

Math 3A03 Handout

The (Volterra) Integral Operator T on $C[a, b]$, Continuity of T with respect to the Standard p -Norms, and Failure for More Exotic Structures

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This handout gives details about a few issues discussed in class, and digs a bit deeper. I prepared the handout by prompting ChatGPT to write a \LaTeX document, iterating until I had most of the content of the document I wanted, and then editing for a couple of hours on my own.

–David Earn

Introduction

Let

$$C[a, b] := \{f : [a, b] \rightarrow \mathbb{R} : f \text{ is continuous}\}$$

be the vector space of all real-valued continuous functions on the closed interval $[a, b]$.

In this handout we study the *Volterra integral operator*

$$(Tf)(x) := \int_a^x f(t) dt.$$

This is one of the simplest and most important examples of an integral operator: it assigns to each function f the accumulated area under the graph of f from a to x .

Our goals are:

1. to prove that T is continuous with respect to every usual p -norm on $C[a, b]$;
2. to explain that if one allows arbitrary metrics, then producing a metric for which T is not continuous is very easy;
3. to show that even if one insists on a *norm*, there still exists a norm on $C[a, b]$ for which T is not continuous;

4. to explain the algebraic idea behind that norm, namely a *Hamel basis*.

Result 1 is standard analysis. Since $f(\alpha f + \beta g) = \alpha f f + \beta f g$, points 2 and 3 imply that a linear operator on $C[a, b]$ need not be continuous (which is what you might guess for something that is “linear”). Point 4 is a good illustration of how infinite-dimensional vector spaces can behave in ways that are very different from finite-dimensional spaces.

1 The usual p -norms on $C[a, b]$

For $1 \leq p < \infty$, define

$$\|f\|_p := \left(\int_a^b |f(x)|^p dx \right)^{1/p},$$

and define the sup norm by

$$\|f\|_\infty := \max_{x \in [a, b]} |f(x)|.$$

Because f is continuous on a closed interval, all of these quantities are finite.

The norm $\|\cdot\|_\infty$ measures the largest value of $|f|$, while the norms $\|\cdot\|_p$ for $1 \leq p < \infty$ measure the average size of $|f|^p$.

2 The Volterra operator

Definition 1. *The Volterra integral operator is the map $T : C[a, b] \rightarrow C[a, b]$ defined by*

$$(Tf)(x) := \int_a^x f(t) dt.$$

If $f \in C[a, b]$, then Tf is again continuous (by the “integrals are continuous” theorem), and in fact the Fundamental Theorem of Calculus tells us that Tf is differentiable and

$$(Tf)'(x) = f(x).$$

3 Continuity of T with respect to every p -norm

We first record a standard fact about linear operators between normed spaces.

Lemma 1. *Let X and Y be normed vector spaces, and let $S : X \rightarrow Y$ be linear. If there exists a constant $K \geq 0$ such that*

$$\|Sf\|_Y \leq K \|f\|_X \quad \text{for all } f \in X,$$

then S is continuous.

Proof. To prove continuity at an arbitrary point $f_0 \in X$, let $\varepsilon > 0$ be given and choose

$$\delta := \frac{\varepsilon}{K + 1}.$$

If $\|f - f_0\|_X < \delta$, then by linearity,

$$\|Sf - Sf_0\|_Y = \|S(f - f_0)\|_Y \leq K \|f - f_0\|_X < K\delta < \varepsilon.$$

Thus S is continuous at f_0 . Since f_0 was arbitrary, S is continuous on X . □

Remark 1. *The inequality*

$$\|Sf\|_Y \leq K \|f\|_X \quad \text{for all } f \in X$$

says that the operator S can enlarge vectors by at most the fixed factor K . For this reason, one says that S is a bounded operator.

In more advanced courses in analysis, one defines a norm on the space of linear operators from X to Y , namely

$$\|S\|_{\text{op}} := \sup_{\|f\|_X \leq 1} \|Sf\|_Y,$$

and this terminology becomes even more natural: the condition above is exactly the statement that this supremum is finite. In other words, a linear operator is called bounded precisely when it has finite operator norm.

