We normally assume implicitly that the endpoints of intervals are distinct. If the endpoints are the same then the interval is **degenerate**, *e.g.*, $(a, a) = \varnothing$ and $[a, a] = \{a\}$.

Topology of $\mathbb{R}$

Intervals

Open interval:

$$(a, b) = \{x : a < x < b\}$$

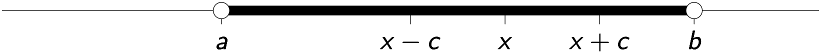
Closed interval:

$$[c, d] = \{x : c \leq x \leq d\}$$

Half-open interval:

$$(e, f] = \{x : e < x \leq f\}$$

Interior point



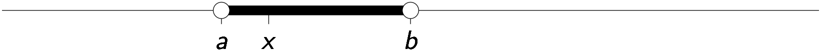
Definition (Interior point)

If $E \subseteq \mathbb{R}$ then x is an **interior point** of E if x lies in an open interval that is contained in E, *i.e.*, $\exists c > 0$ such that $(x - c, x + c) \subset E$.

Interior point examples

Set E	**Interior points?**
$(-1,1)$	Every point
$[0,1]$	Every point *except the endpoints*
$\mathbb{N}$	$\nexists$
$\mathbb{R}$	Every point
$\mathbb{Q}$	$\nexists$
$(-1,1)\cup[0,1]$	Every point *except 1*
$(-1,1)\setminus\{\frac{1}{2}\}$	Every point

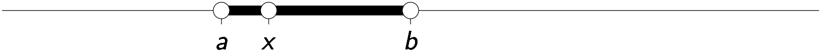
Neighbourhood



Definition (Neighbourhood)

A **neighbourhood** of a point $x \in \mathbb{R}$ is an open interval containing x.

Deleted neighbourhood

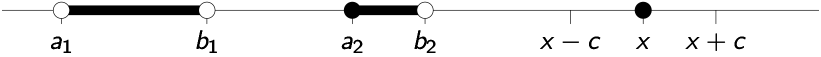


Definition (Deleted neighbourhood)

A **deleted neighbourhood** of a point $x \in \mathbb{R}$ is a set formed by removing x from a neighbourhood of x.

$$(a, b) \setminus \{x\}$$

Isolated point



$$E = (a_1, b_1) \cup [a_2, b_2) \cup \{x\}$$

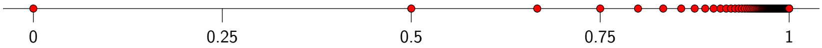
Definition (Isolated point)

If $x \in E \subseteq \mathbb{R}$ then x is an **isolated point** of E if there is a neighbourhood of x for which the only point in E is x itself, *i.e.*, $\exists c > 0$ such that $(x - c, x + c) \cap E = \{x\}$.

Isolated point examples

Set E	**Isolated points?**
$(-1,1)$	$\nexists$
$[0,1]$	$\nexists$
$\mathbb{N}$	Every point
$\mathbb{R}$	$\nexists$
$\mathbb{Q}$	$\nexists$
$(-1,1)\cup[0,1]$	$\nexists$
$(-1,1)\setminus\{\frac{1}{2}\}$	$\nexists$

Accumulation point



$$E = \left\{ 1 - \frac{1}{n} : n \in \mathbb{N} \right\}$$