Math 3A03 Handout

The p-Norms on $C[a,b]$

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The next natural step after $\mathbb{R}^n$ and ℓ^p would be to consider function spaces such as L^p. Since we have not discussed the Lebesgue integral, a good intermediate step is to work with continuous functions on a closed interval, where everything can be done using the ordinary Riemann integral. I asked ChatGPT to produce a handout giving a complete proof, but without repeating more than necessary from the earlier handouts. I coached it to add some remarks at the end about the inadequacy of the Riemman integral in this context, and roughly speaking how the Lebesgue theory handled the issues that the Riemann theory cannot.

–David Earn

In $\mathbb{R}^n$, the p-norms are defined by finite sums. In ℓ^p, they are defined by infinite series. The next natural analogue is to replace sums by integrals.

Because we have not yet discussed the Lebesgue integral, we work on the space $C[a,b]$ of continuous real-valued functions on a closed interval $[a,b]$. Since continuous functions on $[a,b]$ are Riemann integrable, all the integrals below are well defined.

Definition 1. Let

$$C[a,b] := \{f : [a,b] \to \mathbb{R} : f \text{ is continuous}\}.$$

For $1 \le p < \infty$ and $f \in C[a,b]$, define

$$\|f\|_p := \left(\int_a^b |f(x)|^p \, dx\right)^{1/p}.$$

Also define

$$\|f\|_\infty := \max_{x\in[a,b]} |f(x)|.$$

Definition 2. A *norm* on a real vector space V is a function $\|\cdot\| : V \to [0,\infty)$ such that for all $u, v \in V$ and all $c \in \mathbb{R}$, the following properties hold:

1. *Positive definiteness:*

$$\|u\| \geq 0, \qquad \|u\| = 0 \iff u = 0.$$

2. *Absolute homogeneity:*

$$\|cu\| = |c| \, \|u\| \, .$$

3. *Triangle inequality:*

$$\|u + v\| \leq \|u\| + \|v\| \, .$$

Our goal is to prove that for every $p \in [1, \infty]$, the function $\|\cdot\|_p$ is a norm on $C[a, b]$.

As in the finite-dimensional and sequence cases, the only genuinely difficult part is the triangle inequality for $1 < p < \infty$. The key ingredient is Hölder's inequality for integrals. Its proof is exactly the same as before: one applies Young's inequality pointwise and then integrates.

Lemma 1 (Hölder's inequality for integrals)**.** *Let $1 < p < \infty$, and let q be defined by*

$$\frac{1}{p} + \frac{1}{q} = 1.$$

If $f, g \in C[a, b]$, then

$$\int_a^b |f(x)g(x)| \, dx \leq \left(\int_a^b |f(x)|^p \, dx \right)^{1/p} \left(\int_a^b |g(x)|^q \, dx \right)^{1/q} .$$

Proof. If $f = 0$ or $g = 0$, the inequality is immediate. So assume $f \neq 0$ and $g \neq 0$, and set

$$A := \left(\int_a^b |f(x)|^p \, dx \right)^{1/p}, \qquad B := \left(\int_a^b |g(x)|^q \, dx \right)^{1/q} .$$

Then $A > 0$ and $B > 0$.

For each $x \in [a, b]$, define

$$u(x) := \frac{|f(x)|}{A}, \qquad v(x) := \frac{|g(x)|}{B}.$$

By Young's inequality,

$$u(x)v(x) \leq \frac{u(x)^p}{p} + \frac{v(x)^q}{q} \qquad \text{for all } x \in [a, b].$$

Multiplying by AB, we obtain

$$|f(x)g(x)| \leq AB \left(\frac{u(x)^p}{p} + \frac{v(x)^q}{q} \right).$$

Integrating over $[a, b]$, we get

$$\int_a^b |f(x)g(x)| \, dx \leq AB \left(\frac{1}{p} \int_a^b u(x)^p \, dx + \frac{1}{q} \int_a^b v(x)^q \, dx \right).$$

Now

$$\int_a^b u(x)^p\,dx = \int_a^b \frac{|f(x)|^p}{A^p}\,dx = \frac{1}{A^p}\int_a^b |f(x)|^p\,dx = 1,$$

and similarly

$$\int_a^b v(x)^q\,dx = 1.$$

Hence

$$\int_a^b |f(x)g(x)|\,dx \le AB\left(\frac{1}{p}+\frac{1}{q}\right) = AB.$$

Substituting the definitions of A and B proves the result. $\square$

Theorem 1. *For every $p \in [1,\infty]$, the function $\|\cdot\|_p$ is a norm on $C[a,b]$.*

Proof. We consider separately the cases $1 \le p < \infty$ and $p = \infty$.

