Math 3A03 Handout

From $\mathbb{R}^n$ to ℓ^p

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In the previous handout *The p-Norms on* $\mathbb{R}^n$, we proved that the p-norms are norms on $\mathbb{R}^n$. The proof for sequence spaces is very similar, but there are two new issues: first, one must define the correct domain, and second, one must justify the transition from finite sums to infinite series. I asked ChatGPT to write this handout, emphasizing only the new points.

–David Earn

This handout assumes that you have just read the proof that the p-norms are norms on $\mathbb{R}^n$. We will therefore not repeat the whole argument. Instead, we explain what changes in the infinite-dimensional setting.

1. The first new point: the domain

In $\mathbb{R}^n$, every vector has only finitely many coordinates, so the quantity

$$\left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p}$$

is automatically finite. For an infinite sequence, this is no longer true. Thus the first issue is that for $1 \le p < \infty$, the p-norm is *not* defined on all bounded sequences.

Definition 1. Let $1 \le p < \infty$. We define

$$\ell^p := \left\{ x = (x_1, x_2, \dots) \in \mathbb{R}^{\mathbb{N}} : \sum_{n=1}^{\infty} |x_n|^p < \infty \right\}.$$

For $x \in \ell^p$, we define

$$\|x\|_p := \left(\sum_{n=1}^{\infty} |x_n|^p\right)^{1/p}.$$

We also define

$$\ell^\infty := \left\{ x = (x_1, x_2, \dots) \in \mathbb{R}^{\mathbb{N}} : \sup_{n \geq 1} |x_n| < \infty \right\},$$

and for $x \in \ell^\infty$,

$$\|x\|_\infty := \sup_{n \geq 1} |x_n|.$$

Remark. The constant sequence

$$(1, 1, 1, \dots)$$

belongs to ℓ^∞, but not to ℓ^p for any finite p, since

$$\sum_{n=1}^{\infty} 1 = \infty.$$

So for $1 \leq p < \infty$, the formula for $\|x\|_p$ defines a norm on ℓ^p, not on all of ℓ^∞.

2. The second new point: finite sums versus infinite series

In the $\mathbb{R}^n$ proof, the key tool was Hölder's inequality for finite sums. The corresponding statement for sequences is obtained by applying the finite-dimensional version to partial sums and then letting $N \to \infty$.

Lemma 1 (Hölder's inequality for series)**.** *Let $1 < p < \infty$, and let q be defined by*

$$\frac{1}{p} + \frac{1}{q} = 1.$$

If $x = (x_n) \in \ell^p$ and $y = (y_n) \in \ell^q$, then

$$\sum_{n=1}^{\infty} |x_n y_n| \leq \left(\sum_{n=1}^{\infty} |x_n|^p \right)^{1/p} \left(\sum_{n=1}^{\infty} |y_n|^q \right)^{1/q}.$$

Proof. For each $N \in \mathbb{N}$, apply the finite-dimensional Hölder inequality to the first N terms:

$$\sum_{n=1}^{N} |x_n y_n| \leq \left(\sum_{n=1}^{N} |x_n|^p \right)^{1/p} \left(\sum_{n=1}^{N} |y_n|^q \right)^{1/q}.$$

Since the partial sums are increasing,

$$\sum_{n=1}^{N} |x_n|^p \leq \sum_{n=1}^{\infty} |x_n|^p, \qquad \sum_{n=1}^{N} |y_n|^q \leq \sum_{n=1}^{\infty} |y_n|^q,$$

so

$$\sum_{n=1}^{N} |x_n y_n| \leq \left(\sum_{n=1}^{\infty} |x_n|^p \right)^{1/p} \left(\sum_{n=1}^{\infty} |y_n|^q \right)^{1/q}.$$

Thus the partial sums of $\sum |x_n y_n|$ are increasing and uniformly bounded above, so the series converges and

$$\sum_{n=1}^{\infty} |x_n y_n| \leq \left(\sum_{n=1}^{\infty} |x_n|^p \right)^{1/p} \left(\sum_{n=1}^{\infty} |y_n|^q \right)^{1/q}.$$

□

3. What stays the same

Once Lemma 1 is available, the proof that $\|\cdot\|_p$ is a norm on ℓ^p is almost identical to the proof on $\mathbb{R}^n$.