For linear maps between normed spaces, the important fact is that

$$\text{boundedness} \iff \text{continuity}.$$

So in practice, proving an estimate of the form

$$\|Sf\|_Y \leq K \|f\|_X$$

is the standard way to prove that a linear operator is continuous.

We now prove the main analytic fact, exploiting the Lemma above and the fact that the integral is a linear operator.

Theorem 1. *For every $1 \leq p \leq \infty$, the Volterra operator*

$$T : (C[a, b], \|\cdot\|_p) \rightarrow (C[a, b], \|\cdot\|_p)$$

is continuous. More precisely,

$$\|Tf\|_p \leq (b - a) \|f\|_p \quad \text{for all } f \in C[a, b].$$

Proof. We treat the cases $p = \infty$ and $1 \leq p < \infty$ separately.

First suppose $p = \infty$. For each $x \in [a, b]$,

$$|(Tf)(x)| = \left| \int_a^x f(t) dt \right| \leq \int_a^x |f(t)| dt \leq \int_a^x \|f\|_\infty dt \leq (b - a) \|f\|_\infty.$$

Taking the maximum over x gives

$$\|Tf\|_\infty \leq (b - a) \|f\|_\infty.$$

Now suppose $1 \leq p < \infty$. For each $x \in [a, b]$,

$$|(Tf)(x)| \leq \int_a^x |f(t)| dt \leq \int_a^b |f(t)| dt = \|f\|_1.$$

Hence

$$\|Tf\|_\infty \leq \|f\|_1.$$

On the finite interval $[a, b]$, Hölder's inequality (see the handout on p -norms) implies

$$\|f\|_1 \leq (b-a)^{1-1/p} \|f\|_p.$$

Therefore

$$\|Tf\|_\infty \leq (b-a)^{1-1/p} \|f\|_p.$$

Using the elementary inequality

$$\|g\|_p \leq (b-a)^{1/p} \|g\|_\infty \quad \text{for all } g \in C[a, b],$$

we get

$$\|Tf\|_p \leq (b-a)^{1/p} \|Tf\|_\infty \leq (b-a)^{1/p} (b-a)^{1-1/p} \|f\|_p = (b-a) \|f\|_p.$$

This proves the claim for all $1 \leq p \leq \infty$. □

Remark 2. *The theorem shows that the Volterra operator is continuous for every familiar L^p -type norm on $C[a, b]$. This is why, in ordinary analysis, one thinks of T as a very well-behaved operator.*

4 A simpler question: what if we only ask for a metric?

Before looking for a norm with respect to which T is not continuous, it is worth observing that if one allows *arbitrary metrics*, the question becomes much easier.

Definition 2. *The discrete metric on a set X is defined by*

$$d(x, y) = \begin{cases} 0, & x = y, \\ 1, & x \neq y. \end{cases}$$

Proposition 1. *If the domain $C[a, b]$ is given its usual sup-norm metric and the codomain $C[a, b]$ is given the discrete metric, then the Volterra operator T is not continuous.*

Proof. Consider the sequence of constant functions

$$f_n(x) = \frac{1}{n}.$$

Then $f_n \rightarrow 0_{C[a, b]}$ in the sup norm (where $0_{C[a, b]}$ is the zero function, $0_{C[a, b]}(x) = 0 \forall x$), because

$$\|f_n - 0_{C[a, b]}\|_\infty = \frac{1}{n} \rightarrow 0.$$

However,

$$(Tf_n)(x) = \int_a^x \frac{1}{n} dt = \frac{x-a}{n},$$

which is not the zero function for any n . Thus, in the discrete metric on the codomain,

$$d(Tf_n, T0_{C[a, b]}) = 1 \quad \text{for all } n.$$

So $Tf_n \not\rightarrow T0_{C[a, b]}$. Therefore T is not continuous. □

Alternative proof using inverse images of open sets. Let

$$U := \{0_{C[a,b]}\} \subseteq C[a,b],$$

where the codomain is equipped with the discrete metric. Since every subset of a discrete metric space is open, U is open.

Now

$$T^{-1}(U) = T^{-1}(\{0_{C[a,b]}\}) = \{f \in C[a,b] : Tf = 0_{C[a,b]}\}.$$

By the Fundamental Theorem of Calculus, if $Tf = 0_{C[a,b]}$, then

$$0_{C[a,b]} = (Tf)' = f,$$

so $f = 0_{C[a,b]}$. Thus

$$T^{-1}(U) = \{0_{C[a,b]}\}.$$

But $\{0_{C[a,b]}\}$ is not open in $C[a,b]$ with the sup-norm metric: every sup-norm ball centered at $0_{C[a,b]}$ contains nonzero constant functions. Therefore the inverse image of the open set U is not open, so T is not continuous. \square

Remark 3. *This example is very simple, but it is not very interesting analytically, because the discrete metric has nothing to do with the vector-space structure of $C[a,b]$ or the usual analytic structures on it (e.g., the integral operator). The more interesting question is whether there exists a norm on $C[a,b]$ for which T is not continuous.*

5 Why the norm question is subtler

In finite-dimensional vector spaces, all norms induce the same notion of continuity. Thus, in finite dimensions, if a linear operator is continuous for one norm, then it is continuous for every norm.

In infinite-dimensional spaces this is false. Different norms can give genuinely different topologies, so continuity can depend strongly on the chosen norm.

For the usual norms $\|\cdot\|_p$, the Volterra operator is continuous. But there are much more exotic norms on $C[a,b]$, and with respect to one of those norms the operator T can fail to be continuous.

To build such a norm, we need one algebraic idea.

6 Hamel bases

Definition 3. *Let V be a vector space over \mathbb{R} . A Hamel basis for V is a subset $\mathcal{B} \subseteq V$ such that every vector $v \in V$ can be written in one and only one way as a finite linear combination of elements of \mathcal{B} .*

This is the same notion of basis that you will have seen in finite-dimensional linear algebra; the only difference is that in an infinite-dimensional vector space the basis itself is infinite.

Example 1. In the vector space of polynomials, the set

$$\{1, x, x^2, x^3, \dots\}$$

is a Hamel basis, because every polynomial is a finite linear combination of powers of x .

For the space $C[a, b]$, the existence of a Hamel basis is a standard theorem from abstract linear algebra (usually proved using the axiom of choice). In practice one cannot write such a basis down explicitly, but one can still use it to build examples.

The only fact we need is the following.

Proposition 2. Every linearly independent subset of $C[a, b]$ can be extended to a Hamel basis of $C[a, b]$.

We will not prove this here; it is a standard basis extension theorem.

7 A linearly independent family adapted to the Volterra operator

Define, for $n = 0, 1, 2, \dots$,

$$e_n(x) := \left(\frac{x-a}{b-a}\right)^{2n}.$$

Then each e_n belongs to $C[a, b]$, and

$$\|e_n\|_\infty = 1.$$

Applying the Volterra operator gives

$$(Te_n)(x) = \int_a^x \left(\frac{t-a}{b-a}\right)^{2n} dt = \frac{b-a}{2n+1} \left(\frac{x-a}{b-a}\right)^{2n+1}.$$

Thus e_n is an even power of $\frac{x-a}{b-a}$, while Te_n is a constant multiple of an odd power.

Lemma 2. The family

$$\{e_0, e_1, e_2, \dots\} \cup \{Te_0, Te_1, Te_2, \dots\}$$

is linearly independent.

Proof. Suppose a finite linear combination is zero:

$$\sum_{k=0}^N \alpha_k e_k + \sum_{k=0}^N \beta_k Te_k = 0.$$

Substituting the formulas for e_k and Te_k gives a polynomial identity in the variable

$$y := \frac{x-a}{b-a}.$$

The first sum involves only even powers y^{2k} , while the second sum involves only odd powers y^{2k+1} . A polynomial is identically zero only if all its coefficients are zero, so all the α_k and β_k must vanish. Hence the family is linearly independent. \square

8 The general Hamel-basis strategy

The Hamel-basis construction is not tied specifically to the sup norm. In fact, once one fixes any norm N on $C[a, b]$, one can often build a new norm of the form

$$\|f\|_{N,*} := N(f) + |L(f)|$$

for which a given linear operator fails to be continuous.

For the Volterra operator, this works starting from each of the usual p -norms. To keep the presentation concrete, we first carry out the construction using $\|\cdot\|_\infty$, and then explain the modification needed for a general p -norm.

9 Constructing an exotic norm using a Hamel basis

Since the family from the lemma is linearly independent, it can be extended to a Hamel basis \mathcal{B} of $C[a, b]$.

We now define a linear functional on that basis.

Definition 4. Define $L : C[a, b] \rightarrow \mathbb{R}$ on the Hamel basis \mathcal{B} by prescribing

$$L(e_n) = 0, \quad L(Te_n) = n \quad (n = 0, 1, 2, \dots),$$

and setting

$$L(b) = 0 \quad \text{for every other basis vector } b \in \mathcal{B}.$$

Then extend L linearly to all of $C[a, b]$.

This is well-defined because every element of $C[a, b]$ is a finite linear combination of basis vectors.

Now define

$$\|f\|_* := \|f\|_\infty + |L(f)|.$$

Proposition 3. The formula

$$\|f\|_* := \|f\|_\infty + |L(f)|$$

defines a norm on $C[a, b]$.

Proof. Nonnegativity and homogeneity are immediate. The triangle inequality follows from the triangle inequality for $\|\cdot\|_\infty$ and the linearity of L :

$$\|f + g\|_* = \|f + g\|_\infty + |L(f + g)| \leq \|f\|_\infty + \|g\|_\infty + |L(f)| + |L(g)| = \|f\|_* + \|g\|_*.$$

Finally, if $\|f\|_* = 0$, then in particular $\|f\|_\infty = 0$, so $f = 0$. □

10 The counterexample norm based on $\|\cdot\|_\infty$

We now show that the Volterra operator is not continuous with respect to $\|\cdot\|_*$.

Theorem 2. *There exists a norm on $C[a, b]$ such that the Volterra operator*

$$T : C[a, b] \rightarrow C[a, b]$$

is not continuous.

Proof. Take the norm $\|\cdot\|_*$ constructed above.

For each n , we have $L(e_n) = 0$ and $\|e_n\|_\infty = 1$, so

$$\|e_n\|_* = 1.$$

On the other hand,

$$\|Te_n\|_* = \|Te_n\|_\infty + |L(Te_n)|.$$

Now

$$L(Te_n) = n,$$

and from the explicit formula for Te_n we get

$$\|Te_n\|_\infty = \frac{b-a}{2n+1}.$$

Therefore

$$\|Te_n\|_* = \frac{b-a}{2n+1} + n.$$

Hence

$$\frac{\|Te_n\|_*}{\|e_n\|_*} = \frac{b-a}{2n+1} + n \rightarrow \infty.$$

So there cannot exist a constant C such that

$$\|Tf\|_* \leq C \|f\|_* \quad \text{for all } f \in C[a, b].$$

Thus T is unbounded, and since T is linear, it is not continuous. □

11 The same Hamel-basis idea works for every usual p -norm

The preceding construction used the sup norm as the “honest” analytic norm and then added the algebraically defined term $|L(f)|$. The same idea works if one starts from any other usual p -norm on $C[a, b]$.

Fix $1 \leq p < \infty$. Define

$$u_n(x) := \left(\frac{2np+1}{b-a}\right)^{1/p} \left(\frac{x-a}{b-a}\right)^{2n}.$$

Then a direct calculation shows that

$$\|u_n\|_p^p = \frac{2np+1}{b-a} \int_a^b \left(\frac{x-a}{b-a}\right)^{2np} dx = 1,$$

so

$$\|u_n\|_p = 1.$$

Moreover, each u_n is just a nonzero scalar multiple of e_n , so the family

$$\{u_0, u_1, u_2, \dots\} \cup \{Tu_0, Tu_1, Tu_2, \dots\}$$

is still linearly independent. Extend it to a Hamel basis, define a linear functional L_p by

$$L_p(u_n) = 0, \quad L_p(Tu_n) = n,$$

and let L_p vanish on all the other basis vectors. Then define

$$\|f\|_{p,*} := \|f\|_p + |L_p(f)|.$$

This is a norm on $C[a, b]$, and

$$\|u_n\|_{p,*} = 1, \quad \|Tu_n\|_{p,*} \geq |L_p(Tu_n)| = n.$$

Therefore T is not continuous with respect to $\|\cdot\|_{p,*}$.

So the Hamel-basis construction can be carried out starting from any of the standard norms $\|\cdot\|_p$ on $C[a, b]$, not only from $\|\cdot\|_\infty$.

12 What this example means

The moral is not that the Volterra operator is badly behaved. Quite the opposite: with respect to all the familiar analytic norms, it is very well behaved.

Rather, the lesson is that in infinite-dimensional spaces there are many norms besides the familiar ones, and some of them can be tailored to force a given linear operator to fail continuity.

The norm $\|\cdot\|_*$ above is not natural from the viewpoint of analysis. It is built by means of a Hamel basis, which is an algebraic object rather than an analytic one. This is why such counterexamples rarely appear in introductory courses: they are conceptually useful, but they do not reflect the usual geometric or analytic structure of function spaces.

13 The space $C_b(I)$ on an arbitrary interval

It is natural to ask what changes if one replaces $C[a, b]$ by the space

$$C_b(I) := \{f : I \rightarrow \mathbb{R} : f \text{ is continuous and bounded}\}$$

on an arbitrary interval I .

Bounded intervals

If $I = [a, b]$ is a bounded interval, then $C_b(I) = C[a, b]$, so everything in this handout applies without change.

Unbounded intervals

If I is unbounded, the situation changes immediately. For a fixed base point $x_0 \in I$, define formally

$$(Tf)(x) := \int_{x_0}^x f(t) dt.$$

Even when $f \in C_b(I)$, the function Tf need not be bounded. For example, if $f \equiv 1$, then

$$(Tf)(x) = x - x_0,$$

which is unbounded on an unbounded interval. Thus, on such an interval, the Volterra operator is generally *not even a map*

$$T : C_b(I) \rightarrow C_b(I).$$

So the basic boundedness question already fails before one asks about continuity.

There is also an important difference for the finite p -norms. On an unbounded interval, the expression

$$\|f\|_p = \left(\int_I |f(x)|^p dx \right)^{1/p}$$

need not be finite for every bounded continuous function. For instance, the constant function 1 belongs to $C_b(I)$, but if I has infinite length, then

$$\|1\|_p = \left(\int_I 1^p dx \right)^{1/p}$$

is not finite. Thus, for unbounded intervals, $\|\cdot\|_p$ with $p < \infty$ is generally *not* a norm on all of $C_b(I)$.

By contrast, the sup norm

$$\|f\|_\infty := \sup_{x \in I} |f(x)|$$

is always a norm on $C_b(I)$. On a bounded interval one has the estimate

$$\|Tf\|_\infty \leq |I| \|f\|_\infty,$$

but on an unbounded interval there is no such uniform estimate, because the “length” of the interval is infinite. This is exactly why the example $f \equiv 1$ causes trouble.

14 Summary

We have shown the following.

1. For every $1 \leq p \leq \infty$, the Volterra operator satisfies

$$\|Tf\|_p \leq (b - a) \|f\|_p,$$

so it is continuous on $(C[a, b], \|\cdot\|_p)$.

2. If one allows arbitrary metrics, then there are trivial counterexamples: for example, if the codomain is given the discrete metric, T is not continuous.

3. If one insists on a norm, the question becomes subtler. Nevertheless, by using a Hamel basis one can build a norm $\|\cdot\|_*$ on $C[a, b]$ such that T is not continuous.

Thus the correct conclusion is:

The Volterra operator is continuous for all the standard p -norms on $C[a, b]$, but not for all possible norms or metrics.