Case 1: $1 \le p < \infty$. Let $f \in C[a,b]$. Since $|f(x)|^p \ge 0$ for all $x \in [a,b]$, we have

$$\int_a^b |f(x)|^p\,dx \ge 0,$$

and therefore $\|f\|_p \ge 0$.

Positive definiteness. Suppose first that $\|f\|_p = 0$. Then

$$\int_a^b |f(x)|^p\,dx = 0.$$

We claim that this implies $f = 0$.

Assume, for contradiction, that $f \ne 0$. Then there exists $x_0 \in [a,b]$ such that $f(x_0) \ne 0$. Hence $|f(x_0)| > 0$. By continuity, there exists $\delta > 0$ and a constant $c > 0$ such that

$$|f(x)|^p \ge c$$

for all $x \in [a,b] \cap (x_0-\delta, x_0+\delta)$. Therefore

$$\int_a^b |f(x)|^p\,dx \ge \int_{[a,b]\cap(x_0-\delta,x_0+\delta)} |f(x)|^p\,dx \ge c\cdot \text{length}([a,b]\cap(x_0-\delta,x_0+\delta)) > 0,$$

which is a contradiction. Thus $f = 0$.

Conversely, if $f = 0$, then clearly

$$\int_a^b |f(x)|^p\,dx = 0,$$

so $\|f\|_p = 0$. Hence

$$\|f\|_p = 0 \iff f = 0.$$

Absolute homogeneity. For $c \in \mathbb{R}$,

$$\|cf\|_p = \left(\int_a^b |cf(x)|^p \, dx \right)^{1/p} = \left(\int_a^b |c|^p |f(x)|^p \, dx \right)^{1/p} = |c| \left(\int_a^b |f(x)|^p \, dx \right)^{1/p} = |c| \; \|f\|_p \, .$$

Triangle inequality when $p = 1$. For all $x \in [a, b]$,

$$|f(x) + g(x)| \leq |f(x)| + |g(x)|.$$

Integrating, we obtain

$$\|f + g\|_1 = \int_a^b |f(x) + g(x)| \, dx \leq \int_a^b |f(x)| \, dx + \int_a^b |g(x)| \, dx = \|f\|_1 + \|g\|_1 \, .$$

Triangle inequality when $1 < p < \infty$. Let q be the conjugate exponent, so that

$$\frac{1}{p} + \frac{1}{q} = 1.$$

Then

$$\|f + g\|_p^p = \int_a^b |f(x) + g(x)|^p \, dx = \int_a^b |f(x) + g(x)| \, |f(x) + g(x)|^{p-1} \, dx.$$

Using $|f(x) + g(x)| \leq |f(x)| + |g(x)|$, we get

$$\begin{aligned} \|f + g\|_p^p \leq & \int_a^b |f(x)| \, |f(x) + g(x)|^{p-1} \, dx \\ & + \int_a^b |g(x)| \, |f(x) + g(x)|^{p-1} \, dx. \end{aligned}$$

Apply Hölder's inequality, Lemma 1, to the first integral:

$$\int_a^b |f(x)| \, |f(x) + g(x)|^{p-1} \, dx \leq \left(\int_a^b |f(x)|^p \, dx \right)^{1/p} \left(\int_a^b |f(x) + g(x)|^{(p-1)q} \, dx \right)^{1/q} .$$

Since

$$(p - 1)q = (p - 1)\frac{p}{p - 1} = p,$$

this becomes

$$\int_a^b |f(x)| \, |f(x) + g(x)|^{p-1} \, dx \leq \|f\|_p \; \|f + g\|_p^{p/q} \, .$$

But

$$\frac{p}{q} = p - 1,$$

so

$$\int_a^b |f(x)| \, |f(x) + g(x)|^{p-1} \, dx \leq \|f\|_p \; \|f + g\|_p^{p-1} \, .$$

Similarly,

$$\int_a^b |g(x)|\,|f(x)+g(x)|^{p-1}\,dx \le \|g\|_p \,\|f+g\|_p^{p-1}.$$

Therefore

$$\|f+g\|_p^p \le \left(\|f\|_p + \|g\|_p\right)\|f+g\|_p^{p-1}.$$

If $\|f+g\|_p = 0$, then there is nothing to prove. Otherwise divide by $\|f+g\|_p^{p-1}$ to obtain

$$\|f+g\|_p \le \|f\|_p + \|g\|_p.$$

Thus $\|\cdot\|_p$ is a norm on $C[a,b]$ for every $1 \le p < \infty$.

Case 2: $p = \infty$**.** Now let

$$\|f\|_\infty = \max_{x\in[a,b]} |f(x)|.$$

Because f is continuous on the closed interval $[a,b]$, the Extreme Value Theorem implies that $|f|$ attains a maximum, so $\|f\|_\infty$ is well defined.

Positive definiteness. Clearly $\|f\|_\infty \ge 0$. Also,

$$\|f\|_\infty = 0 \iff |f(x)| = 0 \text{ for all } x\in[a,b] \iff f = 0.$$

Absolute homogeneity. For $c \in \mathbb{R}$,

$$\|cf\|_\infty = \max_{x\in[a,b]} |cf(x)| = |c| \max_{x\in[a,b]} |f(x)| = |c|\,\|f\|_\infty.$$

Triangle inequality. For each $x \in [a,b]$,

$$|f(x)+g(x)| \le |f(x)| + |g(x)| \le \|f\|_\infty + \|g\|_\infty.$$

Taking the maximum over $x \in [a,b]$ gives

$$\|f+g\|_\infty \le \|f\|_\infty + \|g\|_\infty.$$

This completes the proof. □

Why the Riemann integral is not enough for L^p

At first glance, one might wonder why we do not simply define $L^p[a,b]$ to be the set of all functions $f : [a,b] \to \mathbb{R}$ for which

$$\int_a^b |f(x)|^p\,dx$$

exists as a Riemann integral.

The problem is that the class of Riemann integrable functions is too small and too fragile for this purpose.

First issue: many useful functions are not Riemann integrable. A bounded function on $[a, b]$ is Riemann integrable if and only if its set of discontinuities has measure zero. So there are many functions for which $|f|^p$ should be integrable in a reasonable sense, but which are not Riemann integrable. For example, the characteristic function of the rational numbers in $[0, 1]$,

$$\chi_{\mathbb{Q}\cap[0,1]}(x) = \begin{cases} 1, & x \in \mathbb{Q}, \\ 0, & x \notin \mathbb{Q}, \end{cases}$$

is not Riemann integrable, because it is discontinuous at every point. However, from the Lebesgue point of view it is perfectly natural to assign it integral 0, since the rationals have measure zero.

Second issue: L^p is meant to ignore changes on sets of measure zero. In L^p theory, functions that differ only on a set of measure zero are regarded as essentially the same. The Lebesgue integral is built to respect that idea. The Riemann integral is much less well suited to it.

For instance, consider

$$f(x) = 0 \quad \text{for all } x \in [0, 1],$$

and

$$g(x) = \begin{cases} 1, & x \in \mathbb{Q} \cap [0, 1], \\ 0, & x \notin \mathbb{Q}. \end{cases}$$

These functions differ only on the rationals, which form a set of measure zero. In L^p theory they represent the same element. But g is not Riemann integrable, so a framework based only on the Riemann integral cannot even accommodate this basic example properly.

Third issue: completeness fails in the Riemann world. One of the main reasons L^p spaces are so important is that they are complete: every Cauchy sequence converges in the space. If one tried to build a space using only Riemann integrable functions, this completeness property would fail. Limits of sequences of Riemann integrable functions can easily produce functions that are not Riemann integrable, even though they are perfectly acceptable from the Lebesgue point of view.

A comment about L^p

For $1 \leq p < \infty$, the Lebesgue-space version of the p-norm is

$$\|f\|_p = \left(\int |f(x)|^p \, dx \right)^{1/p},$$

where the integral is understood in the Lebesgue sense.

Almost all of the proof above still works in that setting: nonnegativity, absolute homogeneity, Hölder's inequality, and the triangle inequality all go through.

The subtle point is *positive definiteness*. For continuous functions on $[a, b]$, we proved that

$$\int_a^b |f(x)|^p \, dx = 0$$

implies $f = 0$, because continuity forces a nonzero value at one point to persist on a whole interval.

For arbitrary functions, this is false. A function can be nonzero on a set of measure zero and still have integral 0. For example,

$$f(x) = \begin{cases} 1, & x = 0, \\ 0, & x \neq 0, \end{cases}$$

is not the zero function, but

$$\int_0^1 |f(x)|^p \, dx = 0.$$

So the formula does not define a norm on the set of all functions.

Instead, in L^p theory one identifies functions that are equal *almost everywhere*, meaning equal except on a set of measure zero. The space L^p is the set of equivalence classes of such functions. On that space, the formula

$$\|f\|_p = \left(\int |f(x)|^p \, dx \right)^{1/p}$$

really is a norm.

So the picture is:

- on $C[a, b]$, the p-norm is a norm on actual functions;
- on L^p, the same formula becomes a norm only after identifying functions that are equal almost everywhere.

Remark. As in the previous settings, the restriction $p \geq 1$ is essential. For $0 < p < 1$, the formula

$$\left(\int_a^b |f(x)|^p \, dx \right)^{1/p}$$

does not satisfy the triangle inequality, so it is not a norm.