Theorem 1. *For every p with $1 \leq p < \infty$, the function $\|\cdot\|_p$ is a norm on ℓ^p.*

Proof. We check the three norm properties.

Positive definiteness. If $x = (x_n) \in \ell^p$, then each term $|x_n|^p$ is nonnegative, so

$$\|x\|_p \geq 0.$$

Also,

$$\|x\|_p = 0 \iff \sum_{n=1}^{\infty} |x_n|^p = 0.$$

Since this is a series of nonnegative terms, it can equal 0 only if every term is 0. Hence $x_n = 0$ for all n, so $x = 0$.

Absolute homogeneity. For $a \in \mathbb{R}$,

$$\|ax\|_p = \left(\sum_{n=1}^{\infty} |ax_n|^p \right)^{1/p} = |a| \left(\sum_{n=1}^{\infty} |x_n|^p \right)^{1/p} = |a| \, \|x\|_p .$$

Triangle inequality when $p = 1$. For each n,

$$|x_n + y_n| \leq |x_n| + |y_n|.$$

Summing over n gives

$$\|x + y\|_1 = \sum_{n=1}^{\infty} |x_n + y_n| \leq \sum_{n=1}^{\infty} |x_n| + \sum_{n=1}^{\infty} |y_n| = \|x\|_1 + \|y\|_1 .$$

Triangle inequality when $1 < p < \infty$. Let q be the conjugate exponent. Exactly as in the $\mathbb{R}^n$ proof,

$$\|x + y\|_p^p = \sum_{n=1}^{\infty} |x_n + y_n|^p = \sum_{n=1}^{\infty} |x_n + y_n| \, |x_n + y_n|^{p-1}.$$

Using $|x_n + y_n| \le |x_n| + |y_n|$, we obtain

$$\|x+y\|_p^p \le \sum_{n=1}^{\infty} |x_n|\, |x_n+y_n|^{p-1} + \sum_{n=1}^{\infty} |y_n|\, |x_n+y_n|^{p-1}.$$

Apply Lemma 1 to each sum. Since

$$(p-1)q = p, \qquad \frac{p}{q} = p-1,$$

we get

$$\sum_{n=1}^{\infty} |x_n|\, |x_n+y_n|^{p-1} \le \|x\|_p \, \|x+y\|_p^{p-1},$$

and similarly

$$\sum_{n=1}^{\infty} |y_n|\, |x_n+y_n|^{p-1} \le \|y\|_p \, \|x+y\|_p^{p-1}.$$

Therefore

$$\|x+y\|_p^p \le \left(\|x\|_p + \|y\|_p\right) \|x+y\|_p^{p-1}.$$

If $\|x+y\|_p = 0$, there is nothing to prove. Otherwise, divide by $\|x+y\|_p^{p-1}$ to obtain

$$\|x+y\|_p \le \|x\|_p + \|y\|_p.$$

□

4. The sup norm

For ℓ^∞, nothing essentially new happens. The proof is exactly the same as in $\mathbb{R}^n$, except that max is replaced by sup.

Theorem 2. *The function $\|\cdot\|_\infty$ is a norm on ℓ^∞.*

Proof. Let $x = (x_n) \in \ell^\infty$.

Positive definiteness. Clearly $\|x\|_\infty = \sup_{n\ge 1} |x_n| \ge 0$. Also,

$$\|x\|_\infty = 0 \iff |x_n| = 0 \text{ for every } n \iff x = 0.$$

Absolute homogeneity. For $a \in \mathbb{R}$,

$$\|ax\|_\infty = \sup_{n\ge1} |ax_n| = |a| \sup_{n\ge1} |x_n| = |a| \, \|x\|_\infty.$$

Triangle inequality. For each n,

$$|x_n + y_n| \le |x_n| + |y_n| \le \|x\|_\infty + \|y\|_\infty.$$

Taking the supremum over $n \ge 1$ gives

$$\|x+y\|_\infty \le \|x\|_\infty + \|y\|_\infty.$$

□

Remark. So the main new ideas are these:

1. for $1 \le p < \infty$, the natural domain is ℓ^p, not ℓ^∞;
2. finite-sum inequalities must be converted into statements about infinite series, usually by applying them to partial sums and passing to the limit.

Other than that, the proof is essentially the same as in $\mathbb{R}^n